Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 1, 1–21; https://doi.org/10.14232/ejqtde.2021.1.1 www.math.u-szeged.hu/ejqtde/

Existence, uniqueness and qualitative properties of heteroclinic solutions to nonlinear second-order ordinary differential equations

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Received 30 July 2020, appeared 10 January 2021
Communicated by Gennaro Infante

Abstract. By means of the shooting method together with the maximum principle and the Kneser–Hukahara continuum theorem, the authors present the existence, uniqueness and qualitative properties of solutions to nonlinear second-order boundary value problem on the semi-infinite interval of the following type:

$$\begin{cases} y'' = f(x,y,y'), & x \in [0,\infty), \\ y'(0) = A, & y(\infty) = B \end{cases}$$

and

$$\begin{cases} y'' = f(x,y,y'), & x \in [0,\infty), \\ y(0) = A, & y(\infty) = B, \end{cases}$$

where $A, B \in \mathbb{R}$, $f(x,y,z)$ is continuous on $[0,\infty) \times \mathbb{R}^2$. These results and the matching method are then applied to the search of solutions to the nonlinear second-order non-autonomous boundary value problem on the real line

$$\begin{cases} y'' = f(x,y,y'), & x \in \mathbb{R}, \\ y(-\infty) = A, & y(\infty) = B, \end{cases}$$

where $A \neq B$, $f(x,y,z)$ is continuous on $\mathbb{R}^3$. Moreover, some examples are given to illustrate the main results, in which a problem arising in the unsteady flow of power-law fluids is included.

Keywords: semi-infinite interval, heteroclinic solution, shooting method, maximum principle, Kneser–Hukahara continuum theorem, matching method.

2020 Mathematics Subject Classification: 34B40, 34C37.

1 Introduction

The study of heteroclinic solutions for second-order ordinary differential equations can be applied to various biological, physical, and chemical models, for instance, phase-transition,

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physical processes in which the variable transits from an unstable equilibrium to a stable one, or front-propagation in reaction-diffusion equations, and has been intensively studied by many authors, see [6,16,28–31,34,38,42,44] and references therein. In particular, we mention that in [29], by means of a suitable fixed point technique, Malaguti and Marcelli proved the existence of a one-parameter family of solutions of the nonautonomous problem

$$\begin{cases} u'' = h(t,u,u') & \text{on } \mathbb{R}, \\ u(-\infty) = 0, & u(\infty) = 1, \end{cases}$$

where $h : \mathbb{R}^3 \to \mathbb{R}$ is continuous, and $h(t,u,v)/v$ is monotone nondecreasing in v for each $(t,u) \in \mathbb{R} \times (0,1)$.

In [34], Marcelli and Papalini considered the following problem

$$\begin{cases} u'' = f(t,u,u'), & \text{a.e. on } \mathbb{R}, \\ u(-\infty) = 0, & u(\infty) = 1, \end{cases}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function satisfying the condition $f(t,0,0) = f(t,1,0) = 0$ for a.e. $t \in \mathbb{R}$. Under suitable assumptions on f, the authors proved some existence and non-existence results for the problem which become operative criteria in the case that the function $f(t,u,u')$ has a product structure.

In [31], deriving from the comparison-type theory, Malaguti et al. obtained the expressive sufficient conditions for the solvability of the following problem

$$\begin{cases} u'' = f(t,u,u') & \text{on } \mathbb{R}, \\ x(-\infty) = 0, & x(\infty) = 1, \\ 0 \le u(t) \le 1 & \text{for } t \in \mathbb{R}, \end{cases}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous, $f(t,0,0) = f(t,1,0) = 0$ for $t \in \mathbb{R}$.

In recent years, due to the applications in various sciences, heteroclinic solutions of second-order ordinary differential equations governed by nonlinear differential operators, such as the classical p-Laplacian, Φ-Laplacian, singular Φ-Laplacian and some mixed differential operators, received more attractions see [8–11,13,14,25,32,33,35] and references therein. The main tools used in these works are the upper and lower solution method together with diagonalization process, and the fixed point theorem in cone.

Inspired by the above works and [19,39], the main aim of the present paper is to establish the new results on the existence, uniqueness, and qualitative properties of heteroclinic solutions to nonlinear second-order ordinary differential equations

$$y'' = f(x,y,y') \quad \text{on } \mathbb{R} \tag{1.1}$$

by the matching method, where $f(x,y,z)$ is continuous on $\mathbb{R}^3$. To this end, we needs to consider the following second-order semi-infinite interval problems

$$\begin{cases} y'' = f(x,y,y') & \text{on } [0,\infty), \\ y'(0) = A, & y(\infty) = B, \end{cases} \tag{1.2}$$

and

$$\begin{cases} y'' = f(x,y,y') & \text{on } [0,\infty), \\ y(0) = A, & y(\infty) = B, \end{cases} \tag{1.3}$$

where $A, B \in \mathbb{R}$, $f(x, y, z)$ is continuous on $[0, \infty) \times \mathbb{R}^2$.

Second-order semi-infinite interval problems arise in the modeling of a great variety of physical phenomena such as the unsteady flow of a gas through semi-infinite porous medium, the heat transfer in radial flow between circular disks, plasma physics, the mass transfer on a rotating disk in a non-Newtonian fluid, the travelling waves in reaction-diffusion equations, et cetera [1, 36], and have been studied by many papers, for instance, see [2–5, 7, 9, 12, 15, 17, 18, 21–24, 26, 27, 37, 40, 43, 45, 46] and references therein. Among the above references, the main research methods they used are the fixed point theorems in cones [15, 21, 24, 27, 46], fixed point index theorems in cones [23, 37], upper and lower solutions method [2, 5, 22, 43], diagonalization process [3, 4, 26], variational methods [17, 18], Banach contraction mapping principle [40, 45], shooting method [7], etc.

The paper is organized as follows. In Section 2, we give some preparatory lemmas, including maximum principle, Kneser–Hukahara continuum theorem, comparison principle, continuum result and global existence of initial value problems for equation (1.1). In Section 3, using shooting method together with maximum principle and Kneser–Hukahara continuum theorem, we obtain the existence, uniqueness and qualitative properties of solutions to semi-infinite interval problems (1.2) and (1.3). In Section 4, by matching techniques we establish new results on existence, uniqueness and qualitative properties of solutions of full-infinite interval problem

$$\begin{cases} y'' = f(x, y, y') & \text{on } \mathbb{R}, \\ y(-\infty) = A, & y(\infty) = B, \end{cases} \tag{1.4}$$

where $A \neq B$. In Section 5, we demonstrate the importance of our results through some illustrative examples, which contain a problem that arises in the unsteady flow of power-law fluids.

To the best of our knowledge, the results presented in this paper are new. Compared with the recent results, we obtain not only the existence and uniqueness of the heteroclinic solutions, but also the monotonicity, convex-concave property, and asymptotic properties of the heteroclinic solutions, which are rarely considered in the literature. Moreover, the hypotheses used in this paper are different from those in recent literature, for instance, our monotonicity condition is different from those in [28, 29]. It is worth to note that one important feature of our work is that the nonlinearity $f(x, y, z)$ in Theorem 4.5 may be super-quadratic with respect to z, which are not studied by [13, 14, 32, 33, 35]. In addition, our Theorem 3.4 for problem (1.2) complements theorem 4.2 in [7].

2 Some preliminaries

In this section, as preliminaries we shall present some lemmas, which are useful in the proof of our main results.

Throughout this paper we shall use the following conditions:

(H_1) $f(x, y, z)$ is continuous on $I \times \mathbb{R}^2$;

(H_2) $f(x, y, z)$ is nondecreasing in y for each fixed pair $(x, z) \in I \times \mathbb{R}$;

(H_3) $f(x, y, z)$ satisfies a uniform Lipschitz condition on each compact subset of $I \times \mathbb{R}^2$ with respect to z, i.e., for each compact subset $E \subset I \times \mathbb{R}^2$, there exists a constant $L_E > 0$ such that

$$|f(x, y, z_1) - f(x, y, z_2)| \leq L_E |z_1 - z_2|, \quad \forall (x, y, z_1), (x, y, z_2) \in E;$$

(H_4) $zf(x,y,z) \le 0$ for $(x,y,z) \in I \times \mathbb{R}^2$,

where $I = [0,b](b > 0)$ or $[0,\infty)$.

Lemma 2.1 (Maximum principle [41]). *Let $u = u(x)$ be a nonconstant solution of the differential inequality*

$$u'' + \alpha(x)u' + \beta(x)u \ge 0 \quad \text{in } J = (a,b),$$

where $\alpha(x)$ and $\beta(x)$ are bounded function in J, and $\beta(x) \le 0$ in J. Then a nonnegative maximum of $u = u(x)$ can only occur on ∂J, and the outward derivative $\frac{du}{dn} > 0$ there.

Lemma 2.2 ([7]). *Assume f satisfies assumptions (H_1), (H_2) and (H_3) with $I = [0,b]$. Suppose $\phi_1(x), \phi_2(x)$ have continuous second derivatives on an interval $[a_1,b_1) \subset I$ and satisfy*

$$\begin{aligned} \phi_1''(x) &\le f(x,\phi_1(x),\phi_1'(x)), \quad a_1 \le x < b_1; \\ \phi_2''(x) &\ge f(x,\phi_2(x),\phi_2'(x)), \quad a_1 \le x < b_1. \end{aligned}$$

Suppose further that

$$\phi_1(a_1) \le \phi_2(a_1), \quad \phi_1'(a_1) \le \phi_2'(a_1)$$

and

$$\phi_1(a_1) + \phi_1'(a_1) < \phi_2(a_1) + \phi_2'(a_1).$$

Then

$$\phi_1'(x) \le \phi_2'(x), \quad \phi_1(x) \le \phi_2(x) \quad \text{for } a_1 \le x < b_1.$$

Lemma 2.3 ([7]). *Suppose f satisfies assumptions (H_1), (H_2), (H_3) and (H_4) with $I = [0,b]$. Then every solution $\phi(x)$ of the initial value problem*

$$\begin{cases} y'' = f(x,y,y'), & 0 \le x \le b, \\ y(0) = y_0, & y'(0) = y_1 \end{cases}$$

can be continued to the entire interval $[0,b]$.

Lemma 2.4 (Kneser–Hukahara Continuum Theorem [20]). *Consider the system $y' = f(x,y), y \in \mathbb{R}^n$. Suppose that the function $f(x,y)$ is continuous and bounded on $D = \{(x,y) : a \le x \le b, y \in \mathbb{R}^n\}$. Let C be a compact and connected subset of D and $\mathfrak{F}(C)$ be the set of solutions which start in C. Then $\mathfrak{F}(C)$ is a compact and connected subset of $C([a,b],\mathbb{R}^n)$.*

Consider the following initial value problems

$$\begin{cases} y'' = f(x,y,y'), & 0 \le x \le b, \\ y(0) = \lambda, & y'(0) = A \end{cases} \tag{IVP$_0(\lambda)$}$$

and

$$\begin{cases} y'' = f(x,y,y'), & 0 \le x \le b, \\ y(0) = A, & y'(0) = \lambda. \end{cases} \tag{IVP$_1(\lambda)$}$$

Now, we introduce some notations:

$$\mathfrak{F}_0 := \{\phi : \phi(x) \text{ is a solution of } \mathrm{IVP}_0(\lambda), \lambda \in \mathbb{R}\}$$

and

$$\mathfrak{F}_1 := \{\phi : \phi(x) \text{ is a solution of } \mathrm{IVP}_1(\lambda), \lambda \in \mathbb{R}\}.$$

Lemma 2.5. *Suppose that* (H_1)*,* (H_2)*,* (H_3) *and* (H_4) *with* $I = [0, b]$ *hold. Let* $\lambda_1, \lambda_2 \in \mathbb{R}$ *with* $\lambda_1 < \lambda_2$*. Then*

$$F_0 = \{\phi \in \mathfrak{F}_0 : \lambda_1 \le \lambda \le \lambda_2\}$$

is a compact and connected subset of $C^1[0, b]$*.*

Proof. Let $y_0 = y, y_1 = y_0'$. Then $\mathrm{IVP}_0(\lambda)$ is equivalent to the following initial value problem of system

$$\begin{cases} \dfrac{dY}{dx} = G(x, y_0, y_1), \\ Y(0) = (\lambda, A), \end{cases} \tag{2.1}$$

where $Y = (y_0, y_1)$, $G(x, y_0, y_1) = (y_1, f(x, y_0, y_1))$. Consider a set of solutions of (2.1), denoted by S as follows:

$$S := \{(y_0(x, \lambda), y_1(x, \lambda)) : \lambda_1 \le \lambda \le \lambda_2\}.$$

From Lemma 2.2 and 2.3, for $\lambda_1 \le \lambda \le \lambda_2$ and $i = 0, 1$, we have

$$y_i(x, \lambda_1 - 1) \le y_i(x, \lambda) \le y_i(x, \lambda_2 + 1) \quad \text{on } [0, b],$$

then there exists $M > 0$ such that

$$|y_i(x, \lambda)| \le M, \quad i = 0, 1, \ (x, \lambda) \in [0, b] \times [\lambda_1, \lambda_2].$$

Let

$$H := \{(x, y_0, y_1) : 0 \le x \le b, |y_i| \le M + 1, i = 0, 1\}.$$

Then $G(x, y_0, y_1)$ is continuous and bounded on H, and can be extended to a bounded continuous function $G^*(x, y_0, y_1)$ on $D = [0, b] \times \mathbb{R}^2$ such that

$$G^*(x, y_0, y_1) \equiv G(x, y_0, y_1) \quad \text{for } (x, y_0, y_1) \in H.$$

Now, we consider an initial value problem of system

$$\begin{cases} \dfrac{dY}{dx} = G^*(x, y_0, y_1), \\ Y(0) = (\lambda, A). \end{cases} \tag{2.2}$$

We note that

$$C := \{(0, \lambda, A) : \lambda_1 \le \lambda \le \lambda_2\}$$

is a compact and connected subset of D, and then by Lemma 2.4 the set of solutions of initial value problem of system (2.2)

$$\mathfrak{F}_0(C) := \{(y_0(x, \lambda), y_1(x, \lambda)) : \lambda_1 \le \lambda \le \lambda_2\}$$

is a compact and connected subset of $C([0, b], \mathbb{R}^2)$. Since $\mathfrak{F}_0(C) = S$, it follows that F_0 is a compact and connected subset of $C^1[0, b]$. This completes the proof of the lemma. □

The following lemma can be readily obtained by using Lemma 2.2 and 2.4.

Lemma 2.6. *Suppose that* (H_1)*,* (H_2)*,* (H_3) *and* (H_4) *with* $I = [0, b]$ *hold. Let* $\lambda_1, \lambda_2 \in \mathbb{R}$ *with* $\lambda_1 < \lambda_2$*. Then*

$$F_1 = \{\phi \in \mathfrak{F}_1 : \lambda_1 \le \lambda \le \lambda_2\}$$

is a compact and connected subset of $C^1[0, b]$*.*

Lemma 2.7 ([7])**.** *Suppose f satisfies assumptions* (H_1)*,* (H_2)*,* (H_3) *and* (H_4) *with $I = [0,\infty)$. Suppose also that*

(H_5) *there exist constants γ, r, ρ, M_1, K for which $\gamma \geq 0$, $0 \leq r < \gamma + 1$, $\rho \geq 1$, $\gamma > \rho - 2$, $M_1 > 0$, $K > 0$, and*

$$|f(x,y,z)| \geq \frac{M_1 x^\gamma |z|^\rho}{|y|^r} \quad \text{for } |y| \geq K,\ (x,z) \in [0,\infty) \times \mathbb{R}.$$

Then every solution of the initial value problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x < \infty, \\ y(0) = y_0, & y'(0) = y_1 \end{cases}$$

can be continued to the entire interval $[0,\infty)$. Moreover, this global solution $\phi(x)$ is bounded and monotone and hence $\lim_{x\to\infty} \phi(x)$ exists and is finite.

3 Semi-infinite interval problems

In this section, we begin with the study of the finite interval case for problem (1.2) and (1.3) by shooting method.

Theorem 3.1. *Suppose that* (H_1)*,* (H_2)*,* (H_3) *and* (H_4) *with $I = [0,b]$ hold. Then the finite interval problem*

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x \leq b, \\ y'(0) = A, & y(b) = B \end{cases} \tag{3.1}$$

has a unique solution.

Proof. **Existence.** Let $\phi(x,\lambda)$ be a solution of the initial value problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x \leq b, \\ y(0) = \lambda, & y'(0) = A. \end{cases}$$

Then, by Lemma 2.3, $\phi(x,\lambda)$ can be extended to the entire interval $[0,b]$. From Lemma 2.2, it follows that

$$\phi'(x,\lambda) \leq \phi'(x,0) \quad \text{for } \lambda < 0$$

and

$$\phi(b,\lambda) - \phi(b,0) = \lambda + \int_0^b (\phi'(x,\lambda) - \phi'(x,0))\mathrm{d}x \leq \lambda.$$

Therefore

$$\phi(b,\lambda) \to -\infty \quad \text{as } \lambda \to -\infty.$$

Hence, there exists $\lambda_1 < 0$ such that $\phi(b,\lambda_1) < B$. Similarly, there exists $\lambda_2 > 0$ such that $\phi(b,\lambda_2) > B$.

From Lemma 2.5, the set

$$F_0 = \{\phi(x,\lambda) \in \mathfrak{F}_0 : \lambda_1 \leq \lambda \leq \lambda_2\}$$

is a compact and connected subset of $C^1[0,b]$.

Now, we define a mapping $T : F_0 \to \mathbb{R}$ as follows:

$$T(\phi(x,\lambda)) = \phi(b,\lambda) - B, \quad \forall \phi(x,\lambda) \in F_0.$$

Then T is continuous on F_0. Since $T(\phi(b,\lambda_1)) < 0$ and $T(\phi(b,\lambda_2)) > 0$, from Bolzano's theorem there exists $\phi(x,\lambda^*) \in F_0$ such that

$$T(\phi(x,\lambda^*)) = \phi(b,\lambda^*) - B = 0,$$

that is, $\phi(b,\lambda^*) = B$. Obviously, $\phi(x,\lambda^*)$ is a solution of problem (3.1).

Uniqueness. Suppose $\phi_1(x), \phi_2(x)$ are solutions of problem (3.1). We consider two cases.

Case 1. $\phi_2(x) - \phi_1(x)$ is a constant on $[0,b]$. In this case, since $\phi_2(b) = \phi_1(b)$, we have $\phi_2(x) \equiv \phi_1(x)$ on $[0,b]$.

Case 2. $\phi_2(x) - \phi_1(x)$ is not a constant on $[0,b]$. In this case, since $\phi_2(b) = \phi_1(b)$, there exists $x_1 \in [0,b)$ such that $\phi_2(x_1) \neq \phi_1(x_1)$. Without loss of generality, we assume that $\phi_2(x_1) > \phi_1(x_1)$. Then there exists $x_2 \in [0,b)$ such that

$$\phi_2(x_2) - \phi_1(x_2) = \max_{x\in[0,b]} (\phi_2(x) - \phi_1(x)) > 0.$$

From the condition $\phi_2'(0) = \phi_1'(0)$, it follows that

$$\phi_2'(x_2) = \phi_1'(x_2).$$

Also since $\phi_2(b) = \phi_1(b)$, there exists $x_3 \in (x_2, b]$ such that

$$\phi_2(x_3) - \phi_1(x_3) = 0, \quad \phi_2(x) - \phi_1(x) > 0, \quad x \in [x_2, x_3).$$

Now, let $\psi(x) = \phi_2(x) - \phi_1(x)$. Then, it is easy to check that $\psi(x)$ is a solution of the differential inequality

$$u'' + \alpha(x)u' + \beta(x)u \geq 0 \quad \text{in } J = (x_2, x_3),$$

where

$$\alpha(x) = \begin{cases} -\dfrac{f(x,\phi_2(x),\phi_2'(x)) - f(x,\phi_2(x),\phi_1'(x))}{\phi_2'(x) - \phi_1'(x)}, & \phi_2'(x) \neq \phi_1'(x); \\ 0, & \phi_2'(x) = \phi_1'(x), \end{cases}$$

and

$$\beta(x) = -\frac{f(x,\phi_2(x),\phi_1'(x)) - f(x,\phi_1(x),\phi_1'(x))}{\phi_2(x) - \phi_1(x)}.$$

Obviously, assumptions (H_1), (H_2) and (H_3) guarantees that $\alpha(x), \beta(x)$ are bounded on (x_2, x_3) and $\beta(x) \leq 0$ on (x_2, x_3). Therefore, by Lemma 2.1 the positive maximum of $\psi(x)$ can only occur on $\partial J = \{x_2, x_3\}$ and $\frac{d\psi}{dn} > 0$ there. Since $\psi(x_3) = 0$, the maximum must occur at x_2 and $\frac{d\psi}{dn}\big|_{x=x_2} = -\psi'(x_2) > 0$, i.e., $\psi'(x_2) < 0$, which is a contradiction to $\psi'(x_2) = \phi_2'(x_2) - \phi_1'(x_2) = 0$.

In summary, $\phi_2(x) \equiv \phi_1(x)$ on $[0,b]$. This completes the proof of the theorem. $\square$

Theorem 3.2. *Suppose that* (H_1), (H_2), (H_3) *and* (H_4) *with* $I = [0,b]$ *hold. Suppose also that*

(H_6) *f satisfies the uniform Nagumo condition on $[0,\infty)\times\mathbb{R}$, i.e., for each compact subset $E\subset[0,\infty)\times\mathbb{R}$, there exists a continuous function $h_E:[0,\infty)\to(0,\infty)$ with $\int_0^\infty \frac{s}{h_E(s)}\mathrm{d}s=\infty$ such that*

$$|f(x,y,z)|\le h_E(|z|),\quad \forall(x,y,z)\in E\times\mathbb{R}.$$

Then the finite interval problem

$$\begin{cases} y''=f(x,y,y'), & 0\le x\le b,\\ y(0)=A, & y(b)=B \end{cases} \tag{3.2}$$

has a unique solution.

Proof. If $A=B$, then from (H_4), $\phi(x)\equiv A$ is a solution of problem (3.2). Without loss of generality, we assume that $A<B$. Let $\phi(x,\lambda)$ be a solution of the initial value problem

$$\begin{cases} y''=f(x,y,y'), & 0\le x\le b,\\ y(0)=A, & y'(0)=\lambda. \end{cases}$$

Then, by Lemma 2.3, $\phi(x,\lambda)$ can be extended to the entire interval $[0,b]$. Furthermore, by Lemma 2.2, for each $\lambda>0$, $\phi(x,\lambda)$ is monotone nondecreasing on $[0,b]$. Let

$$\Sigma=\{\phi(b,\lambda):\lambda\in(0,\infty)\}.$$

We assert that $\sup\Sigma>B$. Indeed, suppose by contradiction, that $\sup\Sigma\le B$. Then there exists $R>0$ such that for each $\lambda\in(0,\infty)$,

$$\phi'(x,\lambda)\le R,\quad \forall x\in[0,b].$$

In fact, let $\eta=B-A>0$ and take $r>\eta/b$ such that

$$\int_{\eta/b}^{r}\frac{s}{h_E(s)}\mathrm{d}s\ge B-A,$$

where $E=[0,b]\times[A,B]$. If $\phi'(x,\lambda)>\eta/b$ on $[0,b]$, we get the following contradiction:

$$\eta\ge\phi(b,\lambda)-\phi(0,\lambda)=\int_0^b\phi'(x,\lambda)\mathrm{d}x>\eta.$$

Thus there exists $x_0\in[0,b]$ such that $\phi'(x_0,\lambda)\le\eta/b$. If $\phi'(x,\lambda)\le\eta/b$ on $[0,b]$, it is enough to take $R:=\eta/b$ to finish the proof. Suppose that there exist some $x\in[0,b]$ such that $\phi'(x,\lambda)>\eta/b$. Then by (H_4), for $\lambda>0$, $\phi''(x,\lambda)\le0$ on $[0,b]$. Consider an interval $[x_2,x_1]$ such that $\phi'(x,\lambda)\ge\eta/b$ on $[x_2,x_1]$, $\phi'(x_1,\lambda)=\eta/b$ and $\phi'(x,\lambda)>\eta/b>0$ for every $x\in[x_2,x_1)$. Applying a convenient change of variable, by the fact that $\phi(x,\lambda)$ is monotone nondecreasing on $[0,b]$, we have

$$\begin{aligned}\int_{\phi'(x_1,\lambda)}^{\phi'(x_2,\lambda)}\frac{s}{h_E(s)}\mathrm{d}s&=\int_{x_1}^{x_2}\frac{\phi'(x,\lambda)}{h_E(\phi'(x,\lambda))}\phi''(x,\lambda)\mathrm{d}x\\&=\int_{x_1}^{x_2}\frac{\phi'(x,\lambda)}{h_E(\phi'(x,\lambda))}f(x,\phi(x,\lambda),\phi'(x,\lambda))\mathrm{d}x\\&\le\int_{x_2}^{x_1}\phi'(x,\lambda)\mathrm{d}x=\phi(x_1,\lambda)-\phi(x_2,\lambda)\\&\le\sup\Sigma-A\le\int_{\eta/b}^{r}\frac{s}{h_E(s)}\mathrm{d}s.\end{aligned}$$

Then $\phi'(x_2,\lambda) \le r$ and, by the way as x_1 and x_2 were taken, we have

$$\phi'(x,\lambda) \le r =: R, \quad \forall x \in [0,b],$$

which contradicts $\phi'(0,\lambda) = \lambda \to \infty$ as $\lambda \to \infty$.

In summary, $\sup \Sigma > B$. Therefore there exists $\lambda_1 > 0$ such that $\phi(b,\lambda_1) > B$. Notice that $A < B$, it is clear from Lemma 2.2 that $\phi(b,\lambda_2) < B$ for each $\lambda_2 < 0$.

The remaining part is similar to the proof of Theorem 3.1, therefore it is omitted here. This completes the proof of the theorem. □

Remark 3.3. It is easy to see that if $f(x,y,z)$ satisfies a uniform σ-Lipschitz condition on each compact subset of $I \times \mathbb{R}$ with respect to z, that is, for each compact subset E of $[0,\infty) \times \mathbb{R}$, there exists $L_E > 0$ which depends only on E, such that

$$|f(x,y,z_1) - f(x,y,z_2)| \le L_E |z_1 - z_2|^\sigma, \quad \forall (x,y,z_1),(x,y,z_2) \in E \times \mathbb{R},$$

where $0 < \sigma \le 2$, then f satisfies the condition (H_6).

Now, using Theorem 3.1 and some lemmas in Section 2, we establish here our main results for semi-infinite interval problem (1.2).

Theorem 3.4. *Suppose that* (H_1), (H_2), (H_3), (H_4) *with* $I = [0,\infty)$ *and* (H_5) *hold. Then the semi-infinite interval problem* (1.2) *has a unique solution* $y = \phi(x)$ *satisfying*

(1) *if* $A \ge 0$, *then* $\phi(x)$ *is monotone nondecreasing, concave on* $[0,\infty)$ *and* $\lim_{x\to\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonpositive on* $[0,\infty)$ *when* $B \le 0$;

(2) *if* $A \le 0$, *then* $\phi(x)$ *is monotone nonincreasing, convex on* $[0,\infty)$ *and* $\lim_{x\to\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative on* $[0,\infty)$ *when* $B \ge 0$.

Proof. Firstly, we show the existence of solutions of problem (1.2). Clearly, if $A = 0$, then $\phi(x) \equiv B$ is the solution of problem (1.2). Without loss of generality, we assume that $A > 0$. Then by Theorem 3.1, the finite interval problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \le x \le 1, \\ y'(0) = A, & y(1) = B+1 \end{cases} \tag{3.3}$$

has a unique solution $y = \psi(x)$ on $[0,1]$, and which by Lemma 2.7 can be continued to the entire interval $[0,\infty)$ as a monotone solution of (1.1). Since $\psi'(0) = A > 0$, it follows that $\psi(x)$ is monotone nondecreasing on $[0,\infty)$. Thus from Lemma 2.7, we know that $\psi(\infty) := \lim_{x\to\infty} \psi(x)$ exists, and $\psi(\infty) > B$.

Suppose by contradiction, that problem (1.2) has no solution. Let

$$G = \{\phi \in C^2[0,\infty) : \phi(x) \text{ is solution of (1.1) with } \phi'(0) = A,\ \phi(\infty) < B\}.$$

Then $G \ne \emptyset$. In fact, let $\phi(x,\lambda)$ be a solution of initial value problem

$$\begin{cases} y'' = f(x,y,y'), \\ y(0) = \lambda, \quad y'(0) = A. \end{cases}$$

Then, by Lemma 2.7, $\phi(x,\lambda)$ can be continued to the entire interval $[0,\infty)$ and

$$\phi(\infty,\lambda) = \lim_{x\to\infty} \phi(x,\lambda) < \infty.$$

If $\phi(\infty,0) < B$, then $\phi(x,0) \in G$, and thus $G \neq \emptyset$. If $\phi(\infty,0) > B$, then it follows from Lemma 2.2 that for $\lambda < 0$,

$$\phi'(x,\lambda) \leq \phi'(x,0), \quad \phi(x,\lambda) \leq \phi(x,0), \quad x \in [0,\infty).$$

Hence for each $\lambda < 0$ we have

$$\phi(x,\lambda) - \phi(x,0) = \lambda + \int_0^x (\phi'(t,\lambda) - \phi'(t,0))\mathrm{d}t \leq \lambda, \quad x \in [0,\infty).$$

At the limit, as $x \to \infty$, we obtain

$$\phi(\infty,\lambda) - \phi(\infty,0) \leq \lambda, \quad \lambda < 0,$$

i.e.,

$$\phi(\infty,\lambda) \leq \phi(\infty,0) + \lambda, \quad \lambda < 0.$$

Since $\phi(\infty,0) + \lambda \to -\infty$ as $\lambda \to -\infty$, it follows that there exists $\bar{\lambda} < 0$ such that

$$\phi(\infty,\bar{\lambda}) \leq \phi(\infty,0) + \bar{\lambda} < B.$$

Therefore $\phi(x,\bar{\lambda}) \in G$, and thus $G \neq \emptyset$.

Now, let

$$\Theta = \{\lambda = \phi(0) : \phi \in G\}.$$

Notice that for each $\phi \in G$,

$$\phi(x) \leq \psi(x), \quad \phi'(x) \leq \psi'(x), \quad x \in [0,\infty),$$

then Θ is upper bounded, and $\lambda^* := \sup \Theta < \infty$. Hence there exists $\{\lambda_n\} \subset \Theta$ such that $\lambda_n < \lambda_{n+1} < \lambda^*$ and $\lambda_n \to \lambda^*$ as $n \to \infty$. From Lemma 2.2, for $\phi(x,\lambda_n) \in G, n = 1,2,\ldots,$

$$\phi^{(i)}(x,\lambda_n) \leq \phi^{(i)}(x,\lambda_{n+1}) \leq \phi^{(i)}(x,\lambda^*), \quad i = 0,1,\ x \in [0,\infty).$$

Let $\hat{\phi}(x) = \sup_n \phi(x,\lambda_n)$. Since for each fixed positive number b, the sequence of functions $\{\phi^{(i)}(x,\lambda_n)\}$ $(i = 0,1)$ is equicontinuous on $[0,b]$, then

$$\phi^{(i)}(x,\lambda_n) \to \hat{\phi}^{(i)}(x) \quad (n \to \infty) \quad \text{uniformly on } [0,b],\ i = 0,1.$$

It follows that $\hat{\phi}(x)$ is a solution of (1.1) satisfying $\hat{\phi}'(0) = A$ and $\hat{\phi}(\infty) \leq B$. From the assumption that semi-infinite interval problem (1.2) has no solution, we have $\hat{\phi}(\infty) < B$.

Next, we show that there exists $\check{\phi} \in G$ such that

$$\hat{\phi}(\infty) < \check{\phi}(\infty) < B, \tag{3.4}$$

and thus obtain a contradiction. To do this, choose $b \geq 1$ sufficiently large such that

$$\psi(\infty) - \psi(b) < \frac{1}{2}(B - \hat{\phi}(\infty)).$$

Then by Theorem 3.1, the finite interval problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x \leq b, \\ y'(0) = A, & y(b) = (B + \hat{\phi}(\infty))/2 \end{cases} \tag{3.5}$$

has a unique solution $\check{\phi}(x)$, which by Lemma 2.7 can be continued to $[0,\infty)$ as a monotone nondecreasing solution of (1.1). Thus from (3.5) and (3.3) we obtain

$$\check{\phi}(1) \leq \check{\phi}(b) < B < \psi(1).$$

It follows from Lemma 2.2 that

$$\check{\phi}'(x) \leq \psi'(x), \quad \forall x \in [b,\infty) \subset [1,\infty).$$

Therefore

$$\begin{aligned}
\check{\phi}(\infty) &= \check{\phi}(b) + \int_b^\infty \check{\phi}'(x)\mathrm{d}x \\
&\leq \check{\phi}(b) + \int_b^\infty \psi'(x)\mathrm{d}x \\
&= \frac{1}{2}(B + \hat{\phi}(\infty)) + \psi(\infty) - \psi(b) \\
&< \frac{1}{2}(B + \hat{\phi}(\infty)) + \frac{1}{2}(B - \hat{\phi}(\infty)) = B.
\end{aligned}$$

Also from (3.5) and $\hat{\phi}(\infty) < B$, it follows that $\check{\phi}(b) > \hat{\phi}(\infty)$, then by the monotonicity of $\check{\phi}(x)$ on $[0,\infty)$, we have $\check{\phi}(\infty) \geq \check{\phi}(b) > \hat{\phi}(\infty)$, and so $\check{\phi}(x)$ satisfies (3.4).

Secondly, we show the uniqueness of solutions of problem (1.2). To do this, let $\phi_1(x), \phi_2(x)$ be solutions of problem (1.2). We consider two cases to prove.

Case 1. $\phi_1(0) \neq \phi_2(0)$. Without loss of generality, we assume that $\phi_1(0) < \phi_2(0)$. Then by Lemma 2.2, $\phi_1'(x) \leq \phi_2'(x)$ on $[0,\infty)$, and thus

$$\phi_2(\infty) - \phi_1(\infty) = \phi_2(0) - \phi_1(0) + \int_0^\infty (\phi_2'(x) - \phi_1'(x))\mathrm{d}x > 0,$$

which contradicts $\phi_2(\infty) = \phi_1(\infty)$.

Case 2. $\phi_1(0) = \phi_2(0)$. In this case, we have $\phi_1(x) \equiv \phi_2(x)$ on $[0,\infty)$. In fact, if not, there exists $x_0 \in (0,\infty)$ such that $\phi_1(x_0) \neq \phi_2(x_0)$. We can assume that $\phi_1(x_0) < \phi_2(x_0)$. Then there exists $x_1 \in [0,x_0)$ such that $\phi_1(x_1) = \phi_2(x_1)$ and $\phi_1(x) < \phi_2(x)$ on $(x_1,x_0]$, and so there exists $x_2 \in (x_1,x_0]$ such that $\phi_1'(x_2) < \phi_2'(x_2)$. It follows from Lemma 2.2 that $\phi_1'(x) \leq \phi_2'(x)$ on $[x_2,\infty)$. Therefore

$$0 = \phi_2(\infty) - \phi_1(\infty) = \phi_2(x_2) - \phi_1(x_2) + \int_{x_2}^\infty (\phi_2'(x) - \phi_1'(x))\mathrm{d}x > 0,$$

which is a contradiction. In summary, $\phi_1(x) \equiv \phi_2(x)$ on $[0,\infty)$.

Finally, the qualitative properties of the unique solution is obvious by Lemma 2.7. This completes the proof of the theorem. □

Theorem 3.5. *Suppose that* (H_1), (H_2), (H_3), (H_4) *with* $I = [0,\infty)$, (H_5) *and* (H_6) *hold. Then the semi-infinite interval problem* (1.3) *has a unique solution* $y = \phi(x)$ *satisfying*

(1) *if* $A \leq B$, *then* $\phi(x)$ *is monotone nondecreasing, concave on* $[0,\infty)$ *and* $\lim_{x\to\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative or nonpositive on* $[0,\infty)$ *when* $A \geq 0$ *or* $B \leq 0$, *respectively;*

(2) *if* $A \geq B$, *then* $\phi(x)$ *is monotone nonincreasing, convex on* $[0,\infty)$ *and* $\lim_{x\to\infty} \phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative or nonpositive on* $[0,\infty)$ *when* $B \geq 0$ *or* $A \leq 0$, *respectively.*

Proof. The proof is the same as that for Theorem 3.4 except that Theorem 3.1 is used in place of Theorem 3.2, and we omitted here. This completes the proof of the theorem. □

4 Heteroclinic solutions

In order to obtain the existence, uniqueness and qualitative properties of solutions for full-infinite interval problem (1.4) via matching technique, we first discuss the existence, uniqueness and qualitative properties of solutions to the following semi-infinite interval problems

$$\begin{cases} y'' = f(x,y,y'), & -\infty < x \le 0, \\ y(-\infty) = A, & y'(0) = \eta \end{cases} \tag{4.1}$$

and

$$\begin{cases} y'' = f(x,y,y'), & -\infty < x \le 0, \\ y(-\infty) = A, & y(0) = \eta, \end{cases} \tag{4.2}$$

where $\eta \in \mathbb{R}$.

Let us list the following conditions for convenience.

($\overline{\mathrm{H}}_1$) $f(x,y,z)$ is continuous on $\mathbb{R}^3$;

($\overline{\mathrm{H}}_2$) $f(x,y,z)$ is nondecreasing in y for each fixed $(x,z) \in \mathbb{R}^2$;

($\overline{\mathrm{H}}_3$) $f(x,y,z)$ satisfies a uniform Lipschitz condition on each compact subset of $\mathbb{R}^3$ with respect to z;

($\overline{\mathrm{H}}_4$) $zf(x,y,z) \ge 0$ for $(x,y,z) \in (-\infty,0] \times \mathbb{R}^2$, and $zf(x,y,z) \le 0$ for $(x,y,z) \in [0,\infty) \times \mathbb{R}^2$;

($\overline{\mathrm{H}}_5$) there exist constants γ, r, ρ, M_1, K for which $\gamma \ge 0$, $0 \le r < \gamma+1$, $\rho \ge 1$, $\gamma > \rho - 2$, $M_1 > 0$, $K > 0$, and

$$|f(x,y,z)| \ge \frac{M_1|x|^\gamma |z|^\rho}{|y|^r} \quad \text{for } |y| \ge K,\ (x,z) \in \mathbb{R}^2;$$

($\overline{\mathrm{H}}_6$) f satisfies the uniform Nagumo condition on $\mathbb{R}^2$, i.e., for each compact subset $E \subset \mathbb{R}^2$, there exists a continuous function $h_E : [0,\infty) \to (0,\infty)$ with $\int_0^\infty \frac{s}{h_E(s)} \mathrm{d}s = \infty$ such that

$$|f(x,y,z)| \le h_E(|z|) \quad \text{for } (x,y,z) \in E \times \mathbb{R};$$

($\overline{\mathrm{H}}_6'$) for each $b > 0$, there exists $M = M(b) > 0$ so that

$$|f(x,y,z)| \le M|x|^q|z|^p \quad \text{for } (x,y,z) \in [-b,b] \times \mathbb{R}^2,$$

where $q \ge 0, p \ge 1, q \ge p-2$.

Theorem 4.1. *Suppose that ($\overline{\mathrm{H}}_1$), ($\overline{\mathrm{H}}_2$), ($\overline{\mathrm{H}}_3$), ($\overline{\mathrm{H}}_4$) and ($\overline{\mathrm{H}}_5$) hold. Then problem (4.1) has a unique solution $y = \phi(x)$ satisfying*

(1) *if $\eta \le 0$, then $\phi(x)$ is monotone nonincreasing, concave on $(-\infty,0]$ and $\lim_{x\to-\infty} \phi'(x) = 0$. Furthermore, $\phi(x)$ is nonpositive on $(-\infty,0]$ when $A \le 0$;*

(2) *if $\eta \ge 0$, then $\phi(x)$ is monotone nondecreasing, convex on $(-\infty,0]$ and $\lim_{x\to-\infty} \phi'(x) = 0$. Furthermore, $\phi(x)$ is nonnegative on $(-\infty,0]$ when $A \ge 0$.*

Proof. Let $x = -t$ and $y(x) = u(t)$. Then problem (4.1) is transformed into an equivalent problem

$$\begin{cases} u'' = F(t,u,u'), & 0 \le t < \infty, \\ u'(0) = -\eta, & u(\infty) = A, \end{cases} \tag{4.3}$$

where $F(t,y,z) = f(-t,y,-z)$. It is easy to check that conditions $(\overline{\mathrm{H}}_1)$, $(\overline{\mathrm{H}}_2)$, $(\overline{\mathrm{H}}_3)$, $(\overline{\mathrm{H}}_4)$ and $(\overline{\mathrm{H}}_5)$ imply conditions (H_1), (H_2), (H_3), (H_4) with $I = [0,\infty)$ and (H_5) hold for problem (4.3). Hence by Theorem 3.4, problem (4.3) has a unique solution $u = \psi(t)$, and thus $\phi(x) = \psi(-x)$ is a unique solution of problem (4.1) and satisfies property (1) and (2). This completes the proof of the theorem. □

Applying Theorem 3.5, we can easily obtain the following.

Theorem 4.2. *Suppose that $(\overline{\mathrm{H}}_1)$, $(\overline{\mathrm{H}}_2)$, $(\overline{\mathrm{H}}_3)$, $(\overline{\mathrm{H}}_4)$, $(\overline{\mathrm{H}}_5)$ and $(\overline{\mathrm{H}}_6)$ hold. Then problem (4.2) has a unique solution $y = \phi(x)$ satisfying*

(1) *if $\eta \le A$, then $\phi(x)$ is monotone nonincreasing, concave on $(-\infty,0]$ and $\lim_{x\to-\infty}\phi'(x) = 0$. Furthermore, $\phi(x)$ is nonnegative or nonpositive on $(-\infty,0]$ when $\eta \ge 0$ or $A \le 0$, respectively;*

(2) *if $\eta \ge A$, then $\phi(x)$ is monotone nondecreasing, convex on $(-\infty,0]$ and $\lim_{x\to-\infty}\phi'(x) = 0$. Furthermore, $\phi(x)$ is nonnegative or nonpositive on $(-\infty,0]$ when $A \ge 0$ or $\eta \le 0$, respectively.*

Proof. The proof is similar to that of Theorem 4.1, and is omitted. This completes the proof of the theorem. □

Remark 4.3. Due to Theorem 4.3 of [7], it is easy to see that with the same hypothesis as in Theorem 4.2, except now $(\overline{\mathrm{H}}_6)$ is replaced by $(\overline{\mathrm{H}}_6')$, the conclusion of Theorem 4.2 is still true.

With the above theorems we may now establish our main result of this section on the existence, uniqueness and qualitative properties of solutions for the full-infinite interval problem (1.4).

Theorem 4.4. *Suppose that $(\overline{\mathrm{H}}_1)$, $(\overline{\mathrm{H}}_2)$, $(\overline{\mathrm{H}}_3)$, $(\overline{\mathrm{H}}_4)$, $(\overline{\mathrm{H}}_5)$ and $(\overline{\mathrm{H}}_6)$ hold. Then problem (1.4) has a unique solution $y = \phi(x)$ satisfying*

(1) *if $A < B$, then $\phi(x)$ is monotone nondecreasing on $\mathbb{R}$, convex on $(-\infty,0]$ concave on $[0,\infty)$ and $\lim_{x\to\pm\infty}\phi'(x) = 0$. Furthermore, $\phi(x)$ is nonnegative (nonpositive) on $\mathbb{R}$ when $A \ge 0$ $(B \le 0)$;*

(2) *if $A > B$, then $\phi(x)$ is monotone nonincreasing on $\mathbb{R}$, concave on $(-\infty,0]$, convex on $[0,\infty)$ and $\lim_{x\to\pm\infty}\phi'(x) = 0$. Furthermore, $\phi(x)$ is nonnegative (nonpositive) on $\mathbb{R}$ when $B \ge 0$ $(A \le 0)$.*

Proof. By Theorem 3.4, for any $\eta \in \mathbb{R}$, the following semi-infinite interval problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \le x < \infty, \\ y'(0) = \eta, & y(\infty) = B \end{cases} \tag{4.4}$$

has a unique solution $\phi_1(x,\eta)$.

First it will be shown that $\phi_1(0,\eta)$ is a continuous and strictly decreasing function of η and its range is the set of all real numbers.

Let $\eta_2 > \eta_1$, then $\phi_1(0,\eta_2) < \phi_1(0,\eta_1)$. Indeed, if $\phi_1(0,\eta_2) \geq \phi_1(0,\eta_1)$, then since $\phi_1'(0,\eta_2) = \eta_2 > \eta_1 = \phi_1'(0,\eta_1)$, it follows from Lemma 2.2 that $\phi_1'(x,\eta_2) \geq \phi_1'(x,\eta_1)$ on $[0,\infty)$. Notice that $\phi_1'(0,\eta_2) > \phi_1'(0,\eta_1)$ and $\phi_1(0,\eta_2) \geq \phi_1(0,\eta_1)$, there exists $x^* > 0$ such that $\phi_1(x^*,\eta_2) > \phi_1(x^*,\eta_1)$, and thus

$$\phi_1(x,\eta_2) - \phi_1(x,\eta_1) \geq \phi_1(x^*,\eta_2) - \phi_1(x^*,\eta_1) > 0 \quad \text{on } [x^*,\infty),$$

which contradicts $\phi_1(\infty,\eta_2) = B = \phi_1(\infty,\eta_1)$. Therefore $\phi_1(0,\eta)$ is a strictly decreasing function of η.

Suppose $\phi_1(0,\eta)$ has a jump discontinuity at $\eta = \eta_1$ such that

$$\phi_1(0,\eta_1^-) = \alpha, \quad \phi_1(0,\eta_1) = \beta, \quad \phi_1(0,\eta_1^+) = \gamma,$$

where the monotonicity asserts that $\alpha \geq \beta \geq \gamma$ and $\alpha > \gamma$. Let $\hat{\beta}$ be a real number different from β such that $\alpha \geq \hat{\beta} \geq \gamma$. Then by Theorem 3.5, the following semi-infinite interval problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x < \infty, \\ y(0) = \hat{\beta}, & y(\infty) = B \end{cases}$$

has a unique solution $y = \phi(x)$. Let $\phi'(0) = \hat{\eta}$. Then by Theorem 3.4, $\phi(x) = \phi_1(x,\hat{\eta})$ for all $x \in [0,\infty)$, and thus

$$\phi_1(0,\hat{\eta}) = \phi(0) = \hat{\beta},$$

which is a contradiction. Thus $\phi_1(0,\eta)$ is a continuous function of η.

Suppose that for all real numbers η, $\phi_1(0,\eta)$ is bounded from above, that is, there exists $M_1 > 0$ such that $\phi_1(0,\eta) \leq M_1 < \infty$ for all $\eta \in \mathbb{R}$. By Theorem 3.5, the following semi-infinite interval problem

$$\begin{cases} y'' = f(x,y,y'), & 0 \leq x < \infty, \\ y(0) = M_1 + 1, & y(\infty) = B \end{cases}$$

has a unique solution $y = \psi(x)$. Let $\psi'(0) = \check{\eta}$, then from Theorem 3.4 it follows that $\psi(x) = \phi_1(x,\check{\eta})$ for all $x \in [0,\infty)$, and thus

$$\phi_1(0,\check{\eta}) = \psi(0) = M_1 + 1,$$

which is a contradiction. Thus $\phi_1(0,\eta)$ is unbounded from above. Similarly, it can be shown that $\phi_1(0,\eta)$ is not bounded from below.

We now denote the unique solution of the semi-infinite interval problem (4.1) by $\phi_2(x,\eta)$. Using Theorem 4.1 and 4.2, it can be shown by the same arguments that $\phi_2(0,\eta)$ is a continuous and strictly increasing function of η and its range is the set of all real numbers. Consequently, there exists a unique $\eta^* \in \mathbb{R}$ such that $\phi_1(0,\eta^*) = \phi_2(0,\eta^*)$, and thus $\phi_1^{(i)}(0,\eta^*) = \phi_2^{(i)}(0,\eta^*)$, $i = 0,1$. Therefore $\phi(x)$ defined as

$$\phi(x) := \begin{cases} \phi_1(x,\eta^*), & x \in [0,\infty); \\ \phi_2(x,\eta^*), & x \in (-\infty,0] \end{cases}$$

is a solution of problem (1.4).

We now show the uniqueness. Suppose that $\bar{\phi}(x)$ is another solution of problem (1.4). Let the restrictions of $\bar{\phi}(x)$ to the subinterval $[0,\infty)$ and $(-\infty,0]$ be labeled as $\bar{\phi}_1(x)$ and $\bar{\phi}_2(x)$ respectively. Then from Theorem 3.4 and 4.1, it follows that

$$\bar{\phi}_1(x) \equiv \phi_1(x,\bar{\eta}) \quad \text{on } [0,\infty)$$

and

$$\bar{\phi}_2(x) \equiv \phi_2(x,\bar{\eta}) \quad \text{on } (-\infty,0],$$

where $\bar{\eta} = \bar{\phi}'(0)$. Now, we assert that $\bar{\eta} = \eta^*$. Indeed, if $\bar{\eta} > \eta^*$, then

$$\bar{\phi}_1(0) = \phi_1(0,\bar{\eta}) < \phi_1(0,\eta^*) = \phi_2(0,\eta^*) < \phi_2(0,\bar{\eta}) = \bar{\phi}_2(0),$$

which is a contradiction, and hence $\bar{\eta} \leq \eta^*$. Similarly, $\bar{\eta} \geq \eta^*$. Thus $\bar{\eta} = \eta^*$. Therefore $\bar{\phi}(x) \equiv \phi(x)$ on $\mathbb{R}$, which proves the uniqueness of solution to problem (1.4).

Finally, we show the qualitative properties of the unique solution. We shall consider only the conclusion (1), since the other conclusion is somewhat tricky. Let $\phi(x)$ be the unique solution to problem (1.4), and let $A < B$. It suffices to show that $A \leq \phi(0) \leq B$. Suppose, by contradiction, that $\phi(0) > B$ or $\phi(0) < A$. To make sure, we can assume that $\phi(0) > B$. Then, by Theorem 3.5 and 4.2, $\phi(x)$ is monotone nonincreasing and monotone nondecreasing on $[0,\infty)$ and $(-\infty,0]$, respectively, and thus $\phi'(0) = 0$. By the uniqueness results of solutions of Theorem 3.4, $\phi(x) \equiv B$ on $[0,\infty)$, and hence $\phi(0) = B$, which contradicts $\phi(0) > B$. In summary, $A \leq \phi(0) \leq B$. Consequently, the conclusion (1) holds. This completes the proof of the theorem. □

Theorem 4.5. *Suppose that* $(\overline{\mathrm{H}}_1)$, $(\overline{\mathrm{H}}_2)$, $(\overline{\mathrm{H}}_3)$, $(\overline{\mathrm{H}}_4)$, $(\overline{\mathrm{H}}_5)$ *and* $(\overline{\mathrm{H}}_6')$ *hold. Then problem* (1.4) *has a unique solution* $y = \phi(x)$ *satisfying*

(1) *if* $A < B$, *then* $\phi(x)$ *is monotone nondecreasing on* $\mathbb{R}$, *convex on* $(-\infty,0]$ *concave on* $[0,\infty)$ *and* $\lim_{x\to\pm\infty}\phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative (nonpositive) on* $\mathbb{R}$ *when* $A \geq 0$ $(B \leq 0)$;

(2) *if* $A > B$, *then* $\phi(x)$ *is monotone nonincreasing on* $\mathbb{R}$, *concave on* $(-\infty,0]$, *convex on* $[0,\infty)$ *and* $\lim_{x\to\pm\infty}\phi'(x) = 0$. *Furthermore,* $\phi(x)$ *is nonnegative (nonpositive) on* $\mathbb{R}$ *when* $B \geq 0$ $(A \leq 0)$.

Proof. The proof of this theorem is the same as that for Theorem 4.4 except that Theorem 4.3 of [7] and Remark 4.3 are used in place of Theorem 3.5 and Theorem 4.2, respectively. This completes the proof of the theorem. □

5 Some examples

In this section, as applications, we give five examples to demonstrate our main results.

Example 5.1. Consider nonlinear second-order semi-infinite interval problem

$$y'' + e^{-y}y' = 0, \quad 0 \leq x < \infty, \tag{5.1}$$

$$y'(0) = A, \quad y(\infty) = B, \tag{5.2}$$

where $A \geq 0$ and $B \leq 0$.

We put

$$f(x,y,z) = \begin{cases} -g(0)z, & \text{if } z < 0; \\ -g(y)z, & \text{if } z \geq 0, \end{cases}$$

where

$$g(y) = \begin{cases} e^{-y}, & \text{if } y \leq 0; \\ 1, & \text{if } y > 0. \end{cases}$$

It is easy to verify that f satisfies conditions (H_1), (H_2), (H_3), (H_4) with $I = [0, \infty)$. Also we have

$$|f(x, y, z)| \geq |z| \quad \text{for } (x, y, z) \in [0, \infty) \times \mathbb{R}^2.$$

Then the condition (H_5) is satisfied. Hence from Theorem 3.4, the modified semi-infinite interval problem consisting of

$$y'' = f(x, y, y'), \quad 0 \leq x < \infty$$

and (5.2) has a unique solution ϕ with $\phi'(x) \geq 0$ on $[0, \infty)$ and $\phi(x) \leq 0$ on $[0, \infty)$. Hence by the definitions of f and g, ϕ is the unique solution of problem (5.1), (5.2). Furthermore, ϕ is nonpositive, monotone nondecreasing, concave on $[0, \infty)$ and $\lim_{x \to \infty} \phi'(x) = 0$.

Example 5.2. Consider nonlinear second-order semi-infinite interval problem

$$y'' + xh(y)(y')^{2-q} = 0, \quad 0 \leq x < \infty, \tag{5.3}$$

$$y(0) = A, \quad y(\infty) = B, \tag{5.4}$$

where $0 \leq q \leq 1$, $0 \leq A < B$, $h(y)$ is nonincreasing, continuous and positive on $\mathbb{R}$ with $\inf_{\mathbb{R}} h(y) = m > 0$.

We set

$$f(x, y, z) = -xh(y)|z|^{2-q} \operatorname{sgn} z \quad \text{for } (x, y, z) \in [0, \infty) \times \mathbb{R}^2.$$

It is easy to see that f satisfies conditions (H_1)–(H_4) with $I = [0, \infty)$ and (H_6). Notice that

$$|f(x, y, z)| \geq mx|z|^{2-q} \quad \text{for } (x, y, z) \in [0, \infty) \times \mathbb{R}^2,$$

which implies the condition (H_5) is satisfied. Notice that $A < B$, hence from Theorem 3.5, the modified semi-infinite interval problem consisting of

$$y'' = f(x, y, y'), \quad 0 \leq x < \infty$$

and (5.4) has a unique solution ϕ with $\phi'(x) \geq 0$ on $[0, \infty)$. Therefore by the definition of f, ϕ is the unique solution of problem (5.3), (5.4). Morever, ϕ is positive, nondecreasing, concave on $(0, \infty)$ and $\lim_{x \to \infty} \phi'(x) = 0$.

Note that problem (5.3), (5.4) with $h(y) \equiv m > 0$ and $A = 0, B = 1$ models phenomena in the unsteady flow of power-law fluids (see [36]).

Example 5.3. Consider nonlinear second-order full-infinite interval problem

$$y'' + mx(y')^{2-q} = 0, \quad -\infty < x < \infty, \tag{5.5}$$

$$y(-\infty) = A, \quad y(\infty) = B, \tag{5.6}$$

where $0 \leq q \leq 1$, $m > 0$, $0 \leq A < B$.

We set

$$f(x, y, z) = -mx|z|^{2-q} \operatorname{sgn} z \quad \text{for } (x, y, z) \in \mathbb{R}^3.$$

It is easy to check that $f(x, y, z)$ satisfies conditions $(\overline{\mathrm{H}}_1)$–$(\overline{\mathrm{H}}_6)$. Hence from Theorem 4.4, the modified full-infinite interval problem consisting of

$$y'' = f(x, y, y'), \quad -\infty < x < \infty$$

and (5.6) has a unique solution ϕ which satisfies $\phi'(x) \geq 0$ on $\mathbb{R}$ since $A < B$. Therefore by the definition of f, $y = \phi(x)$ is the unique solution of problem (5.5), (5.6), which is monotone nondecreasing on $\mathbb{R}$, convex on $(-\infty, 0]$, concave on $[0, \infty)$ and $\lim_{x \to \pm\infty} \phi'(x) = 0$. Furthermore, $\phi(x)$ is positive on $\mathbb{R}$.

Example 5.4. Consider nonlinear second-order full-infinite interval problem

$$y'' + mx^3(y')^4 = 0, \quad -\infty < x < \infty, \tag{5.7}$$

$$y(-\infty) = A, \quad y(\infty) = B, \tag{5.8}$$

where $m > 0$, $0 \leq A < B$.

We set

$$f(x,y,z) = -mx^3|z|^4 \operatorname{sgn} z \quad \text{for } (x,y,z) \in \mathbb{R}^3.$$

It is easy to check that $f(x,y,z)$ satisfies conditions $(\overline{\mathrm{H}}_1)$–$(\overline{\mathrm{H}}_5)$ and $(\overline{\mathrm{H}}_6')$. Similar to the discussion of Example 5.3, from Theorem 4.5, problem (5.7), (5.8) has a unique solution, which is monotone nondecreasing on $\mathbb{R}$, convex on $(-\infty, 0]$, concave on $[0,\infty)$ and $\lim_{x\to\pm\infty} \phi'(x) = 0$. Furthermore, $\phi(x)$ is positive on $\mathbb{R}$.

We note here that the results of [13,14,32,33,35] can not be applied to obtain the existence of solutions to problem (5.7), (5.8), since the nonlinearity of the equation (5.7) is super-quadratic with respect to z.

Example 5.5. Consider nonlinear second-order full-infinite interval problem

$$y'' + xy'(\pi - \arctan(xyy')) = 0, \quad -\infty < x < \infty, \tag{5.9}$$

$$y(-\infty) = A, \quad y(\infty) = B, \tag{5.10}$$

where $A, B \in \mathbb{R}$ and $A \neq B$.

We set

$$f(x,y,z) = -xz(\pi - \arctan(xyz)), \quad (x,y,z) \in \mathbb{R}^3.$$

It is easy to check that $f(x,y,z)$ satisfies $(\overline{\mathrm{H}}_1)$, $(\overline{\mathrm{H}}_2)$, $(\overline{\mathrm{H}}_3)$ and $(\overline{\mathrm{H}}_4)$. Also it is easily verified that

$$|f(x,y,z)| \geq \frac{\pi}{2}|x||z|, \quad (x,y,z) \in \mathbb{R}^3$$

and

$$|f(x,y,z)| \leq \frac{3\pi}{2}|x||z|, \quad (x,y,z) \in \mathbb{R}^3.$$

Then $(\overline{\mathrm{H}}_5)$ and $(\overline{\mathrm{H}}_6)$ hold. Hence from Theorem 4.4, problem (5.9), (5.10) has a unique solution $y = \phi(x)$ satisfying

(1) if $A < B$, then $\phi(x)$ is monotone nondecreasing on $\mathbb{R}$, convex on $(-\infty, 0]$ and concave on $[0,\infty)$. Furthermore, $\phi(x)$ is nonnegative (nonpositive) on $\mathbb{R}$ when $A \geq 0$ $(B \leq 0)$;

(2) if $A > B$, then $\phi(x)$ is monotone nonincreasing on $\mathbb{R}$, concave on $(-\infty, 0]$ and convex on $[0,\infty)$. Furthermore, $\phi(x)$ is nonnegative (nonpositive) on $\mathbb{R}$ when $B \geq 0$ $(A \leq 0)$.

Acknowledgements

We thank the referees for useful suggestions that helped us to improve the presentation of the paper. The research was supported by the Education Department of JiLin Province of China (JJKH20200029KJ).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 2, 1–10; https://doi.org/10.14232/ejqtde.2021.1.2 www.math.u-szeged.hu/ejqtde/

Existence of steady-state solutions for a class of competing systems with cross-diffusion and self-diffusion

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Received 10 November 2020, appeared 10 January 2021
Communicated by Eduardo Liz

Abstract. We focus on a system of two competing species with cross-diffusion and self-diffusion. By constructing an appropriate auxiliary function, we derive the sufficient conditions such that there are no coexisting steady-state solutions to the model. It is worth noting that the auxiliary function constructed above is applicable to Dirichlet, Neumann and Robin boundary conditions.

Keywords: cross-diffusion, self-diffusion, steady-state solution, maximum principle.

2020 Mathematics Subject Classification: 35J57, 35B50.

1 Introduction

In this work, we study the steady-state solutions of the following competing systems with cross-diffusion and self-diffusion:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(a_1 - b_1 u - c_1 v), & x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + a_{21}u + a_{22}v)v] + v(a_2 - b_2 u - c_2 v), & x \in \Omega, t > 0, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, u and v are the densities of two competing species, α_i, β_i and a_{ij} $(i,j = 1,2)$ are nonnegative constants, a_i, b_i, c_i and d_i $(i = 1,2)$ are all positive constants, a_{11} and a_{22} stand for the self-diffusion pressures, while a_{12} and a_{21} are the cross-diffusion pressures, a_1, a_2 represent the intrinsic growth rates of the two species, b_1, c_2 describe the intra-specific competitions, while b_2, c_1 describe the inter-specific competitions, and d_1, d_2 are their diffusion rates.

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System (1.1) was first proposed by Shigesada, Kawasaki and Teramoto [10] in 1979 to investigate the spatial segregation of interacting species. In the last several decades, a great deal of mathematical effort has been devoted to the study of the model. For the smooth solutions of the system (1.1) with homogeneous Neumann boundary conditions, [4] and [11] obtained the global existence and boundedness in a bounded convex domain. We refer to [2, 3, 5–7] for the study of the positive steady-state solutions. For instance, Lou and Ni [2] established the sufficient conditions for the existence and nonexistence of nonconstant steady-state solutions in the strong and weak competition case, respectively. When $a_{21} = a_{22} = 0$, Lou et al. [5] provided the parameters ranges such that the system has no nonconstant positive solutions for $a_{11} = 0$ and $a_{11} \neq 0$, respectively.

For literatures about the system (1.1) under homogeneous Dirichlet boundary conditions, see [1, 8, 12, 14] and references therein. In [9], by the decomposing operators and the theory of fixed point, Ryn and Ahn discussed the existence of the positive coexisting steady-state of system (1.1) for two competing species or predator-prey species.

Motivated by [5], we introduce the effect of self-diffusion and consider the model under three boundary conditions. Our purpose is to establish the sufficient conditions such that the following system has no coexisting solutions:

$$\begin{cases} \Delta[(d_1+\alpha v)u] + u(a_1 - b_1 u - c_1 v) = 0, & x \in \Omega, \\ \Delta[(d_2+\beta v)v] + v(a_2 - b_2 u - c_2 v) = 0, & x \in \Omega, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\alpha_i \geq 0, \beta_i \geq 0$ and $\alpha_i + \beta_i > 0$ for $i = 1, 2$. In what follows, we always assume that $\alpha \geq 0, \beta \geq 0, a_i > 0, b_i > 0, c_i > 0$ and $d_i > 0$ for $i = 1, 2$. To achieve that, the main tools we use are the strong maximum principle, Hopf's boundary lemma and the divergence theorem. Since u and v represent species densities, we are interested in the nonnegative classical solution (u, v) of (1.2), which means that $(u, v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$, $u, v \geq 0$ in $\overline{\Omega}$, and satisfies (1.2) in the pointwise sense.

The remainder of this work is organized as follows. In Section 2, we show that the non-negative classical solutions are strictly positive if they are not identically equal to zero, which plays a key role in the proof of main theorems. Section 3 constructs an auxiliary function, which can be used to produce contradictions, and thus parameter ranges for nonexistence of coexisting steady-state solutions will be obtained under three boundary conditions.

2 Preliminaries

Let us first give the following proposition by applying the strong maximum principle, which indicates that nonnegative classical solutions are strictly positive if they are nontrivial.

Proposition 2.1. *Suppose that (u, v) is a nonnegative classical solution of (1.2). Then if $u \not\equiv 0$, we have $u > 0$ in Ω, and if $v \not\equiv 0$, we have $v > 0$ in Ω.*

Proof. We only prove $u > 0$ in Ω whenever $u \not\equiv 0$, since the positivity of v in Ω can be proved in a similar way. Let $w = (d_1 + \alpha v)u$. Due to $d_1 > 0, \alpha \geq 0$ and $v \geq 0$ in $\overline{\Omega}$, it suffices to prove $w > 0$ in Ω. Otherwise, there is $x_0 \in \Omega$ such that $w(x_0) = \min_{x\in\overline{\Omega}} w(x) = 0$.

It follows from the first equation of (1.2) that

$$\Delta w + u(a_1 - b_1 u - c_1 v) = \Delta w + \frac{a_1 - b_1 u - c_1 v}{d_1 + \alpha v} \cdot w = 0.$$

Let

$$Lw = -\Delta w + cw \quad \text{with} \quad c = \frac{b_1 u + c_1 v}{d_1 + \alpha v}.$$

Then

$$c \geq 0 \quad \text{and} \quad Lw = \frac{a_1 w}{d_1 + \alpha v} \geq 0 \quad \text{in } \Omega.$$

So, an application of the strong maximum principle shows that w is constant in Ω, and thus $w = 0$, a contradiction to $u \not\equiv 0$. This completes the proof. □

Remark 2.2. When $\alpha_i = 0$ and $\beta_i > 0$ for $i = 1, 2$, that is, in the case of Neumann boundary conditions, we can get further that $u, v > 0$ in $\overline{\Omega}$ by Hopf's boundary lemma.

Next, we list two lemmas about the existence of positive solutions for single equation under Dirichlet or Robin boundary conditions, which can reveal the existence of semi steady-state solution of system (1.2). The following lemma comes from Theorem 2.1 in [8].

Lemma 2.3. *Consider the following problem:*

$$\begin{cases} -\Delta[(d + \gamma w)w] = w(a - bw), & x \in \Omega, \\ w = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where a, b, d are positive constants and γ is nonnegative constant. Let $\lambda_1^d > 0$ denote the first eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition on $\partial\Omega$. If

$$\lambda_1^d < \frac{a}{d},$$

then problem (2.1) has a unique positive solution.

From now on, if $\lambda_1^d < \frac{a_1}{d_1}$ and $\lambda_1^d < \frac{a_2}{d_2}$, we denote u^* and v^* as the unique positive solution of systems

$$\begin{cases} -d_1 \Delta u + (b_1 u - a_1)u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta[(d_2 + \beta v)v] + (c_2 v - a_2)v = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases}$$

respectively.

For Robin boundary conditions, the corresponding result can be found in Theorem 2.10 of [9].

Lemma 2.4. *Consider the following system:*

$$\begin{cases} -\Delta[(d(x) + \gamma w)w] = w(a(x) - bw), & x \in \Omega, \\ \delta w + \eta \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

where γ is a nonnegative constant, b, δ, η are positive constants and $d(x), a(x) \in C^2(\overline{\Omega})$ with $d(x) > 0$ for all $x \in \overline{\Omega}$. If $\frac{\partial}{\partial \nu}(d(x)) \leq 0$ on $\partial\Omega$ and $\lambda_1(d(x), a(x), \delta, \eta) > 0$, then (2.2) has a unique positive solution, where

$$\lambda_1(d(x), a(x), \delta, \eta) = \frac{\int_\Omega \left(-|\nabla[d(x)\phi_1]|^2 + d(x)a(x)\phi_1^2 \right) - \int_{\partial\Omega} d(x) \left[\frac{\delta}{\eta} d(x) - \frac{\partial d(x)}{\partial \nu} \right] \phi_1^2}{\| \sqrt{d(x)}\phi_1 \|^2_{L^2(\Omega)}}$$

denotes the principal eigenvalue with eigenfunction ϕ_1 of the following eigenvalue problem:

$$\begin{cases} \Delta[(d(x)\phi] + a(x)\phi = \lambda\phi, & x \in \Omega, \\ \delta\phi + \eta \dfrac{\partial \phi}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Similarly, if $\lambda_1(d_1, a_1, \alpha_1, \beta_1) > 0$ and $\lambda_1(d_2, a_2, \alpha_2, \beta_2) > 0$, we write u^{**} and v^{**} as the unique positive solution of systems

$$\begin{cases} -d_1\Delta u + (b_1 u - a_1)u = 0, & x \in \Omega, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta[(d_2 + \beta v)v] + (c_2 v - a_2)v = 0, & x \in \Omega, \\ \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

respectively.

3 Steady-state solutions

Now we give the main theorems of this work. When the intra-competition and inter-competition parameters of one species are greater than inter-competition and intra-competition of the other, respectively, whereas the intrinsic growth rate is less than that of the other, we explore two different sufficient criteria for nonexistence of coexisting solutions of system (1.2).

3.1 Dirichlet boundary conditions

Theorem 3.1. *Let $\alpha_i > 0$, $\beta_i = 0$ for $i = 1, 2$, $\alpha \geq 0$ and $\beta \geq 0$. Assume that (u, v) is a nonnegative classical solution of (1.2). If*

$$\text{(i)} \quad b_1 > b_2,\ c_1 > c_2,\ a_1 < a_2,\ d_1 \geq d_2 \quad \textit{and} \quad \alpha \geq \beta$$

or

$$\text{(ii)} \quad b_1 < b_2,\ c_1 < c_2,\ a_1 > a_2,\ d_1 \leq d_2 \quad \textit{and} \quad \alpha \leq \beta,$$

then we have either

$$(u, v) \equiv (0, 0),$$

or

$$(u, v) = (u^*, 0) \quad \textit{if} \quad \lambda_1^d < \frac{a_1}{d_1},$$

or

$$(u, v) = (0, v^*) \quad \textit{if} \quad \lambda_1^d < \frac{a_2}{d_2}.$$

Proof. (i) By way of contradiction, suppose that $u \not\equiv 0$ and $v \not\equiv 0$. Thus, u and v are positive in Ω by Proposition 2.1. So, it is apparent from system (1.2) that:

$$\begin{cases} \dfrac{\Delta[(d_1+\alpha v)u]}{u} = -a_1 + b_1 u + c_1 v, & x \in \Omega, \\ \dfrac{\Delta[(d_2+\beta v)v]}{v} = -a_2 + b_2 u + c_2 v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Let

$$w = (d_1 + \alpha v)u. \tag{3.2}$$

Then, by (3.1) and conditions $b_1 > b_2,\ c_1 > c_2$ and $a_1 < a_2$, we have

$$\frac{\Delta w}{w} > \frac{\Delta[(d_2+\beta v)v]}{(d_1+\alpha v)v} \quad \text{in } \Omega. \tag{3.3}$$

We now define a function

$$p(s) = s^{\frac{d_2-d_1}{d_1}}(d_1+\alpha s)^{\frac{2\beta-\alpha-\frac{d_2}{d_1}\alpha}{\alpha}} \quad \text{for } s>0. \tag{3.4}$$

It is easy to verify that $p(s) > 0$ for any $s > 0$. Moreover, a direct calculation implies that

$$p'(s) = p(s)\left(\frac{d_2-d_1}{d_1 s} + \frac{2\beta-\alpha-\frac{d_2}{d_1}\alpha}{d_1+\alpha s}\right).$$

This, together with (3.3), yields that

$$\begin{aligned} &\operatorname{div}\left[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\right] \\ &\quad = (d_1+\alpha v)vp(v)\Delta w + \nabla[(d_1+\alpha v)vp(v)]\cdot\nabla w \\ &\qquad - wp(v)\Delta[(d_2+\beta v)v] - \nabla[wp(v)]\cdot\nabla[(d_2+\beta v)v] \\ &\quad > \nabla[(d_1+\alpha v)vp(v)]\cdot\nabla w - \nabla[wp(v)]\cdot\nabla[(d_2+\beta v)v] \quad \text{in } \Omega. \end{aligned}$$

Furthermore, we can see that

$$\begin{aligned} &\nabla[(d_1+\alpha v)vp(v)]\cdot\nabla w - \nabla[wp(v)]\cdot\nabla[(d_2+\beta v)v] \\ &\quad = \Big[\alpha vp(v)\nabla v + (d_1+\alpha v)p(v)\nabla v + (d_1+\alpha v)vp'(v)\nabla v\Big]\cdot\Big[\alpha u\nabla v + (d_1+\alpha v)\nabla u\Big] \\ &\qquad - \Big[\alpha up(v)\nabla v + (d_1+\alpha v)p(v)\nabla u + (d_1+\alpha v)up'(v)\nabla v\Big]\cdot\Big[\beta v\nabla v + (d_2+\beta v)\nabla v\Big] \\ &\quad = |\nabla v|^2\Big[2\alpha^2 uvp(v) + d_1\alpha up(v) + d_1\alpha uvp'(v) + \alpha^2 uv^2 p'(v) - 2\alpha\beta uvp(v) \\ &\qquad - 2d_1\beta uvp'(v) - 2\alpha\beta uv^2 p'(v) - d_2\alpha up(v) - d_1 d_2 up'(v) - d_2\alpha uvp'(v)\Big] \\ &\qquad + \nabla u\cdot\nabla v\Big[3d_1\alpha vp(v) + d_1^2 p(v) + d_1^2 vp'(v) + 2d_1\alpha v^2 p'(v) + 2\alpha^2 v^2 p(v) \\ &\qquad + \alpha^2 v^3 p'(v) - 2d_1\beta vp(v) - 2\alpha\beta v^2 p(v) - d_1 d_2 p(v) - d_2\alpha vp(v)\Big] \\ &\quad \triangleq |\nabla v|^2 M(u,v) + \nabla u\cdot\nabla v N(u,v). \end{aligned}$$

Since

$$
\begin{aligned}
N(u,v) &= p(v)\Big[3d_1\alpha v + d_1^2 + 2\alpha^2 v^2 - 2d_1\beta v - 2\alpha\beta v^2 - d_1 d_2 - d_2\alpha v\Big] + p'(v)v(d_1+\alpha v)^2 \\
&= p(v)\Big[(d_1^2 - d_1 d_2) + (3d_1\alpha - 2d_1\beta - d_2\alpha)v + (2\alpha^2 - 2\alpha\beta)v^2 \\
&\quad + \frac{d_2-d_1}{d_1}(d_1+\alpha v)^2 + v(d_1+\alpha v)(2\beta - \alpha - \frac{d_2}{d_1}\alpha)\Big] \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&M(u,v) \\
&= \alpha u p(v)(d_1 - d_2 + 2\alpha v - 2\beta v) + u p'(v)\big[-d_1 d_2 + (d_1\alpha - 2d_1\beta - d_2\alpha)v + (\alpha^2 - 2\alpha\beta)v^2\big] \\
&= \alpha u p(v)(d_1 - d_2 + 2\alpha v - 2\beta v) + u p'(v)(d_1+\alpha v)(-d_2 + \alpha v - 2\beta v) \\
&= \alpha u p(v)(d_1 - d_2 + 2\alpha v - 2\beta v) + u p(v)\frac{(d_2 - d_1 - 2\alpha v + 2\beta v)}{v}(-d_2 + \alpha v - 2\beta v) \\
&= u p(v)(d_1 - d_2 + 2\alpha v - 2\beta v)\left(\frac{d_2}{v} + 2\beta\right) \\
&\geq 0,
\end{aligned}
$$

we conclude that

$$
\operatorname{div}\Big[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\Big] > 0 \quad \text{in } \Omega. \tag{3.5}
$$

Now, let

$$
\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\} \quad \text{for any small } \varepsilon > 0.
$$

Since $(u,v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$, we know that $\Big[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\Big] \in C^1(\overline{\Omega_\varepsilon})$. Then it follows from divergence theorem that

$$
\begin{aligned}
&\int_\Omega \operatorname{div}\Big[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\Big]\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \operatorname{div}\Big[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\Big]\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \Big[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\Big]\cdot \nu \mathrm{d}S \\
&= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \left[(d_1+\alpha v)vp(v)\frac{\partial w}{\partial \nu} - wp(v)\frac{\partial[(d_2+\beta v)v]}{\partial \nu}\right]\mathrm{d}S \\
&= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \left[\frac{\partial w}{\partial \nu}v^{\frac{d_2}{d_1}}(d_1+\alpha v)^{\frac{2\beta - \frac{d_2}{d_1}\alpha}{\alpha}} - \beta u\frac{\partial v}{\partial \nu}v^{\frac{d_2}{d_1}}(d_1+\alpha v)^{\frac{2\beta - \frac{d_2}{d_1}\alpha}{\alpha}}\right. \\
&\qquad\qquad \left. - \frac{u}{v}(d_2+\beta v)\frac{\partial v}{\partial \nu}v^{\frac{d_2}{d_1}}(d_1+\alpha v)^{\frac{2\beta - \frac{d_2}{d_1}\alpha}{\alpha}}\right]\mathrm{d}S \\
&\triangleq \lim_{\varepsilon\to 0}(I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)).
\end{aligned} \tag{3.6}
$$

Obviously, $I_1(\varepsilon)$ and $I_2(\varepsilon)$ both tend to zero as $\varepsilon \to 0$. To deal with the term $I_3(\varepsilon)$, we take

$$
V = \left\{\varphi(x) \in C^1(\overline{\Omega}) \;\middle|\; \varphi|_\Omega > 0,\ \varphi|_{\partial\Omega} = 0,\ \frac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} < 0\right\}.
$$

Then Hopf's boundary lemma tells us that $\frac{\partial u(x_0)}{\partial \nu} < 0$ and $\frac{\partial v(x_0)}{\partial \nu} < 0$ for any $x_0 \in \partial\Omega$, and thus $u \in V$ and $v \in V$. Define

$$g(x) := \begin{cases} \dfrac{u(x)}{v(x)}, & x \in \Omega, \\ \dfrac{\partial u(x)}{\partial \nu} \Big/ \dfrac{\partial v(x)}{\partial \nu}, & x \in \partial\Omega. \end{cases}$$

Then by applying Lemma 2.4 in [13], we get $g(x) \in C\big(\overline{\Omega}, (0, +\infty)\big)$. Therefore $I_3(\varepsilon)$ also approaches to zero as $\varepsilon \to 0$.

As a result, $\int_\Omega \operatorname{div}\Big[(d_1 + \alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2 + \beta v)v]\Big]\mathrm{d}x = 0$ because of Lebesgue dominated convergence theorem and the boundary conditions in (1.2), which contradicts (3.5). So either $(u, v) = (0, 0)$, or only one of them is equal to zero. When $v \equiv 0$, we have

$$\begin{cases} -\Delta u + \dfrac{1}{d_1}(b_1 u - a_1)u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

By Lemma 2.3, we can see $(u, v) = (u^*, 0)$ if $\lambda_1^d < \frac{a_1}{d_1}$. Similarly, if $u \equiv 0$ and $\lambda_1^d < \frac{a_2}{d_2}$, then $(u, v) = (0, v^*)$. This finishes the proof of the first part.

(ii) Now, we also assume that $u \not\equiv 0$ and $v \not\equiv 0$. Then an application of Proposition 2.1 provides that u and v are positive in Ω. Hence,

$$\frac{\Delta w}{w} < \frac{\Delta[(d_2 + \beta v)v]}{(d_1 + \alpha v)v} \quad \text{in } \Omega,$$

according to the hypotheses $b_1 < b_2$, $c_1 < c_2$ and $a_1 > a_2$, where w is defined by (3.2).

Given $d_1 \le d_2$ and $\alpha \le \beta$, we know that

$$\operatorname{div}\Big[(d_1 + \alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2 + \beta v)v]\Big] < 0 \quad \text{in } \Omega,$$

where $p(v)$ is defined as in (i).

Furthermore, again by divergence theorem, we can prove that

$$\int_\Omega \operatorname{div}\Big[(d_1 + \alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2 + \beta v)v]\Big]\mathrm{d}x = 0,$$

a contradiction. By repeating the argument in (i), we complete the proof of Theorem 3.1. □

3.2 Neumann boundary conditions

In (3.6), if we consider Neumann boundary conditions, we can directly check that (3.6) equals to zero. Consequently, the following theorem is stated without proof.

Theorem 3.2. *Suppose that $\alpha_i = 0, \beta_i > 0$ for $i = 1, 2$ and (u, v) is a nonnegative classical solution of (1.2). If*

$$\text{(i)} \quad b_1 > b_2,\ c_1 > c_2,\ a_1 < a_2,\ d_1 \ge d_2 \quad \textit{and} \quad \alpha \ge \beta$$

or

$$\text{(ii)} \quad b_1 < b_2,\ c_1 < c_2,\ a_1 > a_2,\ d_1 \le d_2 \quad \textit{and} \quad \alpha \le \beta,$$

then either $(u, v) \equiv (0, 0)$, or $(u, v) = (\frac{a_1}{b_1}, 0)$, or $(u, v) = (0, \frac{a_2}{c_2})$.

3.3 Robin boundary conditions

In this subsection, we consider the case in which α_i and β_i $(i = 1, 2)$ are both positive.

Theorem 3.3. *Let $\alpha_i > 0, \beta_i > 0$ for $i = 1, 2$ and (u, v) be a nonnegative classical solution of (1.2). If*

$$\text{(i)} \quad b_1 > b_2,\ c_1 > c_2,\ a_1 < a_2,\ d_1 \geq d_2, \alpha \geq \beta \quad \text{and} \quad \frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2}$$

or

$$\text{(ii)} \quad b_1 < b_2,\ c_1 < c_2,\ a_1 > a_2,\ d_1 \leq d_2, \alpha \leq \beta \quad \text{and} \quad \frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2},$$

then we have either

$$(u, v) \equiv (0, 0),$$

or

$$(u, v) = (u^{**}, 0) \quad \text{if} \quad \lambda_1(d_1, a_1, \alpha_1, \beta_1) > 0,$$

or

$$(u, v) = (0, v^{**}) \quad \text{if} \quad \lambda_1(d_2, a_2, \alpha_2, \beta_2) > 0.$$

Proof. We only prove (i), as (ii) can be proved in a same manner. According to the arguments of the proof of Theorem 3.1, we can obtain from the hypothesis $b_1 > b_2, c_1 > c_2, a_1 < a_2, d_1 \geq d_2$ and $\alpha \geq \beta$ that

$$\operatorname{div}\Big[(d_1 + \alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2 + \beta v)v]\Big] > 0,$$

where the function $p(v)$ is introduced in (3.4).

We mention that $\frac{\partial u}{\partial \nu} = -\frac{\alpha_1}{\beta_1}u$, $\frac{\partial v}{\partial \nu} = -\frac{\alpha_2}{\beta_2}v$ on $\partial\Omega$. The boundary integral becomes that

$$\begin{aligned}
&\int_\Omega \operatorname{div}\Big[(d_1 + \alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2 + \beta v)v]\Big]\mathrm{d}x \\
&= \int_{\partial\Omega} v^{\frac{d_2}{d_1}}(d_1 + \alpha v)^{\frac{2\beta - \frac{d_2}{d_1}\alpha}{\alpha}} \left[\alpha u \frac{\partial v}{\partial \nu} + (d_1 + \alpha v)\frac{\partial u}{\partial \nu} - 2\beta u \frac{\partial v}{\partial \nu} - d_2 \frac{u}{v}\frac{\partial v}{\partial \nu}\right]\mathrm{d}S \\
&= \int_{\partial\Omega} v^{\frac{d_2}{d_1}}(d_1 + \alpha v)^{\frac{2\beta - \frac{d_2}{d_1}\alpha}{\alpha}} \left[uv\left(2\beta\frac{\alpha_2}{\beta_2} - \alpha\frac{\alpha_1}{\beta_1} - \alpha\frac{\alpha_2}{\beta_2}\right) + u\left(d_2\frac{\alpha_2}{\beta_2} - d_1\frac{\alpha_1}{\beta_1}\right)\right]\mathrm{d}S \\
&\leq 0,
\end{aligned}$$

due to $d_1 \geq d_2, \alpha \geq \beta$ and $\frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2}$, a contradiction. Thus, if $v = 0$, we have $(u, v) = (u^{**}, 0)$ when $\lambda_1(d_1, a_1, \alpha_1, \beta_1) > 0$. Similarly, $(u, v) = (0, v^{**})$ if $u = 0$ and $\lambda_1(d_2, a_2, \alpha_2, \beta_2) > 0$. This completes the proof. □

Acknowledgements

The author would like to deeply appreciate Prof. Taishan Yi for valuable guidance and comments. And the author thanks the anonymous referee for her/his comments and suggestions. Research was supported partially by the National Natural Science Foundation of China (Grant No. 11971494).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 3, 1–20; https://doi.org/10.14232/ejqtde.2021.1.3 www.math.u-szeged.hu/ejqtde/

Oscillation and spectral properties of some classes of higher order differential operators and weighted *n*th order differential inequalities

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Received 22 May 2020, appeared 10 January 2021
Communicated by John R. Graef

Abstract. In this paper, we obtain strong oscillation and non-oscillation conditions for a class of higher order differential equations in dependence on an integral behavior of its coefficients in a neighborhood of infinity. Moreover, we establish some spectral properties of the corresponding higher order differential operator. In order to prove these we establish a certain weighted differential inequality of independent interest.

Keywords: higher order differential operator, oscillation, non-oscillation, variational principle, weighted inequality, eigenvalues, spectrum discreteness, spectrum positive definiteness, nuclear operator.

2020 Mathematics Subject Classification: 34C10, 47B25, 26D10.

1 Introduction

Let $I = (0, \infty)$ and u be a continuous and nonnegative function. Suppose that v is a positive function such that it is sufficiently times continuously differentiable on the interval I and for any $a > 0$ the function v^{-1} is integrable on the interval $(0, a)$.

Let $T \geq 0$, $I_T = (T, \infty)$ and $W^n_{2,v} \equiv W^n_{2,v}(I_T)$ be the space of functions $f : I_T \to \mathbb{R}$ having generalized derivatives up to nth order on the interval I_T, for which $\|f^{(n)}\|_{2,v} < \infty$, where $\|g\|_{2,v} = \left(\int_T^\infty v(t)|g(t)|^2 dt\right)^{\frac{1}{2}}$ is the standard norm of the weighted space $L_{2,v}(I) \equiv L_{2,v}$. From the conditions on the function v it easily follows the existence of the finite limit $\lim_{t\to T^+} f^{(i)}(t) \equiv f^{(i)}(T)$, $i = 0, 1, \ldots, n-1$, for any $f \in W^n_{2,v}$. Therefore, the space $W^n_{2,v}$ is a normalized space with the norm

$$\|f\|_{W^n_{2,v}} = \|f^{(n)}\|_{2,v} + \sum_{i=0}^{n-1} |f^{(i)}(T)|. \tag{1.1}$$

Let $\mathring{M}_2(I_T) = \{f \in W^n_2(I_T) \colon \operatorname{supp} f \subset I_T \text{ and } \operatorname{supp} f \text{ is compact}\}$.

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By the assumptions on the function v we have that $\mathring{M}_2(I_T) \subset W^n_{2,v}$. Denote by $\mathring{W}^n_{2,v} = \mathring{W}^n_{2,v}(I_T)$ the closure of the set $\mathring{M}_2$ with respect to norm (1.1).

In the paper we investigate three related problems.

Problem 1. Establish criteria of strong oscillation and non-oscillation of the $2n$th order differential equation

$$(-1)^n(v(t)y^{(n)}(t))^{(n)} - \lambda u(t)y(t) = 0, \qquad t \in I, \tag{1.2}$$

where $n > 1$ and $\lambda > 0$.

A solution of equation (1.2) is a function $y: \ I \to \mathbb{R}$ that is n times differentiable together with the function $v(t)y^{(n)}(t)$ on the interval I, satisfying equation (1.2) for all $t \in I$.

Equation (1.2) is called [9, p. 6] oscillatory, if for any $T > 0$ there exists a (non-trivial) solution of this equation, having more than one zero with multiplicity n to the right of T. Otherwise equation (1.2) is called non-oscillatory. In the sequel, the expression "solution of equation" will mean "non-trivial solution of equation" unless the opposite is specified.

Equation (1.2) is called strong non-oscillatory (oscillatory), if it is non-oscillatory (oscillatory) for all values $\lambda > 0$.

In the mathematical literature, the most number of works is devoted to the oscillatory properties of linear, semilinear and nonlinear second-order differential equations (see, e.g., [5] and references given there). However, such studies for a higher order equation are relatively rare due to the fact that not all methods of studying a second order equation are extended to a higher order equation (see [6]). One of the more universal methods to study the oscillatory properties of symmetric differential equations is the variational method. However, in the variational method, the problem is reduced to solving Problem 3, which has not yet been completely studied. Another method of studying an equation in the form (1.2) is to transfer from equation (1.2) to the system of Hamilton's equations, but even here it is difficult to find the fundamental solutions of the Hamiltonian system, especially when the coefficients of equation (1.2) are arbitrary functions. Therefore, in the works devoted to the problem of oscillation or strong oscillation of higher order equations in the form (1.2), all or one of the coefficients are power functions (see, [6–8] and references given there). In a more general case, in terms of the coefficients of the equation, criteria for its strong oscillation and non-oscillation are given in [20].

The oscillatory and non-oscillatory properties of higher order differential equations and their relation to the spectral characteristics of the corresponding differential operators are well presented in monograph [9].

Problem 2. Investigate the spectral properties of the self-adjoint differential operator L generated by the differential expression

$$l(y) = (-1)^n \frac{1}{u(t)}(v(t)y^{(n)})^{(n)}, \tag{1.3}$$

in the Hilbert space $L_{2,u} \equiv L_{2,u}(I)$ with inner product $(f,g)_{2,u} = \int_0^\infty f(t)g(t)u(t)dt$, where $u > 0$.

The investigation of the spectral characteristics of the operator L is the subject of many works (see, e.g., [2,3], [9, Chapters 29 and 34] , [10,14,21] and references given there).

Problem 3. Find necessary and sufficient conditions for the validity of the inequality

$$\int_T^\infty u(t)|f(t)|^2 dt \le C_T \int_T^\infty v(t)|f^{(n)}(t)|^2 dt, \qquad f \in \mathring{W}^n_{2,v} \tag{1.4}$$

and the sharp estimate of the constant C_T.

The inequality of the type (1.4) was considered in many works (see, e.g., [1,11,17,18] and references given there). The history of the problem and the main achievements are shortly presented in monographs [12] and [13]. Let us note that in [13, Chapter 4] the corresponding comments are given wider than in [12].

We study all these three problems depending on an integral behavior of the function v in a neighborhood of infinity. Problems 1 and 2 have been already investigated in the strong singular case

$$\int_T^\infty v^{-1}(t)dt = \infty. \tag{1.5}$$

Here we assume that

$$\int_T^\infty v^{-1}(t)dt < \infty \quad \text{and} \quad \int_T^\infty v^{-1}(t)t^2 dt = \infty \tag{1.6}$$

for any $T \ge 0$.

The work is organized as follows. In Section 2 we give necessary and sufficient conditions for the validity of inequality (1.4). In Section 3 on the basis of the results on inequality (1.4) we find necessary and sufficient conditions for the functions u and v, under which equation (1.2) is strong oscillatory or non-oscillatory. In Section 4, some spectral characteristics of the operator L are obtained.

The symbol $A \ll B$ means $A \le CB$ with some constant C. If $A \ll B \ll A$, then we write $A \approx B$. Moreover, χ_M stands for the characteristic function of the set M.

2 Validity of inequality (1.4)

We investigate (1.4) under condition (1.6). First, we present the known results required for the proof of the validity of inequality (1.4).

Let $0 \le a < b \le \infty$. From the paper [13, p. 6 and 7], the following theorem follows.

Theorem A.
(i) The inequality

$$\left(\int_a^b u(x)\left(\int_a^x f(t)dt\right)^2 dx\right)^{\frac{1}{2}} \le C\left(\int_a^b v(t)f^2(t)dt\right)^{\frac{1}{2}}, \qquad f \ge 0, \tag{2.1}$$

holds if and only if

$$A^+ = \sup_{a<z<b}\left(\int_z^b u(x)dx\right)^{\frac{1}{2}}\left(\int_a^z v^{-1}(t)dt\right)^{\frac{1}{2}} < \infty.$$

Moreover, $A^+ \le C \le 2A^+$, where C is the best constant in (2.1).
(ii) The inequality

$$\left(\int_a^b u(x)\left(\int_x^b f(t)dt\right)^2 dx\right)^{\frac{1}{2}} \le C\left(\int_a^b v(t)f^2(t)dt\right)^{\frac{1}{2}}, \qquad f \ge 0, \tag{2.2}$$

holds if and only if

$$A^- = \sup_{a<z<b} \left(\int_a^z u(x)dx \right)^{\frac{1}{2}} \left(\int_z^b v^{-1}(t)dt \right)^{\frac{1}{2}} < \infty.$$

Moreover, $A^- \le C \le 2A^-$*, where* C *is the best constant in* (2.2).

Let

$$A_1 = \sup_{a<z<b} \left(\int_z^b u(x)dx \right)^{\frac{1}{2}} \left(\int_a^z (z-t)^{2(n-1)} v^{-1}(t)dt \right)^{\frac{1}{2}},$$

$$A_2 = \sup_{a<z<b} \left(\int_z^b (x-z)^{2(n-1)} u(x)dx \right)^{\frac{1}{2}} \left(\int_a^z v^{-1}(t)dt \right)^{\frac{1}{2}}.$$

The next statement follows from the results in the work [21].

Theorem B. *The inequality*

$$\int_a^b u(z) \left(\int_a^z (z-t)^{n-1} f(t)dt \right)^2 dz \le C \int_a^b v(t) f^2(t)dt, \qquad f \ge 0, \tag{2.3}$$

holds if and only if $\max\{A_1, A_2\} < \infty$*. Moreover,*

$$C \approx \max\{A_1, A_2\}, \tag{2.4}$$

where C *is the best constant in* (2.3).

Assume that $\lim_{t\to\infty} f^{(n-1)}(t) \equiv f^{(n-1)}(\infty)$ and

$$LR^{(n-1)}W^n_{2,v} = \{f \in W^n_{2,v} : f^{(i)}(T) = 0,\ i = 0,1,\dots,n-1;\ f^{(n-1)}(\infty) = 0\},$$
$$LW^n_{2,v} = \{f \in W^n_{2,v} : f^{(i)}(T) = 0, i = 0,1,\dots,n-1\}.$$

From Theorems 1 and 2 in [15] in view of the conditions on v^{-1} in a neighborhood of zero, it follows the next statement.

Theorem C.

(i) If (1.5) *holds, then*

$$\mathring{W}^n_{2,v} \equiv LW^n_{2,v}; \tag{2.5}$$

(ii) if (1.6) *holds, then*

$$\mathring{W}^n_{2,v} \equiv LR^{(n-1)}W^n_{2,v} \quad \text{and} \quad LW^n_{2,v}(I_{T+1}) \equiv LR^{(n-1)}W^n_{2,v}(I_{T+1}) \oplus P_\infty, \tag{2.6}$$

where $P_\infty = \{P(t) = c\chi_{I_{T+1}}(t)t^{n-1} : c \in R\}$.

Assume that $J(f) = \frac{\int_T^\infty u(t)|f(t)|^2dt}{\int_T^\infty v(t)|f^{(n)}(t)|^2dt}$, $C_L = \sup_{f\in LW^n_{2,v}} J(f)$ and $C_{LR} \equiv C_T = \sup_{f\in LR^{(n-1)}W^n_{2,v}} J(f)$. It is obvious that $C_{LR} \le C_L$. We investigate the estimate of the value C_{LR} under the assumption $C_L = \infty$, that in view of (2.6) is equivalent to the condition

$$\int_\alpha^\infty u(x)x^{2(n-1)}dx = \infty \tag{2.7}$$

for any $\alpha > T$.

Let τ be an arbitrary point of the interval I_T. Assume

$$A_{1,1}(T,\tau) = \sup_{T<z<\tau} \int_z^\tau u(x)dx \int_T^z (z-t)^{2(n-1)} v^{-1}(t)dt,$$
$$A_{1,2}(T,\tau) = \sup_{T<z<\tau} \int_z^\tau u(x)(x-z)^{2(n-1)}dx \int_T^z v^{-1}(t)dt,$$
$$A_{1,3}(T,\tau) = \int_\tau^\infty u(x)(x-\tau)^{2(n-2)}dx \int_T^\tau (\tau-t)^2 v^{-1}(t)dt,$$
$$A_{1,4}(T,\tau) = \int_\tau^\infty u(x)dx \int_T^\tau (\tau-t)^{2(n-1)} v^{-1}(t)dt,$$
$$A_{2,1}(T,\tau) = \sup_{z>\tau} \int_z^\infty u(x)(x-\tau)^{2(n-2)}dx \int_\tau^z (t-\tau)^2 v^{-1}(t)dt,$$
$$A_{2,2}(T,\tau) = \sup_{z>\tau} \int_\tau^z u(x)(x-\tau)^{2(n-1)}dx \int_z^\infty v^{-1}(t)dt,$$
$$A(T,\tau) = \max\{A_{1,1}(T,\tau), A_{1,2}(T,\tau), A_{1,3}(T,\tau), A_{1,4}(T,\tau), A_{2,1}(T,\tau), A_{2,2}(T,\tau)\}.$$

Due to (2.6) inequality (1.4) can be written in the form

$$\int_T^\infty u(t)|f(t)|^2 dt \le C_T \int_T^\infty v(t)|f^{(n)}(t)|^2 dt, \qquad f \in LR^{(n-1)}W^n_{2,v}.$$

In work [18] it is obtained that $A(T,\tau) < \infty$ is necessary and sufficient condition for the validity of this inequality, where $\int_T^\tau v^{-1}(t)dt = \int_\tau^\infty v^{-1}(t)dt$. Here we obtain a simpler criterion that is usable for the application to Problem 1 and 2.

Theorem 2.1. *Let $T \ge 0$. Let (1.6) and (2.7) hold. Inequality (1.4) holds if and only if*

$$\lim_{z\to\infty} \int_z^\infty u(x)(x-\tau)^{2(n-2)}dx \int_\tau^z (t-\tau)^2 v^{-1}(t)dt < \infty \tag{2.8}$$

and

$$\lim_{z\to\infty} \int_\tau^z u(x)(x-\tau)^{2(n-1)}dx \int_z^\infty v^{-1}(t)dt < \infty. \tag{2.9}$$

Moreover, there exists a point $\tau_T : T < \tau_T < \infty$ such that

$$C_T \approx \mathcal{A}(T,\tau_T) = \max\{A_{2,1}(T,\tau_T), A_{2,2}(T,\tau_T)\}, \tag{2.10}$$

where C_T is the best constant in (1.4).

Proof. Sufficiency. Let (2.8) and (2.9) hold. Then, due to the conditions on the weight functions u and v, we get that $A(T,\tau) < \infty$ for any $\tau \in I_T$. Therefore, on the basis of the results in [18], inequality (1.4) holds. Now, let us estimate the constant C_T from above. From (2.6) it follows that $f^{(i)}(T) = 0$, $i = 0,1,\ldots,n-1$, $f^{(n-1)}(\infty) = 0$ for any $f \in \mathring{W}^n_{2,v}$. Hence, we present $f \in \mathring{W}^n_{2,v}$ in the form $f(x) = \frac{1}{(n-2)!}\int_T^x (x-s)^{n-2} f^{(n-1)}(s)ds$, $x > T$, where $f^{(n-1)}(s) = \int_T^s f^{(n)}(t)dt = -\int_s^\infty f^{(n)}(t)dt$, $s > T$. Let $\tau \in I_T$. Next, for $T < s < \tau$ we assume that $f^{(n-1)}(s) = \int_T^s f^{(n)}(t)dt$, and for $s > \tau$ we assume that $f^{(n-1)}(s) = -\int_s^\infty f^{(n)}(t)dt$. Then $f(x) = \frac{1}{(n-2)!}\int_T^x (x-s)^{n-2} \int_T^s f^{(n)}(t)dtds$ for $T < x < \tau$ and

$$\begin{aligned} f(x) &= \frac{1}{(n-2)!}\int_T^x (x-s)^{n-2} f^{(n-1)}(s)ds \\ &= \frac{1}{(n-2)!}\left[\int_T^\tau (x-s)^{n-2} f^{(n-1)}(s)ds + \int_\tau^x (x-s)^{n-2} f^{(n-1)}(s)ds\right] \\ &= \frac{1}{(n-2)!}\left[\int_T^\tau (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds - \int_\tau^x (x-s)^{n-2}\int_s^\infty f^{(n)}(t)dtds\right] \end{aligned}$$

for $x > \tau$. Therefore, we have

$$\begin{aligned}\int_T^\infty u(x)|f(x)|^2dx &= \int_T^\tau u(x)|f(x)|^2dx + \int_\tau^\infty u(x)|f(x)|^2dx \\ &= \frac{1}{[(n-2)!]^2}\int_T^\tau u(x)\left|\int_T^x (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds\right|^2 dx \\ &\quad + \frac{1}{[(n-2)!]^2}\int_\tau^\infty u(x)\left|\int_T^\tau (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds - \int_\tau^x (x-s)^{n-2}\int_s^\infty f^{(n)}(t)dtds\right|^2 dx \\ &= \frac{1}{[(n-2)!]^2}\left[F_1(f^{(n)}) + F_2(f^{(n)})\right],\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}F_1(f^{(n)}) &= \int_T^\tau u(x)\left|\int_T^x (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds\right|^2 dx \\ &= \frac{1}{(n-1)^2}\int_T^\tau u(x)\left|\int_T^x (x-t)^{n-1}f^{(n)}(t)dt\right|^2 dx,\end{aligned}$$

$$\begin{aligned}F_2(f^{(n)}) &= \int_\tau^\infty u(x)\left|\int_T^\tau (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds - \int_\tau^x (x-s)^{n-2}\int_s^\infty f^{(n)}(t)dtds\right|^2 dx \\ &= \int_\tau^\infty u(x)\left|\int_T^\tau (x-s)^{n-2}\int_T^s f^{(n)}(t)dtds - \int_\tau^x (x-s)^{n-2}\int_s^x f^{(n)}(t)dtds\right. \\ &\qquad\qquad \left. - \int_\tau^x (x-s)^{n-2}dx\int_x^\infty f^{(n)}(t)dtds\right|^2 dx.\end{aligned}$$

Assume that $f^{(n)} = g$, then $\int_T^\infty g(t)dt = 0$ and the condition $f \in \mathring{W}_{2,v}^n$ is equivalent to the condition $g \in \widetilde{L}_2(I_T) \equiv \{g \in L_2(I_T) : \int_T^\infty g(t)dt = 0\}$. Therefore, from (2.11) it follows that inequality (1.4) is equivalent to the inequality

$$\frac{1}{[(n-2)!]^2}\left[F_1(g) + F_2(g)\right] \le C_T\int_T^\infty v(t)|g(t)|^2dt, \qquad g \in \widetilde{L}_2(I_T). \tag{2.12}$$

Moreover, the best constants in inequalities (1.4) and (2.12) coincide.

On the basis of Theorem B we have

$$\begin{aligned}F_1(g) &= \frac{1}{(n-1)^2}\int_T^\tau u(x)\left|\int_T^x (x-t)^{n-1}g(t)dt\right|^2 dx \\ &\ll \max\{A_{1,1}(T,\tau), A_{1,2}(T,\tau)\}\int_T^\tau v(t)|g(t)|^2dt.\end{aligned} \tag{2.13}$$

Now, we estimate $F_2(g)$.

$$\begin{aligned}F_2(g) \le &\int_\tau^\infty u(x)\left|\int_T^\tau (x-s)^{n-2}\int_T^s |g(t)|dtds + \int_\tau^x (x-s)^{n-2}\int_s^x |g(t)|dtds\right. \\ &\qquad\qquad \left. + \int_\tau^x (x-s)^{n-2}ds\int_x^\infty |g(t)|dt\right|^2 dx \\ = &\int_\tau^\infty u(x)\left|\int_T^\tau |g(t)|\int_t^\tau (x-s)^{n-2}dsdt + \int_\tau^x |g(t)|\int_\tau^t (x-s)^{n-2}dsdt\right. \\ &\qquad\qquad \left. + \frac{1}{n-1}(x-\tau)^{n-1}\int_x^\infty |g(t)|dt\right|^2 dx\end{aligned}$$

$$\le 3\left[\int_\tau^\infty u(x)\left|\int_T^\tau |g(t)|\int_t^\tau (x-s)^{n-2}dsdt\right|^2 dx\right.$$
$$+\int_\tau^\infty u(x)\left|\int_\tau^x |g(t)|\int_\tau^t (x-s)^{n-2}dsdt\right|^2 dx$$
$$\left.+\frac{1}{(n-1)^2}\int_\tau^\infty u(x)(x-\tau)^{2(n-1)}\left(\int_x^\infty |g(t)|dt\right)^2 dx\right]$$
$$= 3\left[J_0 + J_1 + \frac{J_2}{(n-1)^2}\right], \tag{2.14}$$

where

$$J_0 = \int_\tau^\infty u(x)\left|\int_T^\tau |g(t)|\int_t^\tau (x-s)^{n-2}dsdt\right|^2 dx,$$
$$J_1 = \int_\tau^\infty u(x)\left|\int_\tau^x |g(t)|\int_\tau^t (x-s)^{n-2}dsdt\right|^2 dx,$$
$$J_2 = \int_\tau^\infty u(x)(x-\tau)^{2(n-1)}\left(\int_x^\infty |g(t)|dt\right)^2 dx.$$

Let us estimate J_0, J_1 and J_2 separately. For the estimate of J_0 using $(x-s)^{n-2} = (x-\tau+\tau-s)^{n-2} \approx (x-\tau)^{n-2} + (\tau-s)^{n-2}$ and Hölder's inequality, we get

$$J_0 \approx \int_\tau^\infty u(x)(x-\tau)^{2(n-2)}dx\left(\int_T^\tau (\tau-t)|g(t)|dt\right)^2 + \int_\tau^\infty u(x)dx\left(\int_T^\tau (\tau-t)^{n-1}|g(t)|dt\right)^2$$
$$\ll \max\{A_{1,3}(T,\tau), A_{1,4}(T,\tau)\}\int_T^\tau v(t)|g(t)|^2dt. \tag{2.15}$$

For the estimate of J_1 using $\int_\tau^t (x-s)^{n-2}ds = \frac{1}{n-1}\left((x-\tau)^{n-1} - (x-t)^{n-1}\right) \approx (x-\tau)^{n-2}(t-\tau)$ and Theorem A, we get

$$J_1 \approx \int_\tau^\infty u(x)(x-\tau)^{2(n-2)}\left(\int_\tau^x (t-\tau)|g(t)|dt\right)^2 dx \ll A_{2,1}(T,\tau)\int_\tau^\infty v(t)|g(t)|^2dt. \tag{2.16}$$

By Theorem A we have

$$J_2 \ll A_{2,2}(T,\tau)\int_\tau^\infty v(t)|g(t)|^2dt. \tag{2.17}$$

From (2.11), (2.12), (2.13), (2.14), (2.15), (2.16) and (2.17) it follows that there exist positive numbers α and β such that the estimate

$$\int_T^\infty u(x)|f(x)|^2dx \le \beta\mathcal{A}_0(T,\tau)\int_T^\tau v(t)|f^{(n)}(t)|^2dt + \alpha\mathcal{A}(T,\tau)\int_\tau^\infty v(t)|f^{(n)}(t)|^2dt \tag{2.18}$$

holds, where $\mathcal{A}_0(T,\tau) = \max\{A_{1,1}(T,\tau),\ A_{1,2}(T,\tau),\ A_{1,3}(T,\tau), A_{1,4}(T,\tau)\}$ and $\mathcal{A}(T,\tau) = \max\{A_{2,1}(T,\tau), A_{2,2}(T,\tau)\}$.

In view of (2.8) and (2.9), we have that the value $\mathcal{A}_0(T,\tau)$ satisfies the properties $\lim_{\tau\to T}\mathcal{A}_0(T,\tau) = 0$ and $\lim_{\tau\to\infty}\mathcal{A}_0(T,\tau) = \infty$, and the value $\mathcal{A}(T,\tau)$ is non-increasing in τ and $\lim_{\tau\to\infty}\mathcal{A}(T,\tau) < \infty$. Therefore, the following point

$$\tau_T = \sup\{\tau \in I_T : \beta\mathcal{A}_0(T,\tau) \le \alpha\mathcal{A}(T,\tau)\} \tag{2.19}$$

is defined. Then from (2.18) we have

$$\int_T^\infty u(t)|f(t)|^2dt \ll \mathcal{A}(T,\tau_T)\int_T^\infty v(t)|f^{(n)}(t)|^2dt, \tag{2.20}$$

i.e., inequality (1.4) holds with the estimate

$$C_T \ll \mathcal{A}(T,\tau_T) \tag{2.21}$$

for the best constant C_T in (1.4).

Necessity. Let us use the technique used in works [17] and [18]. Let inequality (1.4) hold with the best constant $C_T > 0$. By condition (1.6) we have that $\int_T^\infty v^{-1}(t)dt < \infty$. Suppose that $\gamma_{\tau_T} = \gamma(\tau_T) > 0$ and the function $\rho : (T,\tau_T) \to (\tau_T,\infty)$ is such that

$$\int_T^{\tau_T} v^{-1}(t)dt = \gamma_{\tau_T}\int_{\tau_T}^\infty v^{-1}(t)dt$$

and

$$\int_T^s v^{-1}(t)dt = \gamma_{\tau_T}\int_{\rho(s)}^\infty v^{-1}(t)dt,\ s \in (T,\tau_T). \tag{2.22}$$

It is obvious that the decreasing function ρ is locally absolutely continuous on the interval (T,τ_T) and $\lim\limits_{s\to T^+}\rho(s) = \infty$, $\lim\limits_{s\to\tau_T}\rho(s) = \tau_T$. The differentiation of the both sides of (2.22) gives

$$v^{-1}(s) = -\gamma_{\tau_T}v^{-1}(\rho(s))\rho'(s) = \gamma_{\tau_T}v^{-1}(\rho(s))|\rho'(s)|$$

or

$$\frac{1}{\gamma_{\tau_T}} = \frac{v^{-1}(\rho(s))|\rho'(s)|}{v^{-1}(s)} \tag{2.23}$$

for almost all $s \in (T,\tau_T)$. Let

$$K^+(T,\tau_T) = \{g \in L_1(T,\tau_T)\cap L_{2,v}(T,\tau_T) : \ g \ge 0, g \not\equiv 0\},$$
$$K^-(\tau_T,\infty) = \{g \in L_1(\tau_T,\infty)\cap L_{2,v}(\tau_T,\infty) : \ g \le 0, g \not\equiv 0\}.$$

Let us show that for every $g_2 \in K^-(\tau_T,\infty)$ there exists $g_{1,2} \in K^+(T,\tau_T)$ such that for the functions $g(t) = g_{1,2}(t)$, $t \in (T,\tau_T)$ and $g(t) = g_2(t)$, $t \in (\tau_T,\infty)$ we have that $g \in \widetilde{L}_{2,v}(T,\infty)$.

For $g_2 \in K^-(\tau_T,\infty)$ we assume that $g_{1,2}(x) = -\gamma_{\tau_T}g_2(\rho^{-1}(x))\frac{v^{-1}(x)}{v^{-1}(\rho^{-1}(x))}$. Then $g_{1,2} \ge 0$. Changing the variables $\rho^{-1}(x) = t$ and using (2.23), we have

$$\begin{aligned}\int_T^{\tau_T} g_{1,2}(x)dx &= \gamma_{\tau_T}\int_T^{\tau_T}|g_2(\rho^{-1}(x))|\frac{v^{-1}(x)}{v^{-1}(\rho^{-1}(x))}dx = -\gamma_{\tau_T}\int_{\tau_T}^\infty |g_2(t)|\frac{v^{-1}(\rho(t))}{v^{-1}(t)}\rho'(t)dt\\ &= \gamma_{\tau_T}\int_{\tau_T}^\infty |g_2(t)|\frac{v^{-1}(\rho(t))}{v^{-1}(t)}|\rho'(t)|dt = \int_{\tau_T}^\infty |g_2(t)|dt < \infty.\end{aligned} \tag{2.24}$$

From (2.24) it follows that $\int_T^{\tau_T} g_{1,2}(x)dx < \infty$ and

$$\int_T^{\tau_T} g_{1,2}(x)dx + \int_{\tau_T}^\infty g_2(t)dt = \int_T^\infty g(t)dt = 0. \tag{2.25}$$

Again, changing the variables $\rho^{-1}(x) = t$ and using (2.23), we have

$$\begin{aligned}\int_T^{\tau_T} |g_{1,2}(t)|^2 v(t)dt &= \gamma_{\tau_T}^2 \int_T^{\tau_T} \left| g_2(\rho^{-1}(x)) \frac{v^{-1}(x)}{v^{-1}(\rho^{-1}(x))} \right|^2 v(x)dx \\ &= \gamma_{\tau_T}^2 \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t) \frac{v^{-1}(\rho(t))}{v^{-1}(t)} |\rho'(t)| dt \\ &= \gamma_{\tau_T} \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt < \infty.\end{aligned}$$

Hence,

$$\begin{aligned}\int_T^{\infty} |g(t)|^2 v(t)dt = \int_T^{\tau_T} |g_{1,2}(t)|^2 v(t)dt + \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt \\ = (1+\gamma_{\tau_T}) \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt < \infty, \quad (2.26)\end{aligned}$$

i.e., $g \in L_{2,v}(I_T)$. The last and (2.25) give that $g \in \widetilde{L}_{2,v}(I_T)$.

Let $g_2 \in K^-(\tau_T, \infty)$ and $g_{1,2} \in K^+(T, \tau_T)$ be a function defined by g_2. Then $g \in \widetilde{L}_{2,v}(I_T)$, where $g(t) = g_{1,2}(t)$, $t \in (T, \tau_T)$ and $g(t) = g_2(t)$, $t \in (\tau_T, \infty)$. Since $g \in \widetilde{L}_2(I_T)$, then replacing the function g in (2.12) for $\tau = \tau_T$ and taking into account that $g_{1,2} \geq 0$, $g_2 \leq 0$, we have

$$\begin{aligned}\frac{1}{[(n-2)!]^2} \Bigg[F_1(g_{1,2}) + \int_{\tau_T}^{\infty} u(x) \left(\int_T^{\tau_T} (x-s)^{n-2} \int_T^s g_{1,2}(t)dtds \right. \\ \left. + \int_{\tau_T}^{x} (x-s)^{n-2} \int_s^{\infty} |g_2(t)|dtds \right)^2 dx \Bigg] \leq C_T \int_T^{\infty} v(t)|g(t)|^2 dt,\end{aligned}$$

that together with (2.26) gives

$$\begin{aligned}\int_{\tau_T}^{\infty} u(x) \left(\int_{\tau_T}^{x} (x-s)^{n-2} \int_s^{\infty} |g_2(t)|dtds \right)^2 dx \\ \ll (1+\gamma_{\tau_T}) C_T \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt, \qquad g_2 \in K^-(\tau_T, \infty). \quad (2.27)\end{aligned}$$

Since

$$\int_{\tau_T}^{x} (x-s)^{n-2} \int_s^{\infty} |g_2(t)|dtds \geq (x-\tau_T)^{n-2} \int_{\tau_T}^{x} (t-\tau_T)|g_2(t)|dt + \frac{1}{n-1}(x-\tau_T)^{n-1} \int_x^{\infty} |g_2(t)|dt,$$

then from (2.27) we have

$$\begin{aligned}\int_{\tau_T}^{\infty} u(x)(x-\tau_T)^{2(n-2)} \left(\int_{\tau_T}^{x} (t-\tau_T)|g_2(t)|dt \right)^2 dx \\ \leq (1+\gamma_{\tau_T}) C_T \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt, \qquad g_2 \in K^-(\tau_T, \infty), \quad (2.28)\end{aligned}$$

$$\begin{aligned}\int_{\tau_T}^{\infty} u(x)(x-\tau_T)^{2(n-1)} \left(\int_x^{\infty} |g_2(t)|dt \right)^2 dx \\ \leq C_T(1+\gamma_{\tau_T}) \int_{\tau_T}^{\infty} |g_2(t)|^2 v(t)dt, \qquad g_2 \in K^-(\tau_T, \infty). \quad (2.29)\end{aligned}$$

For any $\tau_T < z < \infty$ the functions $g_2^+(t) = -\chi_{(\tau_T,z)}(t)(t-\tau_T)v^{-1}(t)$, $g_2^-(t) = -\chi_{(z,\infty)}(t)v^{-1}(t)$ belong to the set $K^-(\tau_T,\infty)$. Replacing the functions g_2^+ and g_2^- into (2.28) and (2.29), respectively, we get

$$\mathcal{A}(T,\tau_T) \ll C_T. \tag{2.30}$$

This relation together with (2.21) gives (2.10). From the finiteness of the value $\mathcal{A}(T,\tau_T) = \max\{A_{2,1}(T,\tau_T), A_{2,2}(T,\tau_T)\}$ we have (2.8) and (2.9). The proof of Theorem 2.1 is complete. □

3 Oscillatory properties of equation (1.2)

The main aim of this Section is the investigation of strong oscillation and non-oscillation of differential equation (1.2) in a neighborhood of infinity. Oscillatory properties of (1.2) we investigate under conditions (1.6) and (2.7). Case (1.5) has been investigated in paper [20].

We consider the inequality

$$\int_T^\infty \lambda u(t)|f(t)|^2 dt \le \lambda C_T \int_T^\infty v(t)|f^{(n)}(t)|^2 dt, \qquad f \in \mathring{W}^n_{2,v}, \tag{3.1}$$

with a constant λC_T, where C_T is the best constant in (1.4).

We investigate the oscillatory properties of equation (1.2) by the variation method, i.e., on the basis of the known variational statement.

Lemma A ([9, Theorem 28]). *Equation (1.2) is non-oscillatory if and only if there exists $T > 0$ such that*

$$\int_T^\infty \left[v(t)|f^{(n)}(t)|^2 - \lambda u(t)|f(t)|^2\right] dt \ge 0 \tag{3.2}$$

for all $f \in \mathring{M}_2(I_T)$.

Due to the compactness of the set $supp f$ for $f \in \mathring{M}_2(I_T)$, inequality (3.2) coincide with the inequality

$$\int_T^\infty \lambda u(t)|f(t)|^2 dt \le \int_T^\infty v(t)|f^{(n)}(t)|^2 dt, \ \ \forall f \in \mathring{M}_2(I_T). \tag{3.3}$$

Lemma 3.1. *Equation (1.2)*

(i) is non-oscillatory if and only if there exists $T > 0$ such that inequality (3.1) holds with the best constant $\lambda C_T : 0 < \lambda C_T \le 1$;

(ii) is oscillatory if and only if for any $T > 0$ the best constant is such that $\lambda C_T > 1$ in (3.1).

Proof. Let us prove the statement (i), the statement (ii) is the opposite of the statement (i). If equation (1.2) is non-oscillatory, then for some $T > 0$ inequality (3.3) holds, which means that inequality (3.1) holds with the best constant $0 < \lambda C_T \le 1$. Inversely, if for some $T > 0$ inequality (3.1) holds with the best constant $0 < \lambda C_T \le 1$, then inequality (3.3) holds and by Lemma A equation (1.2) is non-oscillatory. The proof of Lemma 3.1 is complete. □

On the basis of Lemma 3.1 and Theorem 2.1, we have the following statement.

Theorem 3.2. *Let (1.6) and (2.7) hold. Then equation (1.2) is strong non-oscillatory if and only if*

$$\lim_{z\to\infty} \int_z^\infty u(x)(x-T)^{2(n-2)} dx \int_T^z (t-T)^2 v^{-1}(t) dt = 0 \tag{3.4}$$

and

$$\lim_{z\to\infty} \int_T^z u(x)(x-T)^{2(n-1)} dx \int_z^\infty v^{-1}(t) dt = 0. \tag{3.5}$$

Proof. Let equation (1.2) be strong non-oscillatory. Then by Lemma 3.1 for each $\lambda > 0$ there exists $T_\lambda = T(\lambda) > 0$ such that $\lambda C_{T_\lambda} \leq 1$ in (3.1). This gives that $\lim_{\lambda\to\infty} C_{T_\lambda} = 0$, and from (2.10) we have

$$\lim_{\lambda\to\infty} \mathcal{A}(T_\lambda, \tau_{T(\lambda)}) = 0. \tag{3.6}$$

From $\lambda_2 C_{T(\lambda_2)} \leq 1$ it follows that $\lambda_1 C_{T(\lambda_2)} \leq 1$ for $0 < \lambda_1 \leq \lambda_2$. Therefore, $T(\lambda_2) \geq T(\lambda_1)$, $\tau_{T(\lambda_2)} \geq \tau_{T(\lambda_1)}$ and $\lim_{\lambda\to\infty} T(\lambda) = \lim_{\lambda\to\infty} \tau_{T(\lambda)} = \infty$.

Since the value $\mathcal{A}(T,\tau)$ does not increase in $\tau > 0$, from (3.6) we have $\lim_{\tau\to\infty} \mathcal{A}(T,\tau) = 0$, i.e.,

$$\lim_{\tau\to\infty} \sup_{z>\tau} \int_z^\infty u(x)(x-\tau)^{2(n-2)} dx \int_\tau^z (t-\tau)^2 v^{-1}(t) dt = 0, \tag{3.7}$$

$$\lim_{\tau\to\infty} \sup_{z>\tau} \int_\tau^z u(x)(x-\tau)^{2(n-1)} dx \int_z^\infty v^{-1}(t) dt = 0. \tag{3.8}$$

By the definition of the limit (3.7) for any $\varepsilon > 0$ there exists $T_\varepsilon = T(\varepsilon) > T$ such that

$$\int_z^\infty u(x)(x-T_\varepsilon)^{2(n-2)} dx \int_{T_\varepsilon}^z (t-T_\varepsilon)^2 v^{-1}(t) dt \leq \frac{\varepsilon}{5\cdot 2^{2n-3}} \tag{3.9}$$

for all $z \geq T_\varepsilon$. Then there exists $T_1(\varepsilon) \geq T_\varepsilon$ such that

$$\int_z^\infty u(x)(x-T_\varepsilon)^{2(n-2)} dx \int_T^{T_\varepsilon} (T_\varepsilon - T)^2 v^{-1}(t) dt \leq \frac{\varepsilon}{5\cdot 2^{2n-3}}, \qquad z \geq T_1(\varepsilon). \tag{3.10}$$

From (3.9) and (3.10) we get

$$\int_z^\infty u(x)(x-T_\varepsilon)^{2(n-2)} dx \int_T^z (t-T)^2 v^{-1}(t) dt \leq \frac{4\varepsilon}{5\cdot 2^{2n-3}}, \qquad z \geq T_1(\varepsilon). \tag{3.11}$$

Further, there exists $T_2(\varepsilon) \geq T_1(\varepsilon)$ and

$$\int_z^\infty (T_\varepsilon - T)^{2(n-2)} u(x) dx \int_T^z (t-T)^2 v^{-1}(t) dt \leq \frac{\varepsilon}{5\cdot 2^{2n-3}}, \qquad z \geq T_2(\varepsilon). \tag{3.12}$$

Then from (3.11) and (3.12) we have

$$\int_z^\infty u(x)(x-T)^{2(n-2)} dx \int_T^z (t-T)^2 v^{-1}(t) dt \leq \varepsilon$$

for all $z \geq T_2(\varepsilon)$. It means that (3.4) holds. Similarly, we can prove that from (3.8) it follows (3.5).

Sufficiency. Let (3.4) and (3.5) hold. From (3.5) we have

$$\lim_{z\to\infty} \int_\tau^z u(x)(x-\tau)^{2(n-1)} dx \int_z^\infty v^{-1}(t) dt = 0$$

for any $\tau \geq T$. Thus,

$$\lim_{\tau\to\infty} \sup_{z>\tau} \int_\tau^z u(x)(x-\tau)^{2(n-1)} dx \int_z^\infty v^{-1}(t) dt = \lim_{\tau\to\infty} A_{2,2}(T,\tau) = 0.$$

Similarly, from (3.4) we have that $\lim_{\tau\to\infty} A_{2,1}(T,\tau) = 0$. Then $\lim_{\tau\to\infty} \mathcal{A}(T,\tau) = 0$.

Since $\lim_{T\to\infty} \tau_T = \infty$, then $\lim_{T\to\infty} \mathcal{A}(T,\tau_T) = 0$. Hence, from (2.10) we have $\lim_{T\to\infty} C_T = 0$. Therefore, for any $\lambda > 0$ there exists $T_\lambda \geq T$ such that $\lambda C_{T_\lambda} \leq 1$ and by Lemma 3.1 equation (1.2) is non-oscillatory for any $\lambda > 0$. The proof of Theorem 3.2 is complete. $\square$

Theorem 3.3. *Let* (1.6) *and* (2.7) *hold. Then equation* (1.2) *is strong oscillatory if and only if*

$$\lim_{z\to\infty}\int_z^\infty u(x)(x-T)^{2(n-2)}dx\int_T^z (t-T)^2 v^{-1}(t)dt=\infty \tag{3.13}$$

or

$$\lim_{z\to\infty}\int_T^z u(x)(x-T)^{2(n-1)}dx\int_z^\infty v^{-1}(t)dt=\infty. \tag{3.14}$$

Proof. Necessity. Let equation (1.2) be strong oscillatory. Then by Lemma 3.1 $\lambda C_T > 1$ for any $T\geq 0$ and $\lambda>0$. It means that $C_T>\frac{1}{\lambda}$ and for $\lambda\to 0^+$ it follows that $C_T=\infty$ for any $T>0$. Then from (2.10) we have that $\mathcal{A}(T,\tau_T)=\infty$, i.e., $A_{2,1}(T,\tau_T)=\infty$ or $A_{2,2}(T,\tau_T)=\infty$ for all $T\geq 0$. Therefore, (3.13) or (3.14) holds, respectively.

Sufficiency. Let (3.13) or (3.14) hold. Then $A_{2,1}(T,\tau_T)=\infty$ or $A_{2,2}(T,\tau_T)=\infty$, respectively, i.e., $\mathcal{A}(T,\tau_T)=\infty$ for any $T\geq 0$. Then $\lambda\mathcal{A}_(T,\tau_T)=\infty$ for any $\lambda>0$ and $T\geq 0$. Hence, from (2.10) we have $\lambda C_T>1$ for any $\lambda>0$ and $T\geq 0$. It means that equation (1.2) is oscillatory for any $\lambda>0$. The proof of Theorem 3.3 is complete. □

Next, we suppose that functions v and u are positive and n times continuously differentiable on I. Then on the basis of the reciprocity principle [4] equation (1.2) and the reciprocal equation

$$(-1)^n\left(u^{-1}(t)y^{(n)}\right)^{(n)}-\lambda v^{-1}(t)y=0 \tag{3.15}$$

are simultaneously oscillatory or non-oscillatory.

Suppose that for equation (3.15) the following conditions

$$\int_T^\infty u(t)dt<\infty,\qquad \int_T^\infty u(t)t^2dt=\infty \quad\text{and}\quad \int_\alpha^\infty v^{-1}(t)t^{2(n-1)}dt=\infty \tag{3.16}$$

hold for any $\alpha\geq T$.

Applying the reciprocity principle, on the basis of Theorems 3.2 and 3.3 we get the following theorems.

Theorem 3.4. *Let* $T\geq 0$ *and* (3.16) *hold. Then equation* (1.2) *is strong non-oscillatory if and only if*

$$\lim_{z\to\infty}\int_z^\infty v^{-1}(x)(x-T)^{2(n-2)}dx\int_T^z (t-T)^2 u(t)dt=0, \tag{3.17}$$

$$\lim_{z\to\infty}\int_T^z v^{-1}(x)(x-T)^{2(n-1)}dx\int_z^\infty u(t)dt=0. \tag{3.18}$$

Theorem 3.5. *Let* $T\geq 0$ *and* (3.16) *hold. Then equation* (1.2) *is strong oscillatory if and only if*

$$\lim_{z\to\infty}\int_z^\infty v^{-1}(x)(x-T)^{2(n-2)}dx\int_T^z (t-T)^2 u(t)dt=\infty$$

or

$$\lim_{z\to\infty}\int_T^z v^{-1}(x)(x-T)^{2(n-1)}dx\int_z^\infty u(t)dt=\infty.$$

4 Spectral characteristics of L

Let the minimal differential operator $L_{\min}$ be generated by differential expression (1.3) in the space $L_{2,u}$ with inner product $(f,g)_{2,u} = \int_0^\infty f(t)g(t)u(t)dt$. It means that $L_{\min}(y) = l(y)$ is an operator with the domain

$$D(L_{\min}) = \Big\{ y : I \to R : y^{(i)} \in AC^{\mathrm{loc}}(I),\ \mathrm{supp}\ y^{(i)} \subset I, \\ \mathrm{supp}\ y^{(i)} \ \text{ is compact},\ i = 0, 1, \dots, n-1,\ l(y) \in L_{2,u} \Big\}.$$

It is known that all self-adjoint extensions of the minimal differential operator L have the same spectrums (see [9]).

Let us consider the problem of boundedness from below and discreteness of the operator L in case (1.6). Case (1.5) was considered in [21].

The relation between the oscillatory properties of equation (1.2) and spectral properties of the operator L is given in the following statement.

Lemma 4.1 ([9])**.** *The operator L is bounded from below and has the discrete spectrum if and only if equation* (1.2) *is strong non-oscillatory.*

On the basis of Lemma 4.1, by Theorems 3.2 and 3.4 we have the following theorem.

Theorem 4.2.

(i) If conditions (1.6) *and* (2.7) *hold, then the operator L is bounded from below and has the discrete spectrum if and only if* (3.4) *and* (3.5) *hold;*

(ii) If condition (3.16) *holds, then the operator L is bounded from below and has the discrete spectrum if and only if* (3.17) *and* (3.18) *hold.*

The operator $L_{\min}$ is non-negative. Therefore, it has the Friedrichs extension L_F. By Theorem 4.2 the operator L_F has the discrete spectrum if and only if (i) (3.4) and (3.5) hold in case (1.6) and (2.7); (ii) (3.17) and (3.18) hold in case (3.16).

From Theorem 2.1 we can state Theorem 4.3.

Theorem 4.3. *Let* (1.6) *and* (2.7) *hold. Then the operator L_F is positive defined if and only if $\mathcal{A}(0,\tau_0) = \max\{A_{2,1}(0,\tau_0), A_{2,2}(0,\tau_0)\} < \infty$. Moreover, there exist constants $\alpha, \beta : 0 < \alpha < \beta$ such that the estimate $\alpha\mathcal{A}(0,\tau_0) \le \lambda_1^{-1/2} \le \beta\mathcal{A}(0,\tau_0)$ holds for the smallest eigenvalue λ_1 of the operator L_F.*

On the basis of Rellih's Lemma [16, p. 183] and Theorem 2.1 it follows one more theorem.

Theorem 4.4. *Let* (1.6) *and* (2.7) *hold. Then*

(i) the embedding $\mathring{W}^n_{2,v}(I) \hookrightarrow L_{2,u}(I)$ is compact if and only if (3.4) *and* (3.5) *hold;*

(ii) the operator L_F^{-1} is completely continuous on $L_{2,u}$ if and only if (3.4) *and* (3.5) *hold.*

The next statement is presented in [3].

Lemma B. *Let $H = H(I)$ be a certain Hilbert function space and $C[0,\infty) \cap H$ be dense in it. For any point $x_0 \in I$ we introduce the operator $F_{x_0} f = f(x_0)$ defined on $C[0,\infty) \cap H$, which acts in the space of complex numbers. Let us assume that F_{x_0} is a closure operator. Then the norm of this operator is equal to the value $\left(\sum_{n=1}^\infty |\varphi_n(x_0)|^2\right)^{1/2}$ (finite or infinite), where $\{\varphi_n(\cdot)\}_{n=1}^\infty$ is any complete orthonormal system of continuous functions in H.*

Lemma 4.5. *Let* (1.6) *and* (2.7) *hold. Then for* $x \in I$

$$\frac{\sup_{\tau\in I} D(x,\tau)}{(n-1)!} \le \sup_{f\in \mathring{W}^n_{2,v}} \frac{|f(x)|}{\|f^{(n)}\|_{2,v}} \le \sqrt{2}\frac{\inf_{\tau\in I} D(x,\tau)}{(n-1)!}, \tag{4.1}$$

where $\tau \in I$ *and*

$$\begin{aligned} D(x,\tau) = \Big[& \chi_{(0,\tau)}(x)\int_0^x (x-s)^{2(n-1)} v^{-1}(s)ds \\ & + \chi_{(\tau,\infty)}(x)(n-1)^2 \int_0^\tau \left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\ & + \chi_{(\tau,\infty)}(x)(n-1)^2 \int_\tau^x \left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\ & + \chi_{(\tau,\infty)}(x)(x-\tau)^{2(n-1)} \int_x^\infty v^{-1}(s)ds \Big]^{1/2}. \end{aligned}$$

Proof. Let $f \in \mathring{W}^n_{2,v}$. Then due to (1.6) we have $f \in LR^{(n-1)}W^n_{2,v}$. Let $\tau \in I$. Similarly as in the proof of sufficiency of Theorem 2.1, we get

$$f(x) = \frac{1}{(n-1)!}\begin{cases} \int_0^x (x-s)^{n-1} f^{(n)}(s)ds & \text{if } 0 < x < \tau; \\ (n-1)\int_0^\tau (x-t)^{n-2}\int_0^t f^{(n)}(s)dsdt & \\ -(n-1)\int_\tau^x (x-t)^{n-2}\int_t^\infty f^{(n)}(s)dsdt & \text{if } x > \tau \end{cases}$$

or

$$f(x) = \frac{1}{(n-1)!}\begin{cases} \int_0^x (x-s)^{n-1} f^{(n)}(s)ds & \text{if } 0 < x < \tau; \\ (n-1)\int_0^\tau f^{(n)}(s)\int_s^\tau (x-t)^{n-2}dtds & \\ -(n-1)\int_\tau^x f^{(n)}(s)\int_\tau^s (x-t)^{n-2}dtds & \\ -(x-\tau)^{n-1}\int_x^\infty f^{(n)}(s)ds & \text{if } x > \tau, \end{cases}$$

for all $\tau \in I$. The last expression can be rewritten in the form

$$\begin{aligned} f(x) = \frac{1}{(n-1)!}\Big[& \chi_{(0,\tau)}(x)\int_0^x (x-s)^{n-1} f^{(n)}(s)ds \\ & + \chi_{(\tau,\infty)}(x)(n-1)\int_0^\tau f^{(n)}(s)\int_s^\tau (x-t)^{n-2}dtds\Big] \\ & - \chi_{(\tau,\infty)}(x)\left[(n-1)\int_\tau^x f^{(n)}(s)\int_\tau^s (x-t)^{n-2}dtds + (x-\tau)^{n-1}\int_x^\infty f^{(n)}(s)ds\right]. \end{aligned} \tag{4.2}$$

Using Hölder's inequality, we have

$$\begin{aligned} |f(x)| \le \frac{1}{(n-1)!}\Bigg\{ & \Bigg[\chi_{(0,\tau)}(x)\left(\int_0^x (x-s)^{2(n-1)} v^{-1}(s)ds\right)^{1/2} \\ & + \chi_{(\tau,\infty)}(x)(n-1)\left(\int_0^\tau \left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds\right)^{1/2}\Bigg]\left(\int_0^\tau v(t)|f^{(n)}(t)|^2dt\right)^{1/2} \\ & + \chi_{(\tau,\infty)}(x)\Bigg[(n-1)\left(\int_\tau^x \left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds\right)^{1/2} \\ & + (x-\tau)^{n-1}\left(\int_x^\infty v^{-1}(s)ds\right)^{1/2}\Bigg]\left(\int_\tau^\infty v(t)|f^{(n)}(t)|^2dt\right)^{1/2}\Bigg\}. \end{aligned} \tag{4.3}$$

One more time using Hölder's inequality for sums in (4.3), we obtain

$$
\begin{aligned}
|f(x)| \leq \frac{1}{(n-1)!} \Bigg\{ \Bigg[& \chi_{(0,\tau)}(x) \left(\int_0^x (x-s)^{2(n-1)} v^{-1}(s) ds \right)^{1/2} \\
& + \chi_{(\tau,\infty)}(x)(n-1) \left(\int_0^\tau \left(\int_s^\tau (x-t)^{n-2} dt \right)^2 v^{-1}(s) ds \right)^{1/2} \Bigg]^2 \\
& + \chi_{(\tau,\infty)}(x) \Bigg[(n-1) \left(\int_\tau^x \left(\int_\tau^s (x-t)^{n-2} dt \right)^2 v^{-1}(s) ds \right)^{1/2} \\
& + (x-\tau)^{n-1} \left(\int_x^\infty v^{-1}(s) ds \right)^{1/2} \Bigg]^2 \Bigg\}^{\frac{1}{2}} \\
& \times \left(\int_0^\tau v(t) |f^{(n)}(t)|^2 dt + \int_\tau^\infty v(t) |f^{(n)}(t)|^2 dt \right)^{\frac{1}{2}} \\
\leq \frac{1}{(n-1)!} \Bigg[& \chi_{(0,\tau)}(x) \int_0^x (x-s)^{2(n-1)} v^{-1}(s) ds \\
& + \chi_{(\tau,\infty)}(x)(n-1)^2 \int_0^\tau \left(\int_s^\tau (x-t)^{n-2} dt \right)^2 v^{-1}(s) ds \\
& + 2\chi_{(\tau,\infty)}(x)(n-1)^2 \int_\tau^x \left(\int_\tau^s (x-t)^{n-2} dt \right)^2 v^{-1}(s) ds \\
& + 2\chi_{(\tau,\infty)}(x)(x-\tau)^{2(n-1)} \int_x^\infty v^{-1}(s) ds \Bigg]^{\frac{1}{2}} \left(\int_0^\infty v(t) |f^{(n)}(t)|^2 dt \right)^{\frac{1}{2}}
\end{aligned}
$$

for any $\tau \in I$. Therefore,

$$
|f(x)| \leq \frac{\sqrt{2}}{(n-1)!} \inf_{\tau \in I} D(x,\tau) \left(\int_0^\infty v(t) |f^{(n)}(t)|^2 dt \right)^{1/2}.
$$

Then

$$
\sup_{f \in \mathring{W}^n_{2,v}} \frac{|f(x)|}{\|f^{(n)}\|_{2,v}} \leq \frac{\sqrt{2}}{(n-1)!} \inf_{\tau \in I} D(x,\tau). \tag{4.4}
$$

Now, we estimate the value $\sup_{f \in \mathring{W}^n_{2,v}} \frac{|f(x)|}{\|f^{(n)}\|_{2,v}}$ from below. In (4.2) we fix $x \in I$, so that we choose a function $f^{(n)}$, depending on x, as follows

$$
f_x^{(n)}(s) = \begin{cases}
\chi_{(0,x)}(s)(x-s)^{n-1} v^{-1}(s) & \text{if } 0 < x < \tau; \\
\chi_{(0,\tau)}(s)(n-1) \int_s^\tau (x-t)^{n-2} dt v^{-1}(s) & \\
-\chi_{(\tau,x)}(s)(n-1) \int_\tau^s (x-t)^{n-2} dt v^{-1}(s) & \\
-\chi_{(x,\infty)}(s)(x-\tau)^{n-1} v^{-1}(s) & \text{if } x > \tau.
\end{cases}
$$

Replacing this function in (4.2), we get the value of the function $f(f_x^{(n)})(t)$ at the point $t = x$:

$$
\begin{aligned}
f_x(x) = \frac{1}{(n-1)!}\Bigg(\chi_{(0,\tau)}(x)\int_0^x (x-s)^{n-1} f_x^{(n)}(s)ds \\
+\chi_{(\tau,\infty)}(x)(n-1)\int_0^\tau f_x^{(n)}(s)\int_s^\tau (x-t)^{n-2}dtds \\
-\chi_{(\tau,\infty)}(x)(n-1)\int_\tau^x f_x^{(n)}(s)\int_\tau^s (x-t)^{n-2}dtds \\
-\chi_{(\tau,\infty)}(x)(x-\tau)^{n-1}\int_x^\infty f_x^{(n)}(s)ds\Bigg).
\end{aligned}
$$

If $0 < x < \tau$, then $\chi_{(\tau,\infty)}(x) = 0$. Hence, all terms of $f_x(x)$, except the first one, are equal to zero. For the first term the variable s changes from 0 to x, i.e., $\chi_{(0,x)}(s) \neq 0$ and we replace $f_x^{(n)}(s)$ with $(x-s)^{n-1}v^{-1}(s)$. If $x > \tau$, then $\chi_{(0,\tau)}(x) = 0$. It means that the first term is equal to zero, so $f_x(x)$ is defined by the other three terms. In this case, we replace $f_x^{(n)}(s)$ with its values in the intervals $(0,\tau)$, (τ, x) and (x,∞), respectively. Thus, we get

$$
\begin{aligned}
f_x(x) = \frac{1}{(n-1)!}\Bigg(\chi_{(0,\tau)}(x)\int_0^x (x-s)^{2(n-1)} v^{-1}(s)ds \\
+\chi_{(\tau,\infty)}(x)(n-1)^2\int_0^\tau \left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\
+\chi_{(\tau,\infty)}(x)(n-1)^2\int_\tau^x \left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\
+\chi_{(\tau,\infty)}(x)(x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)ds\Bigg) = \frac{D^2(x,\tau)}{(n-1)!}
\end{aligned} \tag{4.5}
$$

for any $\tau \in I$.

Let us calculate the norm $L_{2,v}$ of the function $f_x^{(n)}$. For $0 < x < \tau$ we take $f_x^{(n)}(s) = \chi_{(0,x)}(s)(x-s)^{n-1}v^{-1}(s)$ and have

$$
\int_0^\infty v(s)|f_x^{(n)}(s)|^2 ds = \int_0^x v(s)\left((x-s)^{n-1}v^{-1}(s)\right)^2 ds = \int_0^x (x-s)^{2(n-1)} v^{-1}(s)ds. \tag{4.6}
$$

For $x > \tau$ we take the values of $f_x^{(n)}$ on the intervals $(0,\tau)$, (τ, x) and (x,∞), respectively, and get

$$
\begin{aligned}
\int_0^\infty v(s)|f_x^{(n)}(s)|^2 ds &= \int_0^\tau v(s)|f_x^{(n)}(s)|^2 ds + \int_\tau^x v(s)|f_x^{(n)}(s)|^2 ds + \int_x^\infty v(s)|f_x^{(n)}(s)|^2 ds \\
&= (n-1)^2\int_0^\tau \left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\
&\quad + (n-1)^2\int_\tau^x \left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\
&\quad + (x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)ds.
\end{aligned} \tag{4.7}
$$

Then using the functions $\chi_{(0,\tau)}(x)$ and $\chi_{(\tau,\infty)}(x)$, we combine (4.6) and (4.7) and obtain

$$\begin{aligned}\left(\int_0^\infty v(t)|f_x^{(n)}(t)|^2dt\right)^{1/2} = \Big[&\chi_{(0,\tau)}(x)\int_0^x (x-s)^{2(n-1)}v^{-1}(s)ds \\ &+\chi_{(\tau,\infty)}(x)(n-1)^2\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\ &+\chi_{(\tau,\infty)}(x)(n-1)^2\int_\tau^x\left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)ds \\ &+\chi_{(\tau,\infty)}(x)(x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)ds\Big]^{1/2} = D(x,\tau) \qquad (4.8)\end{aligned}$$

for any $\tau\in I$.

From (4.5) and (4.8) we get

$$\sup_{f\in \mathring{W}^n_{2,v}}\frac{|f(x)|}{\|f^{(n)}\|_{2,v}} \geq \frac{|f_x(x)|}{\|f_x^{(n)}\|_{2,v}} = \frac{\sup_{\tau\in I}D(x,\tau)}{(n-1)!}.$$

This relation together with (4.4) gives (4.1). The proof of Lemma 4.5 is complete. □

Let the operator L_F^{-1} be completely continuous on $L_{2,u}$. Let $\{\lambda_k\}_{k=1}^\infty$ be eigenvalues and $\{\varphi_k\}_{k=1}^\infty$ be a corresponding complete orthonormal system of eigenfunctions of the operator L_F^{-1}.

Theorem 4.6. *Let (1.6), (2.7), (3.4) and (3.5) hold. Then*

(i)
$$\frac{\sup_{\tau\in I}D^2(x,\tau)}{[(n-1)!]^2} \leq \sum_{k=1}^\infty \frac{|\varphi_k(x)|^2}{\lambda_k} \leq \sqrt{2}\frac{\inf_{\tau\in I}D^2(x,\tau)}{[(n-1)!]^2}; \qquad (4.9)$$

(ii) the operator L_F^{-1} is nuclear if and only if $\inf_{\tau\in I}\int_0^\infty u(x)D^2(x,\tau)dx<\infty$. Moreover, there exists $\tau=\mu\in I$ and for the nuclear norm $\|L_F^{-1}\|_{\sigma_1}$ of the operator L_F^{-1} the relation

$$\frac{2}{[(n-1)!]^2}D_1(\mu) \leq \|L_F^{-1}\|_{\sigma_1} = \sum_{k=1}^\infty\frac{1}{\lambda_k} \leq \frac{2\sqrt{2}}{[(n-1)!]^2}D_1(\mu) \qquad (4.10)$$

holds, where

$$\begin{aligned}D_1(\mu) = (n-1)^2\int_\mu^\infty u(x)\int_\mu^x\left(\int_\mu^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx \\ +\int_\mu^\infty u(x)(x-\mu)^{2(n-1)}\int_x^\infty v^{-1}(s)dsdx.\end{aligned}$$

Proof. By the condition of Theorem 4.4 we have that the operator L_F^{-1} is completely continuous on $L_{2,u}$. In Lemma B we take $\mathring{W}^n_{2,v}(I)$ with the norm $\left(\int_0^\infty v(t)|f^{(n)}(t)|^2\right)^{1/2}$ as the space $H(I)$. Since the system of functions $\{\lambda_k^{-1/2}\varphi_k\}_{k=1}^\infty$ is complete orthonormal system in the space $\mathring{W}^n_{2,v}(I)$, then by Lemma B we get

$$\|F_x\|^2 = \left(\sup_{f\in\mathring{W}^n_{2,v}}\frac{|f(x)|}{\|f^{(n)}\|_{2,v}}\right)^2 = \sum_{k=1}^\infty\frac{|\varphi_k(x)|^2}{\lambda_k},$$

where $F_x f = f(x)$. This and (4.1) give (4.9).

Since $\inf_{s\in I} D^2(x,s) \le D^2(x,\tau) \le \sup_{s\in I} D^2(x,s)$ for any $\tau \in I$, multiplying both sides of (4.9) by u and integrating them with respect to x from zero to infinity, we get

$$\frac{1}{[(n-1)!]^2}\int_0^\infty u(x)D^2(x,\tau)dx \le \sum_{k=1}^\infty \frac{1}{\lambda_k} \le \frac{\sqrt{2}}{[(n-1)!]^2}\int_0^\infty u(x)D^2(x,\tau)dx \tag{4.11}$$

for all $\tau \in I$. Let us present the integral $\int_0^\infty u(x)D^2(x,\tau)dx$ in the following way

$$\begin{aligned}\int_0^\infty u(x)D^2(x,\tau)dx = &\int_0^\tau u(x)\int_0^x (x-s)^{2(n-1)}v^{-1}(s)dsdx\\ &+(n-1)^2\int_\tau^\infty u(x)\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx\\ &+(n-1)^2\int_\tau^\infty u(x)\int_\tau^x\left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx\\ &+\int_\tau^\infty u(x)(x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)dsdx = D_0(\tau)+D_1(\tau),\end{aligned}$$

where

$$\begin{aligned}D_0(\tau) = &\int_0^\tau u(x)\int_0^x (x-s)^{2(n-1)}v^{-1}(s)dsdx\\ &+(n-1)^2\int_\tau^\infty u(x)\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx,\end{aligned}$$

$$\begin{aligned}D_1(\tau) = &(n-1)^2\int_\tau^\infty u(x)\int_\tau^x\left(\int_\tau^s (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx\\ &+\int_\tau^\infty u(x)(x-\tau)^{2(n-1)}\int_x^\infty v^{-1}(s)dsdx.\end{aligned}$$

The functions $D_0(\tau)$, $D_1(\tau)$ are continuous and the function $D_1(\tau)$ is decreasing on the interval I and $\lim_{\tau\to\infty} D_1(\tau) = 0$. Since

$$\begin{aligned}&\int_\tau^\infty u(x)\int_0^\tau\left(\int_s^\tau (x-t)^{n-2}dt\right)^2 v^{-1}(s)dsdx\\ &\approx \int_\tau^\infty u(x)(x-\tau)^{2(n-2)}dx\int_0^\tau(\tau-s)^2v^{-1}(s)ds\\ &\quad+\int_\tau^\infty u(x)dx\int_0^\tau(\tau-s)^{2(n-1)}v^{-1}(s)ds,\end{aligned}$$

then we get $\lim_{\tau\to 0^+} D_0(\tau) = 0$. Therefore, there exists a point $\tau = \mu$ such that $D_0(\mu) = D_1(\mu)$. Hence, from (4.11) we have (4.10). The proof of Theorem 4.6 is complete. □

Remark 4.7. In Theorems 4.3, 4.4 and 4.6 and in their proofs replacing v^{-1} by u, u by v^{-1} and conditions (3.4) and (3.5) by (3.17) and (3.18) in the required places, we get the similar statements but under condition (3.16).

Acknowledgements

The authors would like to thank Reviewer for generous comments and suggestions, which have improved this paper. The paper was written with financial support of the Ministry of Education and Science of the Republic of Kazakhstan, grant no. AP08856100 in the area "Research in the field of natural sciences".

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **4**, 1–19; https://doi.org/10.14232/ejqtde.2021.1.4 www.math.u-szeged.hu/ejqtde/

Existence of a positive bound state solution for the nonlinear Schrödinger–Bopp–Podolsky system

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Received 10 July 2020, appeared 11 January 2021
Communicated by Patrizia Pucci

Abstract. In this paper, we study a class of Schrödinger–Bopp–Podolsky system. Under some suitable assumptions for the potentials, by developing some calculations of sharp energy estimates and using a topological argument involving the barycenter function, we establish the existence of positive bound state solution.

Keywords: Schrödinger–Bopp–Podolsky system, variational approach, competing potentials, bound state solution.

2020 Mathematics Subject Classification: 35J48, 35Q60.

1 Introduction

In this paper, we consider the following Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = Q(x)|u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $p \in (3,5)$, $V(x)$, $K(x)$ and $Q(x)$ are positive functions such that

$$\lim_{|x|\to\infty} V(x) = V_\infty > 0, \qquad \lim_{|x|\to\infty} Q(x) = Q_\infty > 0, \qquad \lim_{|x|\to\infty} K(x) = 0.$$

This system appears when a Schrödinger field $\psi = \psi(t,x)$ couple with its electromagnetic field in the Bopp–Podolsky electromagnetic theory. The Bopp–Podolsky theory, developed by Bopp [8], and independently by Podolsky [20], is a second order gauge theory for the electromagnetic field. As the Mie theory [19] and its generalizations given by Born and Infeld [7, 9], it was introduced to solve the so called infinity problem that appears in the classical Maxwell theory. From the well known Gauss law (or Poisson equation), the electrostatic potential ϕ for a given charge distribution whose density is ρ satisfies the equation

$$-\Delta\phi = \rho \ \text{ in } \mathbb{R}^3. \tag{1.2}$$

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If $\rho = 4\pi\delta_{x_0}$, with $x_0 \in \mathbb{R}^3$, the fundamental solution of (1.2) is $E(x - x_0)$, where $E(x) = \frac{1}{|x|}$, and the electrostatic energy is $\varepsilon(E) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla E(x)|^2 \mathrm{d}x = +\infty$. Thus, equation (1.2) was replaced by

$$-\mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = \rho \ \text{ in } \mathbb{R}^3$$

in the Born–Infeld theory (see [2]) or replaced by

$$-\Delta\phi + a^2\Delta^2\phi = \rho \ \text{ in } \mathbb{R}^3$$

in the Bopp–Podolsky one, from the reason that, in both case if $\rho = 4\pi\delta_{x_0}$, their energy is finite. In particular, when we consider the operator $-\Delta + \Delta^2$, by [3], we know that $\mathcal{K}(x - x_0)$, where $\mathcal{K}(x) := \frac{1-e^{-|x|}}{|x|}$, is the fundamental solution of the equation

$$-\Delta\phi + \Delta^2\phi = 4\pi\delta_{x_0},$$

it has no singularity in x_0 since it satisfies $\lim_{x\to x_0} \mathcal{K}(x - x_0) = 1$, and its energy satisfies

$$\varepsilon_{BP}(\mathcal{K}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\mathcal{K}|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |\Delta\mathcal{K}|^2 \,\mathrm{d}x < +\infty.$$

In addition, the Bopp–Podolsky theory may be interpreted as an effective theory for short distances and for large distances it is indistinguishable from the Maxwell one. For more physical details we refer the reader to the recent paper[10,11,14] and their references therein. Indeed the operator $-\Delta + \Delta^2$ appears also in other different interesting mathematical and physical situations [5,15].

Recently, P. d'Avenia and G. Siciliano in [3] introduced the following Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where a, $\omega > 0$, $q \neq 0$, they developed the variational framework for system (1.3) and proved that when $p \in (2, 6)$, there exists $q^* > 0$, for every $q \in (-q^*, q^*) \setminus \{0\}$, problem (1.3) admits a nontrivial solution, when $p \in (3, 6)$, for $q \neq 0$, problem (1.3) admits a nontrivial solution. In [22], G. Siciliano and K. Silva proved that the multiplicity and nonexistence of solutions for problem (1.3) by using the fibering method.

If $a = 0$ in problem (1.3), it reduces to the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 4\pi u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.4}$$

In the last decades, there are lots of results about the existence and multiplicity of solutions for system (1.4), we do not review the huge documents, just list some of them for interesting readers to see [1,6,12,21] and the references therein.

The purpose of this paper is to describe some phenomena that can occur when the coefficients $V(x)$, $K(x)$ and $Q(x)$ are competing. In order to describe our main results, we first rewrite problem (1.1) in a more appropriate way to our aim. Let

$$V(x) = V_\infty + a(x), \qquad Q(x) = Q_\infty - b(x),$$

where $a(x)$ and $b(x)$ satisfies the following assumptions:

(H_1) $a(x) \in L^{\frac{3}{2}}(\mathbb{R}^3)$, $a(x) \geq 0$, $a(x) \not\equiv 0$, and $\lim_{|x|\to\infty} a(x) = 0$;

(H_2) $b(x) \in L^\infty(\mathbb{R}^3)$, $0 \leq b(x) < Q_\infty$, $b(x) \not\equiv 0$, and $\lim_{|x|\to\infty} b(x) = 0$;

(H_3) $K \in L^2(\mathbb{R}^3)$ and there exists $R_0 > 0$ such that $K(x) \leq Ce^{-2\sqrt{V_\infty}|x|}$ for all $|x| \geq R_0$.

Clearly, (1.1) becomes the following form

$$\begin{cases} -\Delta u + (V_\infty + a(x))u + K(x)\phi u = (Q_\infty - b(x))|u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = K(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.5}$$

From the variational framework described in Section 2, we find that the difference between problem (1.5) and system (1.4) is the nonlocal kernel $\mathcal{K}(x) = \frac{1-e^{-|x|}}{|x|}$, comparing the Poisson kernel $\mathcal{P}(x) = \frac{1}{|x|}$, $\mathcal{K}(x)$ is nonhomogeneous and not singular at $x = 0$ because $\lim_{|x|\to 0} \mathcal{K}(x) = 1$, which implies that $\mathcal{K} \in L^\infty(\mathbb{R}^3)$. The non-homogeneity of $\mathcal{K}$ makes difficult the use of rescaling of type $t \to u(t^\gamma\cdot)$ and the non-singularity maybe weak some conditions in the estimates.

To the best of knowledge, the system (1.5) is a new one in the field of variational methods, there are only few works about the existence and multiplicity of solutions, such as the ground state. The purpose of this paper is to study the existence of bound state solution for system (1.5). The approach is inspired by the ideas in [4, 12], we explore some calculations of sharp energy estimates and apply a topological argument involving the barycenter function to show that there exists a critical value of the energy functional, in a higher level of energy which can yield a solution of the problem (1.5). The main difficulties of this work are that the problem is given in the whole space, leading to the loss of compactness, and some sharp energy estimates. For dealing with these obstacles, we borrows a global compactness lemma to recover the compactness and use some careful computations to get the energy estimates.

Now we are ready to give the main results of the paper.

Theorem 1.1. *Suppose that (H_1)–(H_3) hold and*

(H_4) $\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)|x|^2 e^{2\sqrt{V_\infty}|x|}\,\mathrm{d}x < +\infty$ *and* $\int_{\mathbb{R}^3} b(x)|x|^2 e^{2\sqrt{V_\infty}|x|}\,\mathrm{d}x < +\infty$.

Then (1.5) *admits a positive bound state solution.*

The paper is organised as follows. In Section 2, we give general preliminaries in order to attack our problem. In Section 3, we prove Theorem 1.1.

2 Preliminaries

In what follows, we will use the following notations:

- Let $H^1(\mathbb{R}^3)$ be the Sobolev space endowed with the inner product and norm

$$(u, v) := \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)dx, \qquad \|u\|^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx.$$

- $\mathcal{D}$ is the completion of $\mathbf{C}_c^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + |\Delta u|^2)\,\mathrm{d}x\right)^{\frac{1}{2}}$$

- $\mathcal{L}^q(\mathcal{O})$, $1 \le q \le \infty$, $\mathcal{O} \subseteq \mathbb{R}^3$ a measurable set, denotes the Lebesgue space, the norm in $\mathcal{L}^q(\mathcal{O})$ is denoted by $|\cdot|_{\mathcal{L}^q(\mathcal{O})}$ when $\mathcal{O}$ is a proper measurable subset of $\mathbb{R}^3$ and by $|\cdot|_q$ when $\mathcal{O} = \mathbb{R}^3$.
- $B_R(y)$ denotes the ball of radius R centered at y, if $y = 0$, we denote it by B_R.
- $c, c_i, C, C_i, \ldots$ denote a number of positive constants.

In what follows, without any loss of generality we assume $V_\infty = Q_\infty = 1$.

From [3], we know that for $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi \in \mathcal{D}$, denoted by ϕ_u^K, such that $-\Delta\phi + \Delta^2\phi = K(x)u^2$ and it possesses the explicit formula

$$\phi_u^K(x) := \phi(x) = \int_{\mathbb{R}^3} \frac{(1 - e^{-|x-z|})}{4\pi|x-z|} K(z)u^2(z)\,\mathrm{d}z, \qquad x \in \mathbb{R}^3. \tag{2.1}$$

Replacing ϕ by ϕ_u^K in the first equation in system (1.5), then this system reduces to a class of Schrödinger equation

$$-\Delta u + (V_\infty + a(x))u + K(x)\phi_u^K u = (Q_\infty - b(x))|u|^{p-1}u \quad \text{in } \mathbb{R}^3. \tag{2.2}$$

The energy functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ corresponding to equation (2.2) is defined as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + (V_\infty + a(x))u^2\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^K u^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^3}(Q_\infty - b(x))|u|^{p+1}\,\mathrm{d}x,$$

clearly, $I \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ and its critical points are weak solutions of problem (2.2). Therefore, in order to find the solutions of system (1.5), we only need to seek the critical points of functional I. In other words, if $u \in H^1(\mathbb{R}^3)$ is a critical point of I, then (u, ϕ_u) is a weak solution for system (1.5).

Now, by Lemma 3.4 in [3] and applying a similar argument as in Proposition 2.2 of [12], we can show some properties of ϕ_u.

Proposition 2.1. *For each $u \in H^1(\mathbb{R}^3)$, the following statements hold:*

(i) $\phi_u^K \in \mathcal{D} \hookrightarrow L^\infty(\mathbb{R}^3)$;

(ii) $\phi_u^K \ge 0$;

(iii) *For every $s \in (3, +\infty]$, $\phi_u^K \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;*

(iv) *For every $s \in (\frac{3}{2}, +\infty]$, $\nabla\phi_u^K \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;*

(v) $\phi_{tu}^K = t^2\phi_u^K$;

(vi) $|\phi_u^K|_6 \le c\|u\|^2$;

(vii) *For every $y \in \mathbb{R}^3$, $\phi_{u(\cdot+y)}^K = \phi_u^{K(\cdot-y)}(\cdot + y)$;*

(viii) *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then*

$$\phi_{u_n}^K \to \phi_u^K \quad \text{in } \mathcal{D}, \qquad \int_{\mathbb{R}^3} K(x)\phi_{u_n}^K u_n^2\,\mathrm{d}x \to \int_{\mathbb{R}^3} K(x)\phi_u^K u^2\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}^K u_n \varphi\,\mathrm{d}x \to \int_{\mathbb{R}^3} K(x)\phi_u^K u\varphi\,\mathrm{d}x \qquad \forall\, \varphi \in H^1(\mathbb{R}^3).$$

Proof. We only need to verify that (iii), (iv) and $(viii)$ hold true.

Observe that $Ku^2 \in L^r(\mathbb{R}^3)$ for $r \in [1, \frac{3}{2})$ owing to $K \in L^2(\mathbb{R}^3)$ and $u \in H^1(\mathbb{R}^3)$. By (ii) of Lemma 3.3 in [3], we know that $\phi_u^K \in L^q(\mathbb{R}^3)$ for $q \in (\frac{3r}{3-2r}, +\infty]$. From $\frac{3r}{3-2r} \in [3, +\infty)$ and using Lemma 2.20 in [18], we see that $\phi_u^K \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$. Similarly, we can get (iv).

$(viii)$ For all $u \in H^1(\mathbb{R}^3)$, consider the linear functional $\mathcal{L}_u : \mathcal{D} \to \mathbb{R}^3$ defined by

$$\mathcal{L}_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v\,\mathrm{d}x,$$

by the definition of ϕ_u^K, we have $\|\phi_u^K\|_{\mathcal{D}} = \|\mathcal{L}_u\|_{\mathcal{L}(\mathcal{D},\mathbb{R})}$. Therefore we intend to show that as $n \to \infty$

$$\|\mathcal{L}_{u_n} - \mathcal{L}_u\|_{\mathcal{L}(\mathcal{D},\mathbb{R})} \to 0.$$

Let $\varepsilon > 0$ be fixed, there exists a positive number R_ε so large that $|K|_{L^2(\mathbb{R}^3\setminus B(0,R_\varepsilon))} < \varepsilon$. Therefore for $v \in \mathcal{D}(\mathbb{R}^3)$, we have

$$\begin{aligned}
|\mathcal{L}_{u_n}(v) - \mathcal{L}_u(v)| &= \int_{\mathbb{R}^3} K(u_n^2 - u^2)v\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^3\setminus B(0,R_\varepsilon)} K|u_n^2 - u^2||v|\,\mathrm{d}x + \int_{B(0,R_\varepsilon)} K|u_n^2 - u^2||v|\,\mathrm{d}x \\
&\le |K|_{L^2(\mathbb{R}^3\setminus B(0,R_\varepsilon))}|u_n^2 - u^2|_3|v|_6 + \left[\int_{B(0,R_\varepsilon)} K^{\frac{6}{5}}|u_n^2 - u^2|^{\frac{6}{5}}\,\mathrm{d}x\right]^{\frac{5}{6}}|v|_6 \\
&\le \left[\varepsilon c + \left[\int_{B(0,R_\varepsilon)} K^{\frac{6}{5}}|u_n+u|^{\frac{6}{5}}|u_n-u|^{\frac{6}{5}}\,\mathrm{d}x\right]^{\frac{5}{6}}\right]\|v\|_{\mathcal{D}}
\end{aligned}$$

Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, we know that $u_n \to u$ in $L^{\frac{12}{5}}_{loc}(\mathbb{R}^3)$. Furthermore, set $B_M = \{x \in B(0,R_\varepsilon) : K(x) > M\}$ and remark being $K \in L^2(\mathbb{R}^3)$, $|B_M| \to 0$ as $M \to \infty$. Therefore, for large M, $\left(\int_{B_M} K^2\right)^{\frac{3}{5}} < \varepsilon$. So we have

$$\begin{aligned}
&\int_{B(0,R_\varepsilon)} K^{\frac{6}{5}}|u_n+u|^{\frac{6}{5}}|u_n-u|^{\frac{6}{5}}\,\mathrm{d}x \\
&= \int_{B_M} K^{\frac{6}{5}}|u_n+u|^{\frac{6}{5}}|u_n-u|^{\frac{6}{5}}\,\mathrm{d}x + \int_{B(0,R_\varepsilon)\setminus B_M} K^{\frac{6}{5}}|u_n+u|^{\frac{6}{5}}|u_n-u|^{\frac{6}{5}}\,\mathrm{d}x \\
&\le \int_{B_M} |K^2|^{\frac{3}{5}}\left(\int_{\mathbb{R}^3}|u_n+u|^6\right)^{\frac{1}{5}}\left(\int_{\mathbb{R}^3}|u_n-u|^6\right)^{\frac{1}{5}}\,\mathrm{d}x \\
&+ M^{\frac{6}{5}}\left(\int_{B(0,R_\varepsilon)}|u_n+u|^{\frac{12}{5}}\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{B(0,R_\varepsilon)}|u_n-u|^{\frac{12}{5}}\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le c\varepsilon + o(1).
\end{aligned}$$

Therefore $\phi_{u_n}^K \to \phi_u^K$ in $\mathcal{D}$. And

$$\int_{\mathbb{R}^3}(K(x)\phi_{u_n}^K u_n^2 - K(x)\phi_u^K u^2)\,\mathrm{d}x = \int_{\mathbb{R}^3} K(x)(u_n^2 - u^2))\phi_{u_n}^K\,\mathrm{d}x + \int_{\mathbb{R}^3} K(x)u^2(\phi_{u_n}^K - \phi_u^K)\,\mathrm{d}x.$$

Similar to the proof of the above, we can show that

$$\int_{\mathbb{R}^3} K(x)(u_n^2 - u^2))\phi_{u_n}^K = o(1).$$

Because when $n \to \infty$, $\phi_{u_n}^K \to \phi_u^K$, and by (H_3) we know that

$$\int_{\mathbb{R}^3} K(x)u^2(\phi_{u_n}^K - \phi_u^K)\,\mathrm{d}x \le |K|_2|u^2|_3|\phi_{u_n}^K - \phi_u^K|_6 = o(1).$$

Finally,

$$\int_{\mathbb{R}^3}(K(x)\phi_{u_n}^K u_n\varphi - K(x)\phi_u^K u\varphi)\,\mathrm{d}x = \int_{\mathbb{R}^3}(K(x)\varphi u_n)(\phi_{u_n}^K - \phi_u^K)\,\mathrm{d}x + \int_{\mathbb{R}^3}(K(x)\varphi\phi_u^K)(u_n - u)\,\mathrm{d}x.$$

This can be easily proved similar to the above. □

It is not difficult to verify that the functional I is bounded neither from above nor from below in $H^1(\mathbb{R}^3)$. Indeed, there exists $t \in \mathbb{R}^+$ such that $tu \in H^1(\mathbb{R}^3)$ satisfies

$$I(tu) = \frac{t^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + (V_\infty + a(x))u^2\,\mathrm{d}x + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u^K u^2\,\mathrm{d}x - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3}(Q_\infty - b(x))|u|^{p+1}\,\mathrm{d}x.$$

Since $p>3$, $\lim_{t\to+\infty} I(tu)=-\infty$. On the other hand, for some $\alpha,\beta\in\mathbb{R}$ and for any $t>0$, we have

$$\begin{aligned}
I(t^\alpha u(t^\beta x)) &= \frac{1}{2}\int_{\mathbb{R}^3} t^{2\alpha+2\beta}|\nabla u(t^\beta x)|^2+(V_\infty+a(x))t^{2\alpha}|u(t^\beta x)|^2\,\mathrm{d}x\\
&\quad+\frac{t^{4\alpha}}{4}\int_{\mathbb{R}^3} K(x)\phi_u^K u(t^\beta x)^2\,\mathrm{d}x-\frac{t^{(p+1)\alpha}}{p+1}\int_{\mathbb{R}^3}(Q_\infty-b(x))|u(t^\beta x)|^{p+1}\,\mathrm{d}x\\
&=\frac{1}{2}\int_{\mathbb{R}^3} t^{2\alpha+2\beta-3\beta}|\nabla u|^2+\Big(V_\infty+a\Big(\frac{x}{t^\beta}\Big)\Big)t^{2\alpha-3\beta}|u|^2\,\mathrm{d}x\\
&\quad+\frac{t^{4\alpha-3\beta}}{4}\int_{\mathbb{R}^3} K\Big(\frac{x}{t^\beta}\Big)\phi_u^{K(\frac{\cdot}{t^\beta})}u^2\,\mathrm{d}x-\frac{t^{(p+1)\alpha-3\beta}}{p+1}\int_{\mathbb{R}^3}\Big(Q_\infty-b\Big(\frac{x}{t^\beta}\Big)\Big)u^{p+1}\,\mathrm{d}x.
\end{aligned}$$

By (H_1)–(H_3), choosing α,β such that $2\alpha-\beta>(p+1)\alpha-3\beta$, that is $2\beta>(p-1)\alpha$. Particularly, we chose that $\alpha=1,\beta=p$, then $\lim_{t\to+\infty} I(tu)=+\infty$.

Naturally, we consider that the functional I restricted in the Nehari manifold $\mathcal{N}$, that contains all the critical points of I, is bounded from below, where

$$\mathcal{N}:=\Big\{u\in H^1(\mathbb{R}^3)\backslash\{0\}:\ I'(u)[u]=0\Big\}.$$

By using a standard argument, we can show the following lemma.

Lemma 2.2. *Suppose that (H_1)–(H_3) hold, the following statements are true:*

(i) *There exists a positive constant $c>0$ such that for all $u\in\mathcal{N}$, there holds*

$$|u|_{p+1}\geq c>0.$$

(ii) *$\mathcal{N}$ is a C^1 regular manifold diffeomorphic to the sphere of $H^1(\mathbb{R}^3)$.*

(iii) *I is bounded from below on $\mathcal{N}$ by a positive constant.*

(iv) *u is a free critical point of I if and only if u is a critical point of I constrained on $\mathcal{N}$.*

Proof. (i) Let $u\in\mathcal{N}$, by (H_1)–(H_3) and Sobolev's embedding theorem, we have that

$$\begin{aligned}
c_1|u|_{p+1}^2\leq\|u\|^2&\leq\|u\|^2+\int_{\mathbb{R}^3}a(x)u^2\,\mathrm{d}x+\int_{\mathbb{R}^3}K(x)\phi_u^K u^2\,\mathrm{d}x\\
&=\int_{\mathbb{R}^3}(Q_\infty-b(x))|u|^{p+1}\,\mathrm{d}x\leq c_2|u|_{p+1}^{p+1},
\end{aligned}\tag{2.3}$$

where $c_1,c_2>0$ independent of u, and owing to $p>3$, this estimate implies that

$$|u|_{p+1}\geq c>0,\qquad \|u\|\geq c>0,\qquad \forall u\in\mathcal{N},\tag{2.4}$$

where $c>0$ independent of u.

(ii) It suffices to show that $G'(u)[u]<0$ for $u\in\mathcal{N}$, where $G(u)=I'(u)[u]$. Clearly, $G\in C^1(H^1(\mathbb{R}^3),\mathbb{R})$. By (2.4), for any $u\in\mathcal{N}$, we deduce that

$$\begin{aligned}
G'(u)[u]&=2\int_{\mathbb{R}^3}|\nabla u|^2+(V_\infty+a(x))u^2\,\mathrm{d}x+4\int_{\mathbb{R}^3}K(x)\phi_u^K u^2\,\mathrm{d}x\\
&\quad-(p+1)\int_{\mathbb{R}^3}(Q_\infty-b(x))|u|^{p+1}\,\mathrm{d}x\\
&=-(p-1)\int_{\mathbb{R}^3}|\nabla u|^2+(V_\infty+a(x))u^2\,\mathrm{d}x-(p-3)\int_{\mathbb{R}^3}K(x)\phi_u^K u^2\,\mathrm{d}x\\
&\leq-(p-3)C<0,
\end{aligned}\tag{2.5}$$

where $C>0$ is dependent of u. By applying the implicit function theorem, we see that $\mathcal{N}$ is of C^1 manifold.

The remaining proofs of (ii), (iii) and (iv) are standard, we omit them. □

Now, the limit equation corresponding to problem (2.2) is defined as

$$-\Delta u + V_\infty u = Q_\infty |u|^{p-1}u \quad \text{in } \mathbb{R}^3. \tag{2.6}$$

The energy functional $I_\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$ associated to problem (2.6) given by

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty |u|^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^3} Q_\infty |u|^{p+1} \, dx$$

and it is easy to verify that $I_\infty \in C^2(H^1(\mathbb{R}^3), \mathbb{R})$. Denote the Nehari manifold of functional I_∞ by

$$\mathcal{N}_\infty := \Big\{ u \in H^1(\mathbb{R}^3)\backslash(0) : I'_\infty(u)[u] = 0 \Big\}.$$

Set

$$m_\infty := \inf_{u \in \mathcal{N}_\infty} I_\infty(u).$$

$m_\infty > 0$ is achieved by a positive radially symmetric function ω, that is unique (up to translations) positive solution of (2.6) (see [17]), decreasing when the radial coordinate increases and such that

$$\lim_{|x|\to\infty} |D^j \omega(x)||x| e^{\sqrt{V_\infty}|x|} = d_j > 0, \qquad j = 0, 1, \ d_j \in \mathbb{R}. \tag{2.7}$$

Moreover, for any sign-changing critical point u of I_∞, by standard argument, the following inequality holds true

$$I_\infty(u) \geq 2m_\infty. \tag{2.8}$$

Now we are ready to consider the constrained minimization problem $m := \inf\{I(u),\ u \in \mathcal{N}\}$, we find that the relation between least energy m and m_∞ holds and it is not achieved, thus we can not look for the ground state.

Proposition 2.3. *Suppose that* (H_1)–(H_3) *hold. Then there holds*

$$m = m_\infty \tag{2.9}$$

and the infimum is not achieved.

Proof. Let $u \in \mathcal{N}$, then there exists $t_u > 0$ such that $t_u u \in \mathcal{N}_\infty$. Thus, we deduce that

$$I(u) \geq I(t_u u) \geq \frac{t_u^2}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2) \, dx - \frac{t_u^{p+1}}{p+1}\int_{\mathbb{R}^3} Q_\infty |u|^{p+1} \, dx = I_\infty(t_u u) \geq m_\infty$$

from which, we get $m \geq m_\infty$.

Next, it suffices to find a sequence $(u_n)_n$, $u_n \in \mathcal{N}$, such that $\lim_{n\to\infty} I(u_n) = m$. For this purpose, let us consider $(y_n)_n$, with $y_n \in \mathbb{R}^3$, $|y_n| \to +\infty$ as $n \to \infty$ and set $u_n = t_n \omega_{y_n} = t_n \omega(x - y_n)$, where $t_n = t(\omega_{y_n})$ is such that $u_n = t_n \omega_{y_n} \in \mathcal{N}$. Since

$$\begin{aligned} I(u_n) &= \frac{t_n^2}{2}\int_{\mathbb{R}^3} |\nabla \omega_{y_n}|^2 + (V_\infty + a(x))\omega_{y_n}^2 \, dx + \frac{t_n^4}{4}\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{y_n}} \omega_{y_n}^2 \, dx \\ &\quad - \frac{t_n^{p+1}}{p+1}\int_{\mathbb{R}^3} (Q_\infty - b(x))|\omega_{y_n}|^{p+1} \, dx \\ &= \frac{t_n^2}{2}\int_{\mathbb{R}^3} |\nabla \omega|^2 + (V_\infty + a(x+y_n))\omega^2 \, dx + \frac{t_n^4}{4}\int_{\mathbb{R}^3} K(x+y_n)\phi_\omega^{K(\cdot+y_n)} \omega^2 \, dx \\ &\quad - \frac{t_n^{p+1}}{p+1}\int_{\mathbb{R}^3} (Q_\infty - b(x+y_n))|\omega|^{p+1} \, dx, \end{aligned}$$

and from (H_1), (H_2) and (H_3), by Lebesgue dominated theorem, we can deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} a(x+y_n)\omega^2 = 0, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^3} b(x+y_n))|\omega|^{p+1} = 0,$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} K(x+y_n)\phi_\omega^{K(\cdot+y_n)}\omega^2\,\mathrm{d}x = 0.$$

Combining with $t_n\omega_{y_n}\in\mathcal{N}$, we have that $1\le t_n\le C$, where $C>0$ is a positive constant. Therefore, from $\omega\in\mathcal{N}_\infty$ and $p\in(3,5)$, it follows that $t_n\to 1$ and thus $I(u_n)\to m_\infty$.

To complete the proof, we argue by contradiction and we assume that $v\in\mathcal{N}$ exists such that $I(v)=m=m_\infty$. Obviously, there exists $t_v>0$ such that $t_v v\in\mathcal{N}_\infty$, we have that

$$\begin{aligned} m_\infty \le I_\infty(t_v v) &= \left(\frac{1}{2}-\frac{1}{p+1}\right)\|t_v v\|^2 \\ &\le \left(\frac{1}{2}-\frac{1}{p+1}\right)\|t_v v\|^2 + \left(\frac{1}{4}-\frac{1}{p+1}\right)\int_{\mathbb{R}^3} K(x)\phi_{t_v v}^K (t_v v)^2\,\mathrm{d}x \\ &\le I(t_v v)\le I(v) = m = m_\infty \end{aligned}$$

which implies that $t_v=1$ and

$$\int_{\mathbb{R}^3} K(x)\phi_v^K v^2\,\mathrm{d}x = 0. \tag{2.10}$$

Hence, $v\in\mathcal{N}_\infty$ and $I_\infty(v)=m_\infty$. By the uniqueness of solution of problem (2.6), there exists $y\in\mathbb{R}^3$ such that $v(x)=\omega(x-y)>0$, for every $x\in\mathbb{R}^3$, which leads to $\int_{\mathbb{R}^3}K(x)\phi_v^K v^2\,\mathrm{d}x>0$, contradicts with (2.10). □

In order to find a bound state in higher energy level in $(m_\infty, 2m_\infty)$, the next results help us to recover the compactness of the bounded (PS) sequence in $(m_\infty, 2m_\infty)$. Following the proof of Lemma 4.5 in [3], we can show the following splitting lemma.

Lemma 2.4 (Splitting lemma). *Suppose that (H_1)–(H_3) hold. Let $(u_n)_n$ be a (PS) sequence of I constrained on $\mathcal{N}$, i.e. $u_n\in\mathcal{N}$, and $I(u_n)$ is bounded, $\nabla I|_{\mathcal{N}}(u_n)\to 0$ strongly in $H^1(\mathbb{R}^3)$. Then, up to a subsequence, there exist a solution $\overline{u}$ of (2.2), a number $k\in\mathbb{N}\cup\{0\}$, k functions $u^1,\dots,u^k$ of $H^1(\mathbb{R}^3)$ and k sequences of points $(y_n^j)_n$, $y_n^j\in\mathbb{R}^3$, $0\le j\le k$ such that, as $n\to+\infty$,*

- *(i)* $u_n-\sum_{j=1}^k u^j(\cdot-y_n^j)\to\overline{u}$ *in* $H^1(\mathbb{R}^3)$;
- *(ii)* $I(u_n)\to I(\overline{u})+\sum_{j=1}^k I_\infty(u^j)$;
- *(iii)* $|y_n^j|\to+\infty$, $|y_n^i-y_n^j|\to+\infty$ *if* $i\ne j$;
- *(iv)* u^j *are weak solution of* (2.6).

Moreover, in the case $k=0$, the above holds without u^j.

In the end of this section, we recall a technical result for some estimates in the next section, its proof is found in [4,12].

Lemma 2.5. *If $g\in L^\infty(\mathbb{R}^3)$ and $h\in L^1(\mathbb{R}^3)$ are such that, for some $\alpha\ge 0$, $b\ge 0$, $\gamma\in\mathbb{R}$*

$$\lim_{|x|\to+\infty} g(x)e^{\alpha|x|}|x|^b=\gamma \quad \text{and} \quad \int_{\mathbb{R}^3}|h(x)|e^{\alpha|x|}|x|^b\,\mathrm{d}x<+\infty,$$

then, for every $z\in\mathbb{R}^3\setminus\{0\}$,

$$\lim_{\rho\to+\infty}\left(\int_{\mathbb{R}^3} g(x+\rho z)h(x)\,\mathrm{d}x\right)e^{\alpha|\rho z|}|\rho z|^b=\gamma\int_{\mathbb{R}^3} h(x)e^{-\alpha(x\cdot z)/|z|}\,\mathrm{d}x.$$

3 Proof of Theorem 1.1

Now, we turn to build tools and topological techniques to prove the existence of an higher energy solution when (1.5) has no ground state solution. First we recall the definition of barycenter $\beta : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}^3$ of a function $u \in H^1(R^3)$, $u \neq 0$, given in [13], set

$$\mu(u)(x) = \frac{1}{|B_1(0)|} \int_{B_1(x)} |u(y)| dy,$$

then $\mu(u) \in L^\infty(\mathbb{R}^3)$ and is continuous in $H^1(\mathbb{R}^3)$. Let

$$\hat{u}(x) = \left[\mu(u)(x) - \frac{1}{2} \max \mu(u)(x)\right]^+,$$

it is easy to check that $\hat{u} \in C_0(\mathbb{R}^3)$ and then define $\beta : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}^3$ as follows

$$\beta(u) = \frac{1}{|\hat{u}|_1} \int_{\mathbb{R}^3} x\hat{u}(x)\, \mathrm{d}x \in \mathbb{R}^3.$$

Since $\hat{u}$ has compact support, β is well defined and it is easy to verify the following properties:

(1) β is continuous in $H^1(\mathbb{R}^3) \setminus \{0\}$;

(2) if u is a radial function, $\beta(u) = 0$;

(3) for all $t \neq 0$ and for all $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, $\beta(tu) = \beta(u)$;

(4) given $z \in \mathbb{R}^3$ and setting $u_z(x) = u(x - z)$, $\beta(u_z) = \beta(u) + z$.

By Proposition 2.3, we see that m can not be achieved, with the help of the barycenter mapping β, we can add some refined constraint in the Nehari manifold $\mathcal{N}$. For this purpose, define the following minimization problem

$$\mathcal{B}_0 := \inf \{I(u) : u \in \mathcal{N}, \beta(u) = 0\}.$$

Clearly, we have $m = m_\infty \leq \mathcal{B}_0$. Furthermore, the strict inequality holds true.

Lemma 3.1. *Suppose that (H_1)–(H_3) hold. Then*

$$m = m_\infty < \mathcal{B}_0$$

Proof. By contradiction, we assume that $\mathcal{B}_0 = m_\infty$, then there exists $u_n \in \mathcal{N}$ such that $\beta(u_n) = 0$, $I(u_n) \to m_\infty$ and $\nabla I|_{\mathcal{N}}(u_n) \to 0$. By the Ekeland's variational principle (see Theorem 8.15 in [24]), a sequence of $(v_n)_n \in H^1(\mathbb{R}^3)$ exists so that

$$v_n \in \mathcal{N}, \qquad I(v_n) = m_\infty + o_n(1) \quad \text{and} \quad |\beta(v_n) - \beta(u_n)| = o(1). \tag{3.1}$$

By Lemma 2.4, we have that

$$m_\infty = I(u_n) + o(1) = I(\overline{u}) + \Sigma_{j=1}^k I_\infty(u^j) + o(1).$$

Owing to $I(\overline{u}) \geq m = m_\infty$ and $I_\infty(u^j) \geq m_\infty$, we have that $k = 0$. Thus, $v_n \to \overline{u}$. Since $\overline{u}$ is a nontrivial solution of (1.5), we deduce that

$$I(\overline{u}) = m_\infty, \qquad I'(\overline{u}) = 0, \qquad \beta(\overline{u}) = 0,$$

which means that $m = m_\infty$ is achieved, contradicts with Proposition 2.3. The proof is completed. □

Lemma 3.2. *The functional I constrained on $\mathcal{N}$ satisfies the Palais–Smale condition in $(m_\infty, 2m_\infty)$.*

Proof. Let $\{u_n\}$ be a Palais–Smale sequence of $I|_{\mathcal{N}}$ such that $I(u_n) \to c \in (m_\infty, 2m_\infty)$. By Lemma 2.4, we have that

$$c = I(u_n) + o(1) = I(\overline{u}) + \Sigma_{j=1}^{k} I_\infty(u^j) + o(1).$$

The conclusion follows observing that any critical point $\bar{u}$ of I is such that $I(\bar{u}) \geq m = m_\infty$, any solution of (2.6) verifies that $I_\infty(u) \geq m_\infty$ and if it changes sign, $I_\infty(u) \geq 2m_\infty$. Whatever any case, we can obtain the sequence $\{u_n\}$ strongly convergence in $H^1(\mathbb{R}^3)$. The compactness is proved. □

Let $\xi \in \mathbb{R}^3$ with $|\xi| = 1$ and $\Sigma = \{z \in \mathbb{R}^3 : |z - \xi| = 2\}$. For $\rho > 0$ and $(z,s) \in \Sigma \times [0,1]$, define

$$\overline{\psi}_\rho[z,s](x) := (1-s)\omega_{\rho z}(x) + s\omega_{\rho\xi}(x) = (1-s)\omega(x - \rho z) + s\omega(x - \rho\xi), \qquad x \in \mathbb{R}^3,$$

where w is a unique radically symmetric positive solution of problem (2.6), then by virtue of standard argument, there exist positive numbers $t_{\rho,z,s} := t_{\overline{\psi}_\rho[z,s]}$ and $\tau_{\rho,z,s} := \tau_{\overline{\psi}_\rho[z,s]}$ such that

$$\psi_\rho[z,s] = t_{\rho,z,s}\overline{\psi}_\rho[z,s] \in \mathcal{N}, \qquad \psi_{\infty,\rho}[z,s] = \tau_{\rho,z,s}\overline{\psi}_\rho[z,s] \in \mathcal{N}_\infty. \tag{3.2}$$

Remark 3.3. Note that $\overline{\psi}_\rho[z,s] \to \omega(x - \rho z)$ as $s \to 0$ and $\overline{\psi}_\rho[z,s] \to \omega(x - \rho\xi)$ as $s \to 1$, moreover, $\tau_{\rho,z,s} \to 1$ as $s \to 0$ or $s \to 1$ due to $\omega(x - \rho z) \in \mathcal{N}_\infty$ and $\omega(x - \rho\xi) \in \mathcal{N}_\infty$.

Lemma 3.4. *For all $\rho > 0$, we have*

$$\mathcal{B}_0 \leq \mathcal{T}_\rho := \max_{\Sigma \times [0,1]} I(\psi_\rho[z,s]).$$

Proof. Observing that $\beta(\psi_\rho[z,0]) = \beta(t_{\rho,z,0}\overline{\psi}_\rho[z,0]) = \beta(t_{\rho,z,0}\omega_{\rho z}) = \beta(\omega_{\rho z}) = \rho z$ and $\beta(\psi_\rho[z,1]) = \rho\xi$. Let

$$\mathcal{G}(z,s) = s\rho\xi + (1-s)\rho z, \tag{3.3}$$

then $\mathcal{G}(z,s) \in C(\Sigma \times (0,1])$. Define a mapping by

$$h(t,z,s) = t\mathcal{G}(z,s) + (1-t)\beta(\psi_\rho[z,s]), \qquad \forall t \in [0,1], \tag{3.4}$$

then $h(t,z,s) \in C([0,1] \times \overline{\Sigma \times (0,1]})$ is continuous and

$$h(t,z,0) = t\rho z + (1-t)\beta(\psi_\rho[z,0]) = t\rho z + (1-t)\rho z = \rho z \neq 0, \qquad \forall z \in \Sigma$$

and

$$h(t,z,1) = t\rho\xi + (1-t)\beta(\psi_\rho[z,1]) = t\rho\xi + (1-t)\rho\xi = \rho\xi \neq 0, \qquad \forall z \in \Sigma$$

which implies that $0 \notin h(t, \partial(\Sigma \times (0,1]))$, for every $t \in [0,1]$. Therefore, by the homotopical invariance of Brouwer degree, we get $\deg(h(t), \Sigma \times (0,1], 0) = constant$. Thus

$$\deg(h(0), \Sigma \times (0,1], 0) = \deg(h(1), \Sigma \times (0,1], 0),$$

that is

$$\deg(\beta(\psi_\rho[z,s]), \Sigma \times (0,1], 0) = \deg(\mathcal{G}(s,z), \Sigma \times (0,1], 0).$$

Clearly, $\deg(\mathcal{G}(z,s), \Sigma \times (0,1], 0) \neq 0$. Thus, it follows from the solvable property of Brouwer degree that there exists $(\overline{z}, \overline{s}) \in \Sigma \times (0,1]$ such that $\beta(\psi_\rho[\overline{z}, \overline{s}]) = 0$. Therefore, by the definition of $\mathcal{B}_0$, we have that

$$\mathcal{B}_0 \leq I(\psi_\rho[\overline{z}, \overline{s}]) \leq \mathcal{T}_\rho.$$

□

In order to show $\mathcal{T}_\rho < 2m_\infty$, we have to give some estimates from the decay of ω and coefficients $a(x)$, $K(x)$ and $b(x)$.

Lemma 3.5. *Suppose that (H_3) holds. There exists $c > 0$ such that for all $\rho > 3R_0$, the following holds*

$$\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho\zeta}}\omega^2_{\rho\zeta}\,\mathrm{d}x \leq ce^{-\frac{4}{3}\sqrt{V_\infty}\rho}, \quad \textit{for all } \zeta \in \mathbb{R}^3 \textit{ with } |\zeta| \geq 1. \tag{3.5}$$

Proof. Let $\rho > 3R_0$, then $\frac{1}{3}\rho > R_0$ and if $|y| \le \frac{1}{3}\rho$ and $|\zeta| \ge 1$, we have that $|y-\rho\zeta| \ge \rho|\zeta| - |y| \ge \rho - \frac{1}{3}\rho = \frac{2}{3}\rho$. Thus, by the exponential decay (2.7) of ω, $\frac{1-e^{-|x|}}{|x|} \in L^s(\mathbb{R}^3)$ for all $s \in (3,+\infty]$ (see Lemma 3.3 in [3]), Hölder's inequality and (H_3), we deduce that

$$\begin{aligned}
\phi^K_{\omega_{\rho\zeta}}(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|} K(y)\omega^2(y-\rho\zeta)\,\mathrm{d}y \\
&= \frac{1}{4\pi}\left(\int_{|y|\le\frac{1}{3}\rho} \frac{1-e^{-|x-y|}}{|x-y|} K(y)\omega^2(y-\rho\zeta)\,\mathrm{d}y + \int_{|y|>\frac{1}{3}\rho} \frac{1-e^{-|x-y|}}{|x-y|} K(y)\omega^2(y-\rho\zeta)\,\mathrm{d}y\right) \\
&\le \frac{1}{4\pi}\left[\left(\int_{|y|\le\frac{1}{3}\rho} K^2(y)\omega^2(y-\rho\zeta)\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{|y|\le\frac{1}{3}\rho} \left(\frac{1-e^{-|x-y|}}{|x-y|}\right)^4 \mathrm{d}y\right)^{\frac{1}{4}} \left(\int_{|y|\le\frac{1}{3}\rho} \omega^4(y-\rho\zeta)\,\mathrm{d}y\right)^{\frac{1}{4}}\right. \\
&\qquad \left. + \int_{|y|>\frac{1}{3}\rho} \frac{1-e^{-|x-y|}}{|x-y|} K(y)\omega^2(y-\rho\zeta)\,\mathrm{d}y\right] \\
&\le c\left(e^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left(\int_{|y|\le(\frac{1}{2}-q)\rho} K^2(y)\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3}\left(\frac{1-e^{-|x-y|}}{|x-y|}\right)^4 \mathrm{d}y\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}^3}\omega^4(y-\rho\zeta)\,\mathrm{d}y\right)^{\frac{1}{4}}\right. \\
&\qquad \left. + e^{-\frac{2}{3}\sqrt{V_\infty}\rho}\int_{|y|>\frac{1}{3}\rho} \frac{1-e^{-|x-y|}}{|x-y|}\omega^2_{\rho\zeta}\,\mathrm{d}y\right) \\
&\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left(C + \int_{|y|>\frac{1}{3}\rho} \frac{1-e^{-|x-y|}}{|x-y|}\omega^2_{\rho\zeta}\,\mathrm{d}y\right) \\
&\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}.
\end{aligned}$$

Thus, a similar computation gives

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho\zeta}}(x)\omega^2_{\rho\zeta}\,\mathrm{d}x &\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\int_{\mathbb{R}^3} K(x)\omega^2_{\rho\zeta}\,\mathrm{d}x \\
&= ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left(\int_{|x|\le\frac{1}{3}\rho} K(x)\omega^2_{\rho\zeta}\,\mathrm{d}x + \int_{|x|>\frac{1}{3}\rho} K(x)\omega^2_{\rho\zeta}\,\mathrm{d}x\right) \\
&\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left[\left(\int_{|x|\le\frac{1}{3}\rho} K^2(x)\omega^2_{\rho\zeta}\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{|x|\le\frac{1}{3}\rho}\omega^2_{\rho\zeta}\,\mathrm{d}x\right)^{\frac{1}{2}} + \int_{|x|>\frac{1}{3}\rho} K(x)\omega^2_{\rho\zeta}\,\mathrm{d}x\right) \\
&\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}e^{-\frac{2}{3}\sqrt{V_\infty}\rho} = ce^{-\frac{4}{3}\sqrt{V_\infty}\rho}.
\end{aligned}$$

□

From (3.5), it is easy to very that

$$\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho\xi}}\omega^2_{\rho\xi}\,\mathrm{d}x = o(\rho^{-1}e^{-\sqrt{V_\infty}\rho}), \qquad \int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho z}}\omega^2_{\rho z}\,\mathrm{d}x = o(\rho^{-1}e^{-\sqrt{V_\infty}\rho}) \tag{3.6}$$

and we denote $\varepsilon_\rho = \rho^{-1}e^{-\sqrt{V_\infty}\rho}$ for convenience. Moreover, we can obtain the following estimate:

Lemma 3.6.
$$\int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}\overline{\psi}^2_\rho[z,s]\,\mathrm{d}x = o(\varepsilon_\rho), \qquad \forall\, s \in [0,1] \text{ and } z \in \Sigma. \tag{3.7}$$

Proof. Since

$$\phi^K_{\overline{\psi}_\rho[z,s]} = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|} K(y)\overline{\psi}^2_\rho[z,s](y)\,\mathrm{d}y \le 2\phi^K_{\omega_{\rho z}} + 2\phi^K_{\omega_{\rho\xi}}$$

and then

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}\overline{\psi}^2_\rho[z,s]\,\mathrm{d}x &\le 2\int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}(\omega^2_{\rho\xi}+\omega^2_{\rho z})\,\mathrm{d}x\\ &\le 4\int_{\mathbb{R}^3} K(x)(\phi^K_{\omega_{\rho z}}+\phi^K_{\omega_{\rho\xi}})(\omega^2_{\rho\xi}+\omega^2_{\rho z})\,\mathrm{d}x\\ &= 4\int_{\mathbb{R}^3} K(x)(\phi^K_{\omega_{\rho z}}\omega^2_{\rho z}+\phi^K_{\omega_{\rho\xi}}\omega^2_{\rho z}+\phi^K_{\omega_{\rho z}}\omega^2_{\rho\xi}+\phi^K_{\omega_{\rho\xi}}\omega^2_{\rho z})\,\mathrm{d}x.\end{aligned}$$

Now, similar to the proof of Lemma 3.5, we have that

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho z}}\omega^2_{\rho\xi}(x)\,\mathrm{d}x \le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\int_{\mathbb{R}^3} K(x)\omega^2_{\rho\xi}(x)\,\mathrm{d}x\\ &= ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left(\int_{|x|\le\frac{1}{3}\rho} K(x)\omega^2_{\rho\xi}(x)\,\mathrm{d}x+\int_{|x|>\frac{1}{3}\rho} K(x)\omega^2_{\rho\xi}(x)\,\mathrm{d}x\right)\\ &\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}\left(\left(\int_{|x|\le\frac{1}{3}\rho} K^2(x)\omega^2_{\rho\xi}(x)\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{|x|\le\frac{1}{3}\rho}\omega^2_{\rho\xi}(x)\,\mathrm{d}x\right)^{\frac{1}{2}}+\int_{|x|>\frac{1}{3}\rho} K(x)\omega^2_{\rho\xi}(x)\,\mathrm{d}x\right)\\ &\le ce^{-\frac{2}{3}\sqrt{V_\infty}\rho}e^{-\frac{2}{3}\sqrt{V_\infty}\rho}=ce^{-\frac{4}{3}\sqrt{V_\infty}\rho}.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho z}}\omega^2_{\rho\xi}(x)\,\mathrm{d}x=o(\varepsilon_\rho),$$

and the same argument leads to

$$\int_{\mathbb{R}^3} K(x)\phi^K_{\omega_{\rho\xi}}\omega^2_{\rho z}(x)\,\mathrm{d}x=o(\varepsilon_\rho).$$

Therefore, from (3.6), the estimate (3.7) follows. □

Next, we give some estimates which are used in the sequel.

Lemma 3.7. *The following estimates hold:*

$$\int_{\mathbb{R}^3}\omega^p_{\rho z}\omega_{\rho\xi}\,\mathrm{d}x=O(\widetilde{\varepsilon_\rho})=o(\varepsilon_\rho),\qquad \int_{\mathbb{R}^3}\omega^p_{\rho\xi}\omega_{\rho z}\,\mathrm{d}x=O(\widetilde{\varepsilon_\rho})=o(\varepsilon_\rho),\tag{3.8}$$

$$\int_{\mathbb{R}^3} a(x)\omega^2_{\rho z}\,\mathrm{d}x=o(\varepsilon_\rho),\qquad \int_{\mathbb{R}^3} a(x)\omega^2_{\rho\xi}\,\mathrm{d}x=o(\varepsilon_\rho),\qquad \int_{\mathbb{R}^3} a(x)\overline{\psi}^2_\rho[z,s]\,\mathrm{d}x=o(\varepsilon_\rho),\tag{3.9}$$

$$\int_{\mathbb{R}^3} b(x)|\overline{\psi}_\rho[z,s]|^{p+1}\,\mathrm{d}x=o(\varepsilon_\rho),\tag{3.10}$$

where $\widetilde{\varepsilon_\rho}=\rho^{-2}e^{-2\sqrt{V_\infty}\rho}$.

Proof. (i) By (2.7), we can deduce that

$$\begin{aligned}\int_{\mathbb{R}^3}\omega^p_{\rho\xi}\omega_{\rho z}\,\mathrm{d}x&=\int_{\mathbb{R}^3}\omega^p(x-\rho\xi)\omega(x-\rho z)\,\mathrm{d}x=\int_{\mathbb{R}^3}\omega^p(y)\omega(y+\rho(\xi-z))\,\mathrm{d}y\\ &\sim c\int_{\mathbb{R}^3}|y+\rho(\xi-z)|^{-1}e^{-\sqrt{V_\infty}|y+\rho(\xi-z)|}\omega^p(y)\,dy\end{aligned}$$

In order to apply Lemma 2.5, let us set $h(x)=\omega^p(x)\in L^1(\mathbb{R}^3)$, $g(x)=|x|^{-1}e^{-\sqrt{V_\infty}|x|}$, taking $\alpha=\sqrt{V_\infty}$ and $b=1$, clearly,

$$\lim_{|x|\to+\infty} g(x)|x|e^{\sqrt{V_\infty}|x|}=1,$$

$$\int_{\mathbb{R}^3}\omega^p(x)e^{\sqrt{V_\infty}|x|}|x|\,dx\le c\int_{\mathbb{R}^3} e^{-(p-1)\sqrt{V_\infty}|x|}|x|^{-(p-1)}\,dx<+\infty.$$

By using Lemma 2.5 and $z \in \Sigma$ ($|\xi - z| = 2$), we get that

$$\begin{aligned}
&\lim_{\rho\to+\infty} e^{\sqrt{V_\infty}|\rho(\xi-z)|}|\rho(\xi-z)| \int_{\mathbb{R}^3} g(x+\rho(\xi-z))h(x)\,\mathrm{d}x \\
&\quad = 2\lim_{\rho\to+\infty} e^{2\sqrt{V_\infty}\rho}\rho \int_{\mathbb{R}^3} g(x+\rho(\xi-z))h(x)\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} \omega^p(x) e^{-\sqrt{V_\infty}\frac{x\cdot(\xi-z)}{|\xi-z|}}\,\mathrm{d}x = c_1,
\end{aligned}$$

which means that

$$\lim_{\rho\to+\infty} \rho^2 e^{2\sqrt{V_\infty}\rho} \int_{\mathbb{R}^3} g(x+\rho(\xi-z))h(x)\,\mathrm{d}x = c_1. \tag{3.11}$$

This means that

$$\int_{\mathbb{R}^3} \omega^p_{\rho\xi}\omega_{\rho z}\,\mathrm{d}x = O(\widetilde{\varepsilon_\rho}) = o(\varepsilon_\rho).$$

Similar argument as above we can show that

$$\int_{\mathbb{R}^3} \omega^p_{\rho z}\omega_{\rho \xi}\,\mathrm{d}x = O(\widetilde{\varepsilon_\rho}) = o(\varepsilon_\rho).$$

(ii) By Hölder's inequality, we have that

$$\begin{aligned}
\int_{\mathbb{R}^3} a(x)\omega^2_{\rho\xi}\,\mathrm{d}x &= \int_{\mathbb{R}^3} a(x)\omega^2(x-\rho\xi)\,\mathrm{d}x \le \left(\int_{\mathbb{R}^3} \omega^2(x-\rho\xi)\,\mathrm{d}x\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} a^{\frac{3}{2}}\omega^2(x-\rho\xi)\,\mathrm{d}x\right)^{\frac{2}{3}} \\
&\le C\left(\int_{\mathbb{R}^3} a^{\frac{3}{2}}\omega^2(x-\rho\xi)\,\mathrm{d}x\right)^{\frac{2}{3}} \\
&\sim \left(\int_{\mathbb{R}^3} a^{\frac{3}{2}}|x-\rho\xi|^{-2}e^{-2\sqrt{V_\infty}|x-\rho\xi|}\,\mathrm{d}x\right)^{\frac{2}{3}}.
\end{aligned}$$

Taking $\alpha = 2\sqrt{V_\infty}$, $b = 2$, $h(x) = a^{\frac{3}{2}}(x)$, $g(x) = |x|^{-2}e^{-2\sqrt{V_\infty}|x|}$ in Lemma 2.5, by (H_4), it is easy to see that

$$\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)|x|^2 e^{2\sqrt{V_\infty}|x|}\,\mathrm{d}x < +\infty \quad \text{and} \quad \lim_{|x|\to+\infty} g(x)|x|^2 e^{2\sqrt{V_\infty}|x|} = 1.$$

Thus

$$\begin{aligned}
\lim_{\rho\to+\infty} e^{2\sqrt{V_\infty}|-\rho\xi|}|-\rho\xi|^2 \int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)g(x-\rho\xi)\,\mathrm{d}x &= \lim_{\rho\to+\infty} \rho^2 e^{2\sqrt{V_\infty}\rho}\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)g(x-\rho\xi)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)e^{-2\sqrt{V_\infty}\frac{x\cdot(-\xi)}{|\xi|}}\,\mathrm{d}x = c_2,
\end{aligned}$$

which yields that

$$\lim_{\rho\to+\infty}\left(\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)g(x-\rho\xi)\,\mathrm{d}x\right)^{\frac{2}{3}} = O(\rho^{-\frac{4}{3}}e^{-\frac{4}{3}\sqrt{V_\infty}\rho}) = o(\rho^{-1}e^{-\sqrt{V_\infty}\rho}) = o(\varepsilon_\rho).$$

Thus, we conclude that

$$\int_{\mathbb{R}^3} a(x)\omega^2_{\rho\xi}\,\mathrm{d}x = o(\varepsilon_\rho).$$

Note that $z \in \Sigma$ implies that $1 \le |z| \le 3$, parallel to the above argument, we can get that

$$\lim_{\rho\to+\infty} \rho^2|z|^2 e^{2\sqrt{V_\infty}\rho|z|}\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)g(x-\rho z)\,\mathrm{d}x = \int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)e^{-2\sqrt{V_\infty}\frac{x\cdot(-z)}{|z|}}\,\mathrm{d}x = c_3,$$

which leads to

$$\lim_{\rho\to+\infty}\left(\int_{\mathbb{R}^3} a^{\frac{3}{2}}(x)g(x-\rho z)\,\mathrm{d}x\right)^{\frac{2}{3}} = O(\rho^{-\frac{4}{3}}e^{-\frac{4}{3}\sqrt{V_\infty}\rho}) = o(\rho^{-1}e^{-\sqrt{V_\infty}\rho}) = o(\varepsilon_\rho).$$

Thus, it follows that
$$\int_{\mathbb{R}^3} a(x)\omega_{\rho z}^2 \,\mathrm{d}x = o(\varepsilon_\rho).$$
Since
$$\int_{\mathbb{R}^3} a(x)\overline{\psi}_\rho^2[z,s]\,\mathrm{d}x = \int_{\mathbb{R}^3} a(x)[(1-s)\omega_{\rho z} + s\omega_{\rho\xi}]^2\,\mathrm{d}x \le 2\int_{\mathbb{R}^3} a(x)[\omega_{\rho z}^2 + \omega_{\rho\xi}^2]\,\mathrm{d}x,$$
according to the two estimates which have been proved, it is easily to get
$$\int_{\mathbb{R}^3} a(x)\overline{\psi}_\rho^2[z,s]\,\mathrm{d}x = o(\varepsilon_\rho).$$

(iii) By (H_2), we can easily check that $b(x)\omega_{\rho\xi}^{p-1} \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $b(x)\omega_{\rho z}^{p-1} \in L^{\frac{3}{2}}(\mathbb{R}^3)$. In view of (H_4), $b(x) \in L^\infty(\mathbb{R}^3)$ and $\omega(x) \in L^\infty(\mathbb{R}^3)$, we have that
$$\int_{\mathbb{R}^3} b^{\frac{3}{2}}(x)\omega_{\rho z}^{\frac{3(p-1)}{2}}|x|^2 e^{2\sqrt{V_\infty}|x|}\,\mathrm{d}x \le C\int_{\mathbb{R}^3} b(x)|x|^2 e^{2\sqrt{V_\infty}|x|}\,\mathrm{d}x < +\infty.$$
Similar argument as the proof of (ii), we can show that
$$\int_{\mathbb{R}^3} b(x)\omega_{\rho\xi}^{p+1}\,\mathrm{d}x = o(\varepsilon_\rho), \qquad \int_{\mathbb{R}^3} b(x)\omega_{\rho z}^{p+1}\,\mathrm{d}x = o(\varepsilon_\rho).$$
Owing to
$$\begin{aligned}\int_{\mathbb{R}^3} b(x)|\overline{\psi}_\rho[z,s]|^{p+1}\,\mathrm{d}x &= \int_{\mathbb{R}^3} b(x)[(1-s)\omega_{\rho z} + s\omega_{\rho\xi}]^{p+1}\,\mathrm{d}x\\ &\le 2^p\int_{\mathbb{R}^3} b(x)(\omega_{\rho z}^{p+1} + \omega_{\rho\xi}^{p+1})\,\mathrm{d}x,\end{aligned}$$
we conclude that (3.10) follows. □

Lemma 3.8. *Suppose that (H_1)–(H_4) hold. Let $t_{\rho,z,s}$ and $\tau_{\rho,z,s}$ be the number defined in (3.2). Then there exists a constant $C > 0$ such that*
$$0 < t_{\rho,z,s} \le C, \qquad \forall\rho > 0,\ \forall(z,s) \in \Sigma\times[0,1]. \tag{3.12}$$
Moreover,
$$t_{\rho,z,s} = \tau_{\rho,z,s} + o(\varepsilon_\rho). \tag{3.13}$$

Proof. Observing that
$$0 < \frac{1}{2}\left(\int_{B_1(0)} |\nabla\omega|^2 + \omega^2\right)^{\frac{1}{2}} \le \|\overline{\psi}_\rho[z,s]\| \le \|\omega_{\rho z}\| + \|\omega_{\rho\xi}\| = 2\|\omega\| \tag{3.14}$$
and
$$0 < \frac{1}{2}\left(\int_{B_1(0)} \omega^{p+1}\right)^{\frac{1}{p+1}} \le |\overline{\psi}_\rho[z,s]|_{p+1} \le 2^p(|\omega_{\rho z}|_{p+1} + |\omega_{\rho\xi}|_{p+1}) = 2^{p+1}|\omega|_{p+1}. \tag{3.15}$$

Since $t_{\rho,z,s}\overline{\psi}_\rho[z,s] \in \mathcal{N}$, we have that
$$\begin{aligned}t_{\rho,z,s}^2\left(\|\overline{\psi}_\rho[z,s]\|^2 + \int_{\mathbb{R}^3} a(x)\overline{\psi}_\rho^2[z,s]\,\mathrm{d}x\right) + t_{\rho,z,s}^4\int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}\overline{\psi}_\rho^2[z,s]\,\mathrm{d}x&\\ - t_{\rho,z,s}^{p+1}\int_{\mathbb{R}^3}(Q_\infty - b(x))|\overline{\psi}_\rho[z,s]|^{p+1}\,\mathrm{d}x = 0.&\end{aligned} \tag{3.16}$$
By (3.14), (3.15) and (3.16), it is easy to check that (3.12) holds. Thus, by (3.6), we have that
$$t_{\rho,z,s}^2\int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}\overline{\psi}_\rho^2[z,s]\,\mathrm{d}x = o(\varepsilon_\rho). \tag{3.17}$$

By (3.16), we have that

$$t_{\rho,z,s}^{p-1} = \frac{\|\overline{\psi}_\rho[z,s]\|^2}{\int_{\mathbb{R}^3}(Q_\infty - b(x))|\overline{\psi}_\rho[z,s]|^{p+1}\,dx} + \frac{\int_{\mathbb{R}^3} a(x)\overline{\psi}_\rho^2[z,s]\,dx + t_{\rho,z,s}^2 \int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}\overline{\psi}_\rho^2[z,s]\,dx}{\int_{\mathbb{R}^3}(Q_\infty - b(x))|\overline{\psi}_\rho[z,s]|^{p+1}\,dx}.$$

Therefore, by (3.17), (3.9), (3.10) and $\tau_{\rho,z,s}\overline{\psi}_\rho[z,s] \in \mathcal{N}_\infty$, we deduce that

$$t_{\rho,z,s}^{p-1} = \frac{\|\overline{\psi}_\rho[z,s]\|^2}{\int_{\mathbb{R}^3} Q_\infty|\overline{\psi}_\rho[z,s]|^{p+1}\,dx} + o(\varepsilon_\rho) = \tau_{\rho,z,s}^{p-1} + o(\varepsilon_\rho)$$

which yields the conclusion. The proof is completed. □

Lemma 3.9. *Suppose that* (H_1)–(H_4) *hold. Then there exists* ρ_0 *such that, for all* $\rho > \rho_0$,

$$\mathcal{T}_\rho = \max_{\Sigma\times[0,1]} I(\psi_\rho[z,s]) < 2m_\infty.$$

Proof. By (3.2), (3.9), (3.13), (3.14) and (3.17), for any $(z,s) \in \Sigma \times [0,1]$, we deduce that

$$\begin{aligned}
I(\psi_\rho[z,s]) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|t_{\rho,z,s}\overline{\psi}_\rho[z,s]\|^2 + \left(\frac{1}{2} - \frac{1}{p+1}\right) t_{\rho,z,s}^2 \int_{\mathbb{R}^3} a(x)|\overline{\psi}_\rho[z,s]|^2\,dx \\
&\quad + \left(\frac{1}{4} - \frac{1}{p+1}\right) t_{\rho,z,s}^4 \int_{\mathbb{R}^3} K(x)\phi^K_{\overline{\psi}_\rho[z,s]}|\overline{\psi}_\rho[z,s]|^2\,dx \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|t_{\rho,z,s}\overline{\psi}_\rho[z,s]\|^2 + o(\varepsilon_\rho) \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\tau_{\rho,z,s}\overline{\psi}_\rho[z,s]\|^2 + \left(\frac{1}{2} - \frac{1}{p+1}\right)(t_{\rho,z,s}^2 - \tau_{\rho,z,s}^2)\|\overline{\psi}_\rho[z,s]\|^2 + o(\varepsilon_\rho) \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\tau_{\rho,z,s}\overline{\psi}_\rho[z,s]\|^2 + o(\varepsilon_\rho) \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{\|\overline{\psi}_\rho[z,s]\|^2}{|\overline{\psi}_\rho[z,s]|_{p+1}^2}\right)^{\frac{p+1}{p-1}} + o(\varepsilon_\rho) \\
&= I_\infty(\psi_{\infty,\rho}[z,s]) + o(\varepsilon_\rho)
\end{aligned} \tag{3.18}$$

By direction computation, we have that

$$\|\overline{\psi}_\rho[z,s]\|^2 = (\overline{\psi}_\rho[z,s], \overline{\psi}_\rho[z,s])_{H^1(\mathbb{R}^3)} = [(1-s)^2 + s^2]\|\omega\|^2 + 2s(1-s)(\omega_{\rho\xi}, \omega_{\rho z})_{H^1(\mathbb{R}^3)}. \tag{3.19}$$

Since $\omega_{\rho\xi}$ is a positive solution of problem (2.6), it follows that

$$(\omega_{\rho\xi}, \omega_{\rho z})_{H^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \omega_{\rho\xi}^p \omega_{\rho z}\,dx := A_\rho.$$

By (3.19), we see that

$$\|\overline{\psi}_\rho[z,s]\|^2 = (\overline{\psi}_\rho[z,s], \overline{\psi}_\rho[z,s])_{H^1(\mathbb{R}^3)} = [(1-s)^2 + s^2]\|\omega\|^2 + 2s(1-s)A_\rho. \tag{3.20}$$

According to the following equality:

$$(a+b)^{p+1} \geq a^{p+1} + b^{p+1} + (p+1)(a^p b + ab^p), \quad \text{for all } a,b \in \mathbb{R}^+ \text{ and } p \geq 2,$$

we have that

$$\begin{aligned}
|\overline{\psi}_\rho[z,s]|_{p+1}^{p+1} &= \int_{\mathbb{R}^3} [(1-s)\omega_{\rho z} + s\omega_{\rho\xi}]^{p+1} \\
&\geq [(1-s)^{p+1} + s^{p+1}]|\omega|_{p+1}^{p+1} + (p+1)[(1-s)^p s + (1-s)s^p]A_\rho.
\end{aligned} \tag{3.21}$$

When s or $(1-s)$ is small enough, $\psi_{\infty,\rho}[z,s]$ tends to $\omega_{\rho z}$ or $\omega_{\rho\xi}$. Then

$$I_\infty(\psi_{\infty,\rho}[z,s]) \to m_\infty.$$

Therefore there exists $\delta > 0$ such that for $\min\{s, 1-s\} \le \delta$,

$$I_\infty(\psi_{\infty,\rho}[z,s]) < 2m_\infty.$$

In what follows we assume that $\min\{s, 1-s\} \ge \delta$, by virtue of (3.20) and (3.21), we get that

$$\begin{aligned}
\frac{\|\overline{\psi}_\rho[z,s]\|^2}{|\overline{\psi}_\rho[z,s]|_{p+1}^2} &\le \frac{[(1-s)^2+s^2]\|\omega\|^2 + 2s(1-s)A_\rho}{([(1-s)^{p+1}+s^{p+1}]|\omega|_{p+1}^{p+1} + (p+1)[(1-s)^p s + (1-s)s^p]A_\rho)^{\frac{2}{p+1}}} \\
&= \frac{[(1-s)^2+s^2]}{((1-s)^{p+1}+s^{p+1})^{\frac{2}{p+1}}}\frac{\|\omega\|^2}{|\omega|_{p+1}^2} \times \frac{1+\frac{2s(1-s)}{((1-s)^2+s^2)\|\omega\|^2}A_\rho}{\left(1+\frac{(p+1)((1-s)^p s+(1-s)s^p)}{(1-s)^{p+1}+s^{p+1}}\frac{A_\rho}{|\omega|_{p+1}^{p+1}}\right)^{\frac{2}{p+1}}} \qquad (3.22)\\
&\le \frac{[(1-s)^2+s^2]}{((1-s)^{p+1}+s^{p+1})^{\frac{2}{p+1}}}\frac{\|\omega\|^2}{|\omega|_{p+1}^2}\frac{1+\frac{2s(1-s)}{(1-s)^2+s^2}\frac{A_\rho}{\|\omega\|^2}}{1+\frac{2((1-s)^p s+(1-s)s^p)}{(1-s)^{p+1}+s^{p+1}}\frac{A_\rho}{|\omega|_{p+1}^{p+1}}}.
\end{aligned}$$

Notice that we have the following inequalities:

$$\frac{s(1-s)}{(1-s)^2+s^2} < \frac{(1-s)^p s + (1-s)s^p}{(1-s)^{p+1}+s^{p+1}} \quad \text{for } 0 < s < 1,$$

$$\frac{[(1-s)^2+s^2]}{((1-s)^{p+1}+s^{p+1})^{\frac{2}{p+1}}} < 2^{\frac{p-1}{p+1}} \quad \text{for } 0 \le s < 1.$$

Then

$$\begin{aligned}
&I_\infty(\psi_{\infty,\rho}[z,s]) \\
&\le \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{[(1-s)^2+s^2]}{((1-s)^{p+1}+s^{p+1})^{\frac{2}{p+1}}}\frac{\|\omega\|^2}{|\omega|_{p+1}^2}\right)^{\frac{p+1}{p-1}}\left(\frac{1+\frac{2s(1-s)}{(1-s)^2+s^2}\frac{A_\rho}{\|\omega\|^2}}{1+\frac{2((1-s)^p s+(1-s)s^p)}{(1-s)^{p+1}+s^{p+1}}\frac{A_\rho}{|\omega|_{p+1}^{p+1}}}\right)^{\frac{p+1}{p-1}} \\
&\le \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{[(1-s)^2+s^2]}{((1-s)^{p+1}+s^{p+1})^{\frac{2}{p+1}}}\frac{\|\omega\|^2}{|\omega|_{p+1}^2}\right)^{\frac{p+1}{p-1}} \\
&< 2\left(\frac{1}{2}-\frac{1}{p+1}\right)|\omega|_{p+1}^{p+1} \\
&= 2m_\infty.
\end{aligned}$$

Therefore we see that the conclusion follows. □

Lemma 3.10. *There exists $\rho_1 > 0$ such that*

$$\mathcal{A}_\rho := \max\left\{I(\psi_\rho[z,0]) : z \in \Sigma\right\} < \mathcal{B}_0, \qquad \forall \rho > \rho_1. \qquad (3.23)$$

Proof. Observing that $\psi_\rho[z,0] = t_{\rho,z,0}\overline{\psi}_\rho[z,0]$ and $\overline{\psi}_\rho[z,0] = \omega_{\rho z}$. We claim that

$$\lim_{\rho\to+\infty} I(t_{\rho,z,0}\omega_{\rho z}) = m_\infty.$$

Indeed, by (3.6), (3.9) and (3.10), we deduce that

$$
\begin{aligned}
I(t_{\rho,z,0}\omega_{\rho z}) &= \frac{t^2_{\rho,z,0}}{2}\int_{\mathbb{R}^3}|\nabla\omega_{\rho z}|^2+(V_\infty+a(x))\omega^2_{\rho z}\,\mathrm{d}x+\frac{t^4_{\rho,z,0}}{4}\int_{\mathbb{R}^3}K(x)\phi^K_{\omega_{\rho z}}\omega^2_{\rho z}\,\mathrm{d}x\\
&\quad-\frac{t^{p+1}_{\rho,z,0}}{p+1}\int_{\mathbb{R}^3}(Q_\infty-b(x))|\omega_{\rho z}|^{p+1}\,\mathrm{d}x\\
&=\frac{t^2_{\rho,z,0}}{2}\int_{\mathbb{R}^3}|\nabla\omega_{\rho z}|^2+V_\infty\omega^2_{\rho z}\,\mathrm{d}x-\frac{t^{p+1}_{\rho,z,0}}{p+1}\int_{\mathbb{R}^3}Q_\infty|\omega_{\rho z}|^{p+1}\,\mathrm{d}x+o(\varepsilon_\rho)\\
&=\frac{\tau^2_{\rho,z,0}}{2}\int_{\mathbb{R}^3}|\nabla\omega_{\rho z}|^2+V_\infty\omega^2_{\rho z}\,\mathrm{d}x-\frac{\tau^{p+1}_{\rho,z,0}}{p+1}\int_{\mathbb{R}^3}Q_\infty|\omega_{\rho z}|^{p+1}\,\mathrm{d}x+o(\varepsilon_\rho)\\
&=I_\infty(\tau_{\rho,z,0}\omega_{\rho z})+o(\varepsilon_\rho).
\end{aligned}
$$

Owing to $\tau_{\rho,z,0}\omega_{\rho z}\in\mathcal{N}_\infty$, thus $I_\infty(\tau_{\rho,z,0}\omega_{\rho z})\to m_\infty$ as $\rho\to+\infty$. By Lemma 3.1, (3.23) follows. □

Proof of Theorem 1.1. From Proposition 2.3, we see that $m = m_\infty$ and m is not achieved and the problem cannot be solved by minimization. However we are now going to prove the existence of a positive solution of (1.3) having energy greater than m_∞, through using the deformation argument. For this purpose, we denote $I^c=\{u\in\mathcal{N}: I(u)\le c\}$, $c\in\mathbb{R}$.

By Lemma 3.1, Lemma 3.4, Lemma 3.9 and Lemma 3.10, the following chain of inequality holds

$$m_\infty<\mathcal{A}_\rho<\mathcal{B}_0\le\mathcal{T}_\rho<2m_\infty,\quad \text{for all } \rho>\max\{3R_0,\rho_0,\rho_1\}.$$

We aim at showing that there exists a Palais–Smale sequence of the functional I constrained on N at level $c^*\in[\mathcal{B}_0,\mathcal{T}_\rho]$. If this is done, the existence of a nontrivial critical point u with $I(u)<2m_\infty$ follows from Lemma 3.2.

Assume by contradiction, that no Palais–Smale sequence exists in $[\mathcal{B}_0,\mathcal{T}_\rho]$. By using the usual deformation arguments ([23]), there exist a number $\delta>0$ and a continuous function $\eta: I^{\mathcal{T}_\rho}\to I^{\mathcal{B}_0-\delta}$ such that $\mathcal{B}_0-\delta>\mathcal{A}_\rho$ and $\eta(u)=u$ for all $u\in I^{\mathcal{B}_0-\delta}$. Let us define the map $\mathcal{H}:\Sigma\times[0,1]\to\mathbb{R}^3$ by $\mathcal{H}(z,s)=\beta\circ\eta\circ\psi_\rho[z,s]$. Lemma 3.10 tells us that $\psi_\rho[z,0]\subset I^{\mathcal{A}_\rho}\subset I^{\mathcal{B}_0-\delta}$, thus $\eta(\psi[z,0])=\psi[z,0]$ and then $\beta\circ\eta\circ\psi_\rho[z,0]=\beta(\psi[z,0])=\rho z$. Define $h(t,z,s)=t\mathcal{G}(z,s)+(1-t)\mathcal{H}(z,s):[0,1]\times\Sigma\times(0,1]\to\mathbb{R}^3$,, where $\mathcal{G}$ is defined in Lemma 3.4. Clearly, $h\in C([0,1]\times\Sigma\times(0,1])$ and for all $t\in[0,1]$, $z\in\Sigma$, we have that $h(t,z,0)=\rho z\neq 0$, that is $0\notin h(t,\partial(\Sigma\times(0,1]))$. Similar to the proof of Lemma 3.4, we get that there exists $(\bar z,\bar s)\in\Sigma\times(0,1]$ such that

$$\beta\circ\eta\circ\psi_\rho[\bar z,\bar s]=0. \tag{3.24}$$

According to Lemma 3.4, we know that $\Psi_\rho[z,s]\in I^{\mathcal{T}_\rho}$ and then by the properties of η, we have that

$$\eta\circ\psi[z,s]\in I^{\mathcal{B}_0-\delta},\qquad \forall(z,s)\in\Sigma\times[0,1]. \tag{3.25}$$

Clearly, $\eta\circ\psi[z,s]\in\mathcal{N}$, $\forall(z,s)\in\Sigma\times[0,1]$, in particularly, $\eta\circ\psi[\bar z,\bar s]\in\mathcal{N}$, combining with (3.24) and by the definition of $\mathcal{B}_0$, we see that $I(\eta\circ\psi_\rho[\bar z,\bar s])\ge\mathcal{B}_0$, contradicts with (3.25).

Let $u\in I^{\mathcal{T}_\rho}$ be a critical point we have found. To show that u is a constant sign function, we assume by contradiction that $u=u^++u^-$ with $u^\pm\neq 0$. Similar to the proof of Lemma 2.1, Lemma 2.2 and Lemma 2.3 in [16], we conclude that there exists $0<t_{u^+}<1$ and $0<t_{u^-}<1$ such that $t_{u^\pm}u^\pm\in\mathcal{N}$. Thus, by Proposition 2.3, we obtain that

$$2m_\infty=2m\le I(t_{u^+}u^+)+I(t_{u^-}u^-)\le I(t_{u^+}u^++t_{u^-}u^-)<I(u^++u^-)=I(u).$$

which is contrary with $I(u)<2m_\infty$. □

Acknowledgements

We thank the anonymous referee whose important comments helped us to improve our work. The work is supported by NSFC grant 11501403, by fund program for the Scientific Activities of Selected Returned Overseas Professionals in Shanxi Province (2018) and by the Natural Science Foundation of Shanxi Province (No. 201901D111085).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 5, 1–20; https://doi.org/10.14232/ejqtde.2021.1.5 www.math.u-szeged.hu/ejqtde/

Volterra–Stieltjes integral equations and impulsive Volterra–Stieltjes integral equations

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Received 20 August 2020, appeared 12 January 2021
Communicated by Michal Fečkan

Abstract. In this paper, we prove existence and uniqueness of solutions of Volterra–Stieltjes integral equations using the Henstock–Kurzweil integral. Also, we prove that these equations encompass impulsive Volterra–Stieltjes integral equations and prove the existence and uniqueness for these equations. Finally, we present some examples to illustrate our results.

Keywords: Volterra–Stieltjes integral, impulsive equations, existence and uniqueness, Henstock–Kurzweil–Stieltjes integral

2020 Mathematics Subject Classification: 34N05, 45D05, 26A42.

1 Introduction

In this paper, we are interested in the study of integral equations that can be modeled in the form

$$x(t) = x_0 + \int_{t_0}^{t} a(t,s) f(x(s),s)\, \mathrm{d}g(s), \qquad t \in [t_0, t_0+\sigma], \tag{1.1}$$

where the integral on the right-hand side is in the sense of Henstock–Kurzweil–Stieltjes [22]. This class of equations plays an important role from the theoretical point of view as well as for applications, since they subsumes many types of well known mathematical models. As a matter of fact, they can be used to model different problems such as anomalous diffusion processes, heat conduction with memory and diffusion of fluids in porous media, among others. See [3, 5, 7, 20, 21] for instance. On the other hand, the subject of Volterra integral equations has been attracting the attention by several researchers, since they represent a powerful tool for applications. See, for instance, [1, 4, 6, 8, 9, 14, 17].

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It is worth noticing that depending on the choice of the kernel $a : [t_0, t_0 + \sigma] \times [t_0, t_0 + \sigma] \to \mathbb{R}$, we can study in an unified way a very general class of problems. For instance, if $a(t, s) = 1$ for all $(t, s) \in [t_0, t_0 + \sigma] \times [t_0, t_0 + \sigma]$, then equation (1.1) reduces to the classical measure differential equation, which is very well-developed in the literature (see [12]). On the other hand, if $a(t, s) = k(t - s)$ for all $(t, s) \in [t_0, t_0 + \sigma] \times [t_0, t_0 + \sigma]$, then the integral equation (1.1) reduces to a Volterra integral equation which have many applications to the study of heat flow in the materials of fading memory type (see [7,20,21]), among others.

In the present paper, our goal is to prove existence and uniqueness results for the integral equation (1.1) under very weak conditions for the functions f, a and g. These results are more general than the ones presented in the literature, since the required conditions allow that either the function f in (1.1) be highly oscillating, or the functions a, f and g that appear in (1.1) may have a countable number of discontinuities. Also, we present three examples to illustrate our results.

Further, we prove that under certain assumptions the integral equation given by (1.1) can be regarded as an impulsive Volterra–Stieltjes integral equation described by

$$x(t) = x(t_0) + \int_{t_0}^{t} a(t, s) f(x(s), s)\, \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \ldots, m\}, \\ t_k < t}} I_k(x(t_k)). \tag{1.2}$$

These last equations can also be regarded as an impulsive Volterra Δ-integral equation on time scales given by

$$x(t) = x(t_0) + \int_{t_0}^{t} a(t, s) f(x(s), s) \Delta s + \sum_{\substack{k \in \{1, \ldots, m\}, \\ t_k < t}} I_k(x(t_k)), \tag{1.3}$$

when $g(t) = \inf\{s \in \mathbb{T} : s \geq t\}$. We only illustrate the first correspondence in this paper, since it brings more complexity due to the kernel from Volterra–Stieltjes integral equation. On the other hand, we have omitted the second one to turn the paper simpler and shorter, but following similar steps from [12], it is possible to prove such correspondence.

This paper is organized as follows. In the second section, we present the basic concepts and properties concerning the Henstock–Kurzweil–Stieltjes integral which is the main tool to prove our results. In the third section, we investigate the Volterra–Stieltjes integral equations and we prove a result concerning the existence and uniqueness of solutions of these equations. The last section is devoted to present a correspondence between Volterra–Stieltjes integral equations and impulsive Volterra–Stieltjes equations and also, to prove a result concerning existence and uniqueness of solutions for these last equations.

2 Henstock–Kurzweil–Stieltjes integral

In this section, we recall some properties concerning the Henstock–Kurzweil–Stieltjes integral. See [22] for more details.

Let $[a, b]$ be an interval of $\mathbb{R}$, $-\infty < a < b < +\infty$. A *tagged division* of $[a, b]$ is a finite collection of point-interval pairs $D = (\tau_i, [s_{i-1}, s_i])$, where $a = s_0 \leqslant s_1 \leqslant \ldots \leqslant s_{|D|} = b$ is a division of $[a, b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, |D|$, where the symbol $|D|$ denotes the number of subintervals in which $[a, b]$ is divided.

A *gauge* on a set $B \subset [a,b]$ is any function $\delta : B \to (0,\infty)$. Given a gauge δ on $[a,b]$, we say that a tagged division $D = (\tau_i, [s_{i-1}, s_i])$ is *δ-fine* if for every $i \in \{1, 2, \dots, |D|\}$, we have

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)).$$

A function $f : [a,b] \to \mathbb{R}$ is called *Henstock–Kurzweil–Stieltjes* integrable on $[a,b]$ with respect to a function $g : [a,b] \to \mathbb{R}$, if there is an element $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to (0,\infty)$ such that

$$\left| \sum_{i=1}^{|D|} f(\tau_i)\left(g(s_i) - g(s_{i-1})\right) - I \right| < \varepsilon,$$

for all δ–fine tagged partition of $[a,b]$. In this case, I is called *Henstock–Kurzweil–Stieltjes* integral of f with respect to g over $[a,b]$ and it will be denoted by $\int_a^b f(s)\,\mathrm{d}g(s)$, or simply $\int_a^b f\,\mathrm{d}g$.

The Henstock–Kurzweil–Stieltjes integral has the usual properties of linearity, additivity with respect to adjacent intervals, integrability on subintervals (see [22]).

We recall the reader that a function $f : [a,b] \to \mathbb{R}$ is called *regulated* if the lateral limits

$$f(t-) = \lim_{s \to t-} f(s), \quad t \in (a,b] \quad \text{and} \quad f(t+) = \lim_{s \to t+} f(s), \quad t \in [a,b)$$

exist. The space of all regulated functions $f : [a,b] \to \mathbb{R}$ will be denoted by $G([a,b],\mathbb{R})$, which is a Banach space when endowed with the usual supremum norm

$$\|f\|_\infty = \sup_{s \in [a,b]} |f(s)|.$$

Given a regulated function $f : [a,b] \to \mathbb{R}$, we will use the notations $\Delta^+ f(t)$ and $\Delta^- f(t)$ throughout this paper to denote

$$\Delta^+ f(t) := f(t+) - f(t) \quad \text{and} \quad \Delta^- f(t) := f(t) - f(t-),$$

respectively.

The next result ensures the existence of the Henstock–Kurzweil–Stieltjes integral. We observe that the inequalities follow from the definition of the Henstock–Kurzweil–Stieltjes integral. This result can be found in [22, Corollary 1.34].

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a regulated function on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ be a nondecreasing function. Then the following conditions hold.*

(i) *The integral $\int_a^b f(s)\mathrm{d}g(s)$ exists;*

(ii) $\left| \int_a^b f(s)\mathrm{d}g(s) \right| \leqslant \int_a^b |f(s)|\,\mathrm{d}g(s) \leqslant \|f\|_\infty \left(g(b) - g(a)\right)$.

The following inequalities follow directly from the definition of the Henstock–Kurzweil–Stieltjes integral. A similar version was proved in [2, Theorem 7.20] for the case of the Riemann–Stieltjes integral. We omit its proof here, since it is similar to the proof of [2].

Theorem 2.2. *Let f_1, $f_2 : [a,b] \to \mathbb{R}$ be Henstock–Kurzweil–Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function $g : [a,b] \to \mathbb{R}$ and such that $f_1(t) \leqslant f_2(t)$, for $t \in [a,b]$. Then*

$$\int_a^b f_1(s)\mathrm{d}g(s) \leqslant \int_a^b f_2(s)\mathrm{d}g(s).$$

The next result is an immediate consequence of Theorem 2.2.

Corollary 2.3. *Let $f : [a,b] \to \mathbb{R}$ be Henstock–Kurzweil–Stieltjes integrable function on the interval $[a,b]$ with respect to a nondecreasing function $g : [a,b] \to \mathbb{R}$ and such that $f(t) \geqslant 0$, for $t \in [a,b]$. Then*

(i) $\int_a^b f(s)\mathrm{d}g(s) \geqslant 0$.

(ii) *The function $[a,b] \ni t \mapsto \int_a^t f(s)\mathrm{d}g(s)$ is nondecreasing.*

In the sequel, we present a Gronwall–type inequality. See [22, Corollary, 1.43].

Lemma 2.4. *Let $g : [a,b] \to [0,\infty)$ be a nondecreasing and left-continuous function, $k > 0$ and $l \geqslant 0$. Assume that $\psi : [a,b] \to [0,\infty)$ is bounded and satisfies*

$$\psi(\xi) \leqslant k + l\int_a^\xi \psi(s)\mathrm{d}g(s), \qquad \xi \in [a,b].$$

Then $\psi(\xi) \leqslant ke^{l(g(\xi)-g(a))}$ for all $\xi \in [a,b]$.

The following result, which describes some properties of the indefinite Henstock–Kurzweil–Stieltjes integral, is a special case of [22, Theorem 1.16].

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be a pair of functions such that g is regulated and $\int_a^b f(s)\mathrm{d}g(s)$ exists. Then the function*

$$h(t) = \int_a^t f(s)\mathrm{d}g(s), \qquad t \in [a,b]$$

is regulated on $[a,b]$ and satisfy

$$\begin{aligned} h(t+) &= h(t) + f(t)\Delta^+ g(t), \qquad & t \in [a,b), \\ h(t-) &= h(t) - f(t)\Delta^- g(t), \qquad & t \in (a,b]. \end{aligned}$$

The following assertion is a Substitution Theorem for the Henstock–Kurzweil–Stieltjes integral. It can be found in [19, Theorem 2.19].

Theorem 2.6. *Assume the function $h : [a,b] \to \mathbb{R}$ is bounded and that the integral $\int_a^b f(s)\mathrm{d}g(s)$ exists. If one of the integrals*

$$\int_a^b h(t)\mathrm{d}\left(\int_a^t f(\xi)\mathrm{d}g(\xi)\right), \qquad \int_a^b h(t)f(t)\mathrm{d}g(t),$$

exists, then the other one exists as well, in which case the equality below holds

$$\int_a^b h(t)\mathrm{d}\left(\int_a^t f(\xi)\mathrm{d}g(\xi)\right) = \int_a^b h(t)f(t)\mathrm{d}g(t).$$

Now we present a result which is a type of the Dominated Convergence Theorem for Henstock–Kurzweil–Stieltjes integrals. See [22, Corollary 1.32].

Theorem 2.7. *Let $g : [a,b] \to \mathbb{R}$ be a nondecreasing function on $[a,b]$. Assume that $\varphi_n : [a,b] \to \mathbb{R}$ are functions such that the integral $\int_a^b \varphi_n(s)\,\mathrm{d}g(s)$ exists for all $n \in \mathbb{N}$. Suppose that for all $s \in [a,b]$, we have $\lim_{n\to\infty} \varphi_n(s) = \varphi(s)$ and that for $n \in \mathbb{N}$, $s \in [a,b]$ the inequalities $\kappa(s) \leqslant \varphi_n(s) \leqslant \omega(s)$ hold, where $\omega, \kappa : [a,b] \to \mathbb{R}$ are functions such that the integrals $\int_a^b \kappa(s)\,\mathrm{d}g(s)$ and $\int_a^b \omega(s)\,\mathrm{d}g(s)$ exist. Then the integral $\int_a^b \varphi(s)\,\mathrm{d}g(s)$ exists and*

$$\lim_{n\to\infty} \int_a^b \varphi_n(s)\,\mathrm{d}g(s) = \int_a^b \varphi(s)\,\mathrm{d}g(s).$$

The following lemma is a direct consequence of $G([a,b],\mathbb{R}^n)$ being a Banach space.

Lemma 2.8. *If a sequence $\{x_k\}_{k=1}^{\infty}$ of regulated functions (from $[a,b]$ to $\mathbb{R}$) converges uniformly on the interval $[a,b]$ to a function $x : [a,b] \to \mathbb{R}$, then this function is also regulated on $[a,b]$.*

We recall the reader that a set $\mathcal{A} \subset G([a,b],\mathbb{R})$ is called *equiregulated*, if it has the following property: for every $\varepsilon > 0$ and $t_0 \in [a,b]$, there is a $\delta > 0$ such that

(1) if $x \in \mathcal{A}$, $s \in [a,b]$ and $t_0 - \delta < s < t_0$, then $|x(t_0-) - x(s)| < \varepsilon$,

(2) if $x \in \mathcal{A}$, $s \in [a,b]$ and $t_0 < s < t_0 + \delta$, then $|x(t_0+) - x(s)| < \varepsilon$.

The next result describes a necessary and sufficient condition for a subset of $G([a,b],\mathbb{R})$ to be relatively compact, which is an immediate consequence of [15, Theorem 2.18]. We remark that even though the result in [15] requires v to be an increasing function, it is enough to assume that v is nondecreasing and let $\vartheta(t) := v(t) + t$, $t \in [a,b]$, to see that the original assumption is satisfied.

Theorem 2.9. *The following conditions are equivalent.*

(i) *$\mathcal{A} \subset G([a,b],\mathbb{R})$ is relatively compact.*

(ii) *The set $\{x(a) : x \in \mathcal{A}\}$ is bounded and there is a nondecreasing function $v : [a,b] \to \mathbb{R}$ such that*

$$|x(\tau_2) - x(\tau_1)| \leqslant v(\tau_2) - v(\tau_1),$$

for all $x \in \mathcal{A}$ and all $a \leqslant \tau_1 \leqslant \tau_2 \leqslant b$.

The following lemma will be crucial to prove that an impulsive Volterra integral equation can always be transformed to a Volterra integral equation without impulses. This result can be found in [12, Lemma 2.4].

Lemma 2.10. *Let $m \in \mathbb{N}$, $a \leqslant t_1 < t_2 < \cdots < t_m \leqslant b$. Consider a pair of functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$, where g is regulated, left-continuous on $[a,b]$, and continuous at $t_1, \ldots, t_m$. Let $\tilde{f} : [a,b] \to \mathbb{R}$ and $\tilde{g} : [a,b] \to \mathbb{R}$ be such that $\tilde{f}(t) = f(t)$ for every $t \in [a,b] \setminus \{t_1, \ldots, t_m\}$ and $\tilde{g} - g$ is constant on each of the intervals $[a,t_1]$, $(t_1,t_2], \ldots, (t_{m-1},t_m]$, $(t_m,b]$. Then the integral $\int_a^b \tilde{f}\,\mathrm{d}\tilde{g}$ exists if and only if the integral $\int_a^b f\,\mathrm{d}g$ exists; in that case, we have*

$$\int_a^b \tilde{f}(s)\,\mathrm{d}\tilde{g}(s) = \int_a^b f(s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\},\\ t_k<b}} \tilde{f}(t_k)\Delta^+\tilde{g}(t_k).$$

The next result will be essential to prove the existence of solution of Volterra–Stieltjes integral equations. It is a classical result of fixed point.

Theorem 2.11 (Schauder Fixed-Point Theorem)**.** *Let $(E, \|\cdot\|)$ be a normed vector space, S a nonempty convex and closed subset of E and $T : S \to S$ is a continuous function such that $T(S)$ is relatively compact. Then T has a fixed point in S.*

3 Volterra–Stieltjes integral equations

In this section, our goal is to study the following type of equation

$$x(t) = x_0 + \int_{t_0}^{t} a(t,s) f(x(s),s)\,\mathrm{d}g(s), \qquad t \in [t_0, t_0+\sigma], \quad t_0 \in \mathbb{R},$$

where the Henstock–Kurzweil–Stieltjes integral on the right–hand side is taken with respect to a nondecreasing function $g : [t_0, t_0+\sigma] \to \mathbb{R}$, $f : \mathbb{R} \times [t_0, t_0+\sigma] \to \mathbb{R}$, $\sigma > 0$, $x_0 \in \mathbb{R}$, and $a : [t_0, t_0+\sigma]^2 \to \mathbb{R}$, where $[t_0, t_0+\sigma]^2 = [t_0, t_0+\sigma] \times [t_0, t_0+\sigma]$.

Throughout this paper, we will use the symbol $G_2([t_0, t_0+\sigma]^2, \mathbb{R})$ to denote the set of all functions $b : [t_0, t_0+\sigma]^2 \to \mathbb{R}$ such that b is regulated with respect to the second variable, that is, for any fixed $t \in [t_0, t_0+\sigma]$, the function

$$b(t,\cdot) : s \in [t_0, t_0+\sigma] \longmapsto b(t,s) \in \mathbb{R}$$

is regulated.

In what follows, we say that $c : [t_0, t_0+\sigma]^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable if for any fixed $s \in [t_0, t_0+\sigma]$, the function

$$c(\cdot,s) : t \in [t_0, t_0+\sigma] \longmapsto c(t,s) \in \mathbb{R}$$

is nondecreasing.

We assume the following conditions are satisfied.

(A1) The function $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is nondecreasing and left-continuous on $(t_0, t_0+\sigma]$.

(A2) The function $a \in G_2([t_0, t_0+\sigma]^2, \mathbb{R})$ is nondecreasing with respect to the first variable.

(A3) The Henstock–Kurzweil–Stieltjes integral

$$\int_{t_0}^{t_0+\sigma} a(t,s) f(x(s),s)\mathrm{d}g(s)$$

exists, for all $x \in G([t_0, t_0+\sigma], \mathbb{R})$ and all $t \in [t_0, t_0+\sigma]$.

(A4) There exists a Henstock–Kurzweil–Stieltjes integrable function $M : [t_0, t_0+\sigma] \to \mathbb{R}^+$ with respect to g such that

$$\left| \int_{\tau_1}^{\tau_2} (c_2 a(\tau_2,s) + c_1 a(\tau_1,s)) f(x(s),s)\mathrm{d}g(s) \right| \leq \int_{\tau_1}^{\tau_2} |c_2 a(\tau_2,s) + c_1 a(\tau_1,s)|\, M(s)\mathrm{d}g(s),$$

for all $x \in G([t_0, t_0+\sigma], \mathbb{R})$, $c_1, c_2 \in \mathbb{R}$ and all $[\tau_1, \tau_2] \subset [t_0, t_0+\sigma]$. In particular, we have that

$$\left| \int_{\tau_1}^{\tau_2} a(\tau,s) f(x(s),s)\mathrm{d}g(s) \right| \leq \int_{\tau_1}^{\tau_2} |a(\tau,s)| M(s)\mathrm{d}g(s),$$

and

$$\left| \int_{\tau_1}^{\tau_2} (a(\tau_2,s) - a(\tau_1,s)) f(x(s),s)\mathrm{d}g(s) \right| \leq \int_{\tau_1}^{\tau_2} |a(\tau_2,s) - a(\tau_1,s)| M(s)\mathrm{d}g(s)$$

for all $x \in G([t_0, t_0+\sigma], \mathbb{R})$, and all $\tau, \tau_1, \tau_2 \in [t_0, t_0+\sigma]$.

(A5) There exists a regulated function $L : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\left| \int_{\tau_1}^{\tau_2} a(\tau_2, s)[f(x(s), s) - f(z(s), s)] \mathrm{d}g(s) \right| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \, L(s) |x(s) - z(s)| \mathrm{d}g(s),$$

for all $x, z \in G([t_0, t_0 + \sigma], \mathbb{R})$ and all $[\tau_1, \tau_2] \subset [t_0, t_0 + \sigma]$.

Remark 3.1. Note that $\int_{t_0}^{t_0+\sigma} |c_2 a(\tau_2, s) + c_1 a(\tau_1, s)| \, M(s) \mathrm{d}g(s)$ and $\int_{t_0}^{t_0+\sigma} |a(t, s)| \, L(s) |x(s) - z(s)| \mathrm{d}g(s)$ exist. Indeed, by Corollary 2.3, $[t_0, t_0 + \sigma] \ni t \mapsto \int_{t_0}^{t} M(s) \mathrm{d}g(s)$ is a nondecreasing function. On the other hand, the function $[t_0, t_0 + \sigma] \ni s \mapsto c_2 a(\tau_2, s) + c_1 a(\tau_1, s)$ is regulated. Then, by Theorem 2.1, the integral $\int_{t_0}^{t_0+\sigma} |c_2 a(\tau_2, s) + c_1 a(\tau_1, s)| \, \mathrm{d} \left(\int_{t_0}^{s} M(\xi) \mathrm{d}g(\xi) \right)$ exists. Using this fact, the boundedness of $c_2 a(\tau_2, \cdot) + c_1 a(\tau_1, \cdot)$ and Theorem 2.6, we have that the integral $\int_{t_0}^{t_0+\sigma} |c_2 a(\tau_2, s) + c_1 a(\tau_1, s)| \, M(s) \mathrm{d}g(s)$ exists. For the second integral, note that the function $s \mapsto |a(t, s)| \, L(s) |x(s) - z(s)|$ is regulated.

Remark 3.2. Note that when $s \mapsto a(\tau, s) f(x(s), s)$ is a regulated function on $[t_0, t_0 + \sigma]$ for $t_0 \leq \tau \leq t_0 + \sigma$ and g is nondecreasing, then (A4) holds by Theorem 2.1.

Remark 3.3. Suppose that g is a nondecreasing function. Then, the condition (A4) is true whenever the function f is bounded in x. Moreover, we observe that condition (A5) holds whenever the following Lipschitz type condition is satisfied:

$$|f(x(s), s) - f(z(s), s)| \leqslant L(s) |x(s) - z(s)|, \qquad t_0 \leqslant s \leqslant t_0 + \sigma,$$

where $L : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ is a regulated function.

Remark 3.4. Suppose that a satisfies condition (A2). Since $a(t_0, y) \leqslant a(x, y) \leqslant a(t_0 + \sigma, y)$ for all $x, y \in [t_0, t_0 + \sigma]$ and the functions $a(t_0, y), a(t_0 + \sigma, y)$ are regulated in y, we have that a is bounded in $[t_0, t_0 + \sigma]^2$.

Next, we present the main result of this section. It ensures the existence and uniqueness of solution of Volterra–Stieltjes integral equations. In order to prove it, we employ the Schauder Fixed Point Theorem and Gronwall's inequality for Stieltjes integral.

Theorem 3.5. *Assume $f : \mathbb{R} \times [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfies the conditions (A3), (A4) and (A5), $a : [t_0, t_0 + \sigma]^2 \to \mathbb{R}$ satisfies condition (A2) and $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfies condition (A1). Then there exists a unique solution $x : [t_0, t_0 + \sigma] \to \mathbb{R}$ of*

$$x(t) = x_0 + \int_{t_0}^{t} a(t, s) f(x(s), s) \, \mathrm{d}g(s), \qquad t \in [t_0, t_0 + \sigma]. \tag{3.1}$$

Proof. Let us define the following constants:

$$c := \sup_{(t,s) \in [t_0, t_0+\sigma]^2} |a(t, s)|, \tag{3.2}$$

$$\beta := \int_{t_0}^{t_0+\sigma} c M(s) \, \mathrm{d}g(s). \tag{3.3}$$

Notice that all these constants are finite and well-defined in view of conditions (A2), (A4) and Remark 3.4.

Existence. Consider the set

$$H := \{ \varphi \in G([t_0, t_0 + \sigma], \mathbb{R}) : \varphi(t_0) = x_0 \text{ and } |\varphi(t) - x_0| \leqslant \beta, \ t \in [t_0, t_0 + \sigma] \}.$$

The set H is nonempty, since

$$\begin{aligned}\varphi : [t_0, t_0+\sigma] &\to \mathbb{R}\\ s &\mapsto \varphi(s) := x_0,\end{aligned}$$

belongs to H. Define $T : H \to H$ given by

$$(Tx)(t) := x_0 + \int_{t_0}^{t} a(t,s) f(x(s),s)\,\mathrm{d}g(s), \qquad x \in H. \tag{3.4}$$

Taking into account the condition (A3), we infer that the integral on the right-hand side of (3.4) is well-defined. Now, given $x \in H$ and $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0+\sigma$, by conditions (A2), (A3), (A4), Theorem 2.2 and Corollary 2.3, we have

$$\begin{aligned}
&|(Tx)(\tau_2) - (Tx)(\tau_1)|\\
&= \left| \int_{t_0}^{\tau_2} a(\tau_2,s) f(x(s),s)\,\mathrm{d}g(s) - \int_{t_0}^{\tau_1} a(\tau_1,s) f(x(s),s)\,\mathrm{d}g(s) \right|\\
&= \left| \int_{t_0}^{\tau_1} a(\tau_2,s) f(x(s),s)\,\mathrm{d}g(s) + \int_{\tau_1}^{\tau_2} a(\tau_2,s) f(x(s),s)\,\mathrm{d}g(s) - \int_{t_0}^{\tau_1} a(\tau_1,s) f(x(s),s)\,\mathrm{d}g(s) \right|\\
&\leqslant \left| \int_{\tau_1}^{\tau_2} a(\tau_2,s) f(x(s),s)\,\mathrm{d}g(s) \right| + \left| \int_{t_0}^{\tau_1} (a(\tau_2,s) - a(\tau_1,s))\, f(x(s),s)\,\mathrm{d}g(s) \right|\\
&\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\, M(s)\,\mathrm{d}g(s) + \int_{t_0}^{\tau_1} |a(\tau_2,s) - a(\tau_1,s)|\, M(s)\,\mathrm{d}g(s)\\
&\overset{\substack{\text{Thm 2.2, (A2), (A4) and (3.2)}\\ \downarrow}}{\leqslant} \int_{\tau_1}^{\tau_2} cM(s)\,\mathrm{d}g(s) + \int_{t_0}^{\tau_1} (a(\tau_2,s) - a(\tau_1,s)) M(s)\,\mathrm{d}g(s)\\
&\leqslant \int_{\tau_1}^{\tau_2} cM(s)\,\mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} (a(\tau_2,s) - a(\tau_1,s)) M(s)\,\mathrm{d}g(s),
\end{aligned}$$

that is,

$$|(Tx)(\tau_2) - (Tx)(\tau_1)| \leqslant \int_{\tau_1}^{\tau_2} cM(s)\,\mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} (a(\tau_2,s) - a(\tau_1,s)) M(s)\,\mathrm{d}g(s). \tag{3.5}$$

Define $v : [t_0, t_0+\sigma] \to \mathbb{R}$ by

$$v(t) := \int_{t_0}^{t} cM(s)\,\mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} a(t,s) M(s)\,\mathrm{d}g(s), \tag{3.6}$$

for every $t \in [t_0, t_0+\sigma]$. Since M is a Henstock–Kurzweil–Stieltjes integrable function, $\int_{t_0}^{t} cM(s)\,\mathrm{d}g(s)$ exists for all $t \in [t_0, t_0+\sigma]$. On the other hand, using the same arguments as in the Remark 3.1, we ensure the existence of $\int_{t_0}^{t_0+\sigma} a(t,s)M(s)\,\mathrm{d}g(s)$ for all $t \in [t_0, t_0+\sigma]$. Then v is well-defined. Also, it is easy to check that v is a nondecreasing function. Using (3.5) and (3.6), we have

$$|(Tx)(\tau_2) - (Tx)(\tau_1)| \leqslant v(\tau_2) - v(\tau_1), \tag{3.7}$$

for all $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0+\sigma$. Note that the limits $(Tx)(t+)$ for $t \in [t_0, t_0+\sigma)$ and $(Tx)(t-)$ for $t \in (t_0, t_0+\sigma]$ exist. Indeed, since v is a nondecreasing function, then the limits $v(t+)$ for $t \in [t_0, t_0+\sigma)$ and $v(t-)$ for $t \in (t_0, t_0+\sigma]$ exist and, therefore, (3.7) ensures the Cauchy condition is satisfied, which implies the existence of the corresponding limits $(Tx)(t+)$ and

$(Tx)(t-)$. From this, we get that $Tx \in G([t_0, t_0 + \sigma], \mathbb{R})$. Also, for $t_0 \leqslant t \leqslant t_0 + \sigma$, by condition (A4), Theorem 2.2 and Corollary 2.3, we obtain

$$\begin{aligned} |(Tx)(t) - x_0| &= \left| \int_{t_0}^{t} a(t,s) f(x(s), s) \, \mathrm{d}g(s) \right| \\ &\leqslant \int_{t_0}^{t} |a(t,s)| M(s) \, \mathrm{d}g(s) \\ &\leqslant \int_{t_0}^{t} c M(s) \, \mathrm{d}g(s) \\ &\leqslant \int_{t_0}^{t_0+\sigma} c M(s) \, \mathrm{d}g(s) \\ &\overset{(3.3)}{=} \beta. \end{aligned}$$

Clearly, $(Tx)(t_0) = x_0$. It implies that $Tx \in H$ for all $x \in H$. Hence, T is well-defined.

Assertion 1. H is convex and closed.
Let $\varphi, \phi \in H$. Then for all $\theta \in [0,1]$, we have $(1-\theta)\phi + \theta\varphi \in G([t_0, t_0 + \sigma])$ and

$$\begin{aligned} |(1-\theta)\phi(t) + \theta\varphi(t) - x_0| &= |(1-\theta)\phi(t) + \theta\varphi(t) - ((1-\theta)x_0 + \theta x_0)| \\ &\leqslant (1-\theta)|\phi(t) - x_0| + \theta|\varphi(t) - x_0| \\ &\leqslant (1-\theta)\beta + \theta\beta = \beta. \end{aligned}$$

This proves that H is convex.

On the other hand, let $\{\varphi_k\}_{k \in \mathbb{N}} \subset H$ be such that $\varphi_k \xrightarrow{\|\cdot\|_\infty} \varphi$ (on $[t_0, t_0 + \sigma]$) as $k \to \infty$. Since each φ_k is regulated and φ_k converges uniformly to φ on $[t_0, t_0 + \sigma]$, Lemma 2.8 guarantees that φ is regulated on $[t_0, t_0 + \sigma]$ and, therefore, $\varphi \in G([t_0, t_0 + \sigma], \mathbb{R})$. Also, given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|\varphi(t) - x_0| \leqslant |\varphi_k(t) - \varphi(t)| + |\varphi_k(t) - x_0| \leqslant \varepsilon + \beta,$$

for all $t \in [t_0, t_0 + \sigma]$ and $k \geqslant N$. Since $\varepsilon > 0$ is arbitrary, we get $|\varphi(t) - x_0| \leqslant \beta$ for all $t \in [t_0, t_0 + \sigma]$. It implies that H is closed.

Assertion 2. $\mathcal{A} := T(H) = \{Tx : x \in H\}$ is relatively compact.
Note that the set $\{y(t_0) : y \in \mathcal{A}\} = \{\underbrace{(Tx)(t_0)}_{x_0} : x \in H\}$ is bounded. On the other hand, for an arbitrary $y = Tx$, $x \in H$ and $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0 + \sigma$, by (3.7), we have

$$|y(\tau_2) - y(\tau_1)| = |(Tx)(\tau_2) - (Tx)(\tau_1)| \leqslant v(\tau_2) - v(\tau_1). \tag{3.8}$$

Hence, by Theorem 2.9, $\mathcal{A} = T(H)$ is relatively compact.

Assertion 3. T is continuous.
By condition (A5), Theorem 2.2 and Corollary 2.3, we have that for $x, z \in H$ and for $t_0 \leqslant t \leqslant t_0 + \sigma$,

$$\begin{aligned}
|(Tx)(t) - (Tz)(t)| &= \left| \int_{t_0}^{t} a(t,s) f(x(s),s) \, \mathrm{d}g(s) - \int_{t_0}^{t} a(t,s) f(z(s),s) \, \mathrm{d}g(s) \right| \\
&= \left| \int_{t_0}^{t} a(t,s) (f(x(s),s) - f(z(s),s)) \, \mathrm{d}g(s) \right| \\
&\leqslant \int_{t_0}^{t} |a(t,s)| L(s) |x(s) - z(s)| \, \mathrm{d}g(s) \\
&\leqslant \int_{t_0}^{t} |x(s) - z(s)| c L(s) \, \mathrm{d}g(s) \\
&\leqslant \int_{t_0}^{t_0+\sigma} |x(s) - z(s)| c L(s) \, \mathrm{d}g(s) \\
&\leqslant \|x - z\|_\infty \left(\int_{t_0}^{t_0+\sigma} c L(s) \, \mathrm{d}g(s) \right).
\end{aligned}$$

From the above estimate, we conclude that T is continuous.

Therefore, all the hypotheses of the Schauder Fixed-Point Theorem (Theorem 2.11) are satisfied, which implies that T has a fixed point in H. Thus, we conclude that the equation (3.1) possesses a solution $x : [t_0, t_0 + \sigma] \to \mathbb{R}$.

It remains to prove the uniqueness of the solution of (3.1).

Uniqueness: Assume that $x, z : [t_0, t_0 + \sigma] \to \mathbb{R}$ are two solutions of Volterra–Stieltjes integral equation (3.1). Fix arbitrarily $t \in [t_0, t_0 + \sigma]$. Then, keeping in mind condition (A5) and Theorem 2.2, we infer the following estimates

$$\begin{aligned}
|x(t) - z(t)| &= \left| \int_{t_0}^{t} a(t,s) f(x(s),s) \, \mathrm{d}g(s) - \int_{t_0}^{t} a(t,s) f(z(s),s) \, \mathrm{d}g(s) \right| \\
&= \left| \int_{t_0}^{t} a(t,s) (f(x(s),s) - f(z(s),s)) \, \mathrm{d}g(s) \right| \\
&\leqslant \int_{t_0}^{t} |a(t,s)| L(s) |x(s) - z(s)| \, \mathrm{d}g(s) \\
&\leqslant c \, \|L\|_\infty \int_{t_0}^{t} |x(s) - z(s)| \, \mathrm{d}g(s) \\
&< \varepsilon + c \, \|L\|_\infty \int_{t_0}^{t} |x(s) - z(s)| \, \mathrm{d}g(s),
\end{aligned}$$

for every $\varepsilon > 0$. Hence, in view of Lemma 2.4, we have

$$|x(t) - z(t)| \leqslant \varepsilon e^{c\|L\|_\infty (g(t) - g(t_0))}.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $x(t) = z(t)$ for all $t \in [t_0, t_0 + \sigma]$, that is, $x = z$. □

Remark 3.6. If $a(t,s) = a_1(t) b_1(s)$ where a_1 is nondecreasing on $[t_0, t_0 + \sigma]$ and b_1 is regulated and positive on $[t_0, t_0 + \sigma]$, then it is clear that a satisfies condition (A2).

Example 3.7. Consider the Volterra–Stieltjes integral equation given by

$$x(t) = x_0 + \int_{t_0}^{t} k(t-s) f(x(s),s) \, \mathrm{d}g(s), \qquad t \in [t_0, t_0 + \sigma],$$

where $t_0, x_0 \in \mathbb{R}$, $\sigma > 0$, $k : [-\sigma, \sigma] \to \mathbb{R}$ is a nondecreasing function, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfies condition (A1) and $f : \mathbb{R} \times [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfies conditions (A3)–(A5) from Theorem 3.5.

Define $a : [t_0, t_0 + \sigma]^2 \to \mathbb{R}$ given by

$$a(t,s) := k(t-s), \qquad (t,s) \in [t_0, t_0+\sigma]^2.$$

In the sequel, we show that a satisfies condition (A2) from Theorem 3.5. Indeed, notice that given $t, s \in [t_0, t_0 + \sigma]$, we have $t - s \in [-\sigma, \sigma] = \mathrm{Dom}(k)$ and, therefore, a is well-defined over $[t_0, t_0 + \sigma]^2$.

Obviously, $a(\cdot, s)$ is nondecreasing for any $s \in [t_0, t_0 + \sigma]$ and $a(t, \cdot)$ is nonincreasing for any $t \in [t_0, t_0 + \sigma]$, getting (A2).

We will present an example of a Volterra–Stieltjes integral equation of the form (3.1) which satisfies all the hypotheses of the previous theorem.

Example 3.8. Consider the Volterra–Stieltjes integral equation given by

$$x(t) = x_0 + \int_0^t a(t,s) f(x(s), s)\, \mathrm{d}g(s), \qquad t \in [0, 3/\delta],$$

where $x_0 \in \mathbb{R}$, $\delta > 0$, $g : [0, \frac{3}{\delta}] \to \mathbb{R}$ is a nondecreasing function, $a : [0, \frac{3}{\delta}]^2 \to \mathbb{R}$ and $f : \mathbb{R} \times [0, \frac{3}{\delta}] \to \mathbb{R}$ are given, respectively, by

$$a(t,s) = st^3 e^{-\delta t}, \qquad (t,s) \in [0, 3/\delta]^2$$

and

$$f(x,t) = \frac{\{t+2\}\cos(2x)}{4^t + [t]}, \qquad (x,t) \in \mathbb{R} \times [0, 3/\delta],$$

where the symbol $[t]$ denotes the integer part of t, and the symbol $\{t\} := t - [t]$ denotes the fractional part of t. We will verify the conditions (A1)–(A5). Indeed, clearly g satisfies condition (A1).

Note that for any fixed $t \in [0, \frac{3}{\delta}]$, the function $[0, \frac{3}{\delta}] \ni s \mapsto a(t,s)$ is regulated on $[0, \frac{3}{\delta}]$. Since $a(t,s) = st^3 e^{-\delta t}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t,s) = st^2 e^{-\delta t}(3 - \delta t) \geqslant 0,$$

for all $t \in [0, \frac{3}{\delta}]$. Thus, a is a nondecreasing function with respect to the first variable, proving condition (A2).

Let $x \in G([0, \frac{3}{\delta}], \mathbb{R})$ and $t \in [0, \frac{3}{\delta}]$ be given. Notice that $[0, \frac{3}{\delta}] \ni s \mapsto a(t,s) f(x(s), s)$ is a regulated function on $[0, \frac{3}{\delta}]$. Thus by Theorem 2.1 (item (i)), $\int_0^{\frac{3}{\delta}} a(t,s) f(x(s), s) \mathrm{d}g(s)$ exists, obtaining condition (A3).

Define $M : [0, \frac{3}{\delta}] \to \mathbb{R}^+$ by $M(s) = \{s+2\}$, for $s \in [0, \frac{3}{\delta}]$. Evidently, M is a Henstock–Kurzweil–Stieltjes integrable function with respect to g and for $x \in G([0, \frac{3}{\delta}], \mathbb{R})$, $c_1, c_2 \in \mathbb{R}$,

$[\tau_1, \tau_2] \subset [0, \frac{3}{\delta}]$ and $b_{\tau_2,\tau_1}(s) := c_1 a(\tau_2, s) + c_1 a(\tau_1, s)$, we have

$$\begin{aligned}\left|\int_{\tau_1}^{\tau_2} b_{\tau_2,\tau_1}(s) f(x(s), s) \mathrm{d}g(s)\right| &\overset{\text{Thm } 2.1}{\underset{\downarrow}{\leqslant}} \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)|\, |f(x(s), s)|\, \mathrm{d}g(s) \\ &= \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)| \left|\frac{\{s+2\}\cos(2x(s))}{4^s + [s]}\right| \mathrm{d}g(s) \\ &\leqslant \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)|\, \{s+2\}\, \mathrm{d}g(s) \\ &= \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)|\, M(s) \mathrm{d}g(s),\end{aligned}$$

proving the condition (A4).

On the other hand, define $L : [0, \frac{3}{\delta}] \to \mathbb{R}^+$ by $L(t) = 2$, for $t \in [0, \frac{3}{\delta}]$. Note that L is a regulated function and for $x,\ y \in G([0, \frac{3}{\delta}], \mathbb{R})$ and $\tau_1,\ \tau_2 \in [0, \frac{3}{\delta}]$, $\tau_1 \leqslant \tau_2$, we get

$$\begin{aligned}\left|\int_{\tau_1}^{\tau_2} a(\tau_2, s)\, [f(x(s), s) - f(y(s), s)]\, \mathrm{d}g(s)\right| &\overset{\text{Thm } 2.1}{\underset{\downarrow}{\leqslant}} \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, |f(x(s), s) - f(y(s), s)|\, \mathrm{d}g(s) \\ &\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \Big|\cos(2x(s)) - \cos(2y(s))\Big| \mathrm{d}g(s) \\ &\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, |2x(s) - 2y(s)|\, \mathrm{d}g(s) \\ &= 2\int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, |x(s) - y(s)|\, \mathrm{d}g(s),\end{aligned}$$

getting the condition (A5). Hence f, a and g fulfill all the hypotheses of Theorem 3.5.

The next example is an adaptation of [18, Example 7.8]. It is a modified version of a model of a single artificial effective neuron with dissipation. See [10, 16].

Example 3.9. Consider the equation

$$x(t) = x_0 + \int_0^t k(s) \tanh(x(s))\, ds, \qquad t \in [0, 1]$$

where k is a nondecreasing function on $[0, 1]$. Define $a(t, s) := k(s)$ for all $(t, s) \in [0, 1]^2$, $f : \mathbb{R} \times [0, 1] \to \mathbb{R}$ by $f(x, t) := \tanh(x)$ for all $(x, t) \in \mathbb{R} \times [0, 1]$, and $g(s) = s$ for all $s \in [0, 1]$.

Observe that, by definition, the function g is left-continuous on $(0, 1]$ and increasing on $[0, 1]$.

Notice that the function a is constant with relation to the first variable. Thus, a is a nondecreasing function with respect to the first variable. Also, since k is a nondecreasing function, we have that for any fixed $t \in [0, 1]$, the function $[0, 1] \ni s \mapsto a(t, s) = k(s)$ is regulated on $[0, 1]$, obtaining the condition (A2). Moreover, $a(t, s) f(x(s), s)$ is a regulated function on $[0, 1]$, for all $x \in G([0, 1], \mathbb{R})$, and all $t \in [0, 1]$. Hence, the integral $\int_0^1 a(t, s) f(x(s), s) \mathrm{d}g(s)$ exists, getting (A3).

On the other hand, define $M : [0, 1] \to \mathbb{R}^+$ by $M(t) = 1$, for $t \in [0, 1]$. By Theorem 2.1, we have

$$\begin{aligned}\left|\int_{\tau_1}^{\tau_2} b_{\tau_2,\tau_1}(s) f(x(s),s)\mathrm{d}g(s)\right| &\leqslant \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)|\,|f(x(s),s)|\,\mathrm{d}g(s)\\ &= \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)|\,|\tanh(x(s))|\,\mathrm{d}g(s)\\ &\leqslant \int_{\tau_1}^{\tau_2} |b_{\tau_2,\tau_1}(s)|\,M(s)\mathrm{d}g(s),\end{aligned}$$

for $x \in G([0,1],\mathbb{R})$, $c_1, c_2 \in \mathbb{R}$, $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant 1$ and $b_{\tau_2,\tau_1}(s) := c_1 a(\tau_2, s) + c_2 a(\tau_1, s)$, where the third inequality follows of the fact that $-1 < \tanh(x) < 1$ for all $x \in \mathbb{R}$.

Finally, define $L : [0,1] \to \mathbb{R}^+$ by $L(t) = 1$, for $t \in [0,1]$. Evidently, L is a regulated function and

$$\begin{aligned}\left|\int_{\tau_1}^{\tau_2} a(\tau_2,s)\,[f(x(s),s) - f(y(s),s)]\,\mathrm{d}g(s)\right| &\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,|f(x(s),s) - f(y(s),s)|\,\mathrm{d}g(s)\\ &\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,|x(s) - y(s)|\,\mathrm{d}g(s),\end{aligned}$$

for x, $y \in G([0,1],\mathbb{R})$ and all $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant 1$, obtaining the condition (A5). Notice that $|\tanh(v) - \tanh(u)| \leqslant |v - u|$ for all $v, u \in \mathbb{R}$. Hence f, a and g fulfill all the hypotheses of Theorem 3.5.

4 Impulsive Volterra–Stieltjes integral equations

In this section, we are interested in the study of impulsive Volterra–Stieltjes integral equations.

Consider a Volterra–Stieltjes integral equation given by:

$$x(t) = x_0 + \int_{t_0}^{t} a(t,s) f(x(s),s)\,\mathrm{d}g(s), \qquad t \in [t_0, t_0 + \sigma],$$

where the Henstock–Kurzweil–Stieltjes integral on the right-hand side is taken with respect to a nondecreasing function $g : [t_0, t_0 + \sigma] \to \mathbb{R}$.

Let the set $D = \{t_1, \dots, t_m\} \subset [t_0, t_0 + \sigma]$ be such that $t_0 \leqslant t_1 < \cdots < t_m < t_0 + \sigma$ and let the functions $I_k : \mathbb{R} \to \mathbb{R}$ be given for $k \in \{1, \dots, m\}$. Assume that $a(\cdot, s)$ and g are continuous at each $\tau \in D$ and consider the problem to determine a function x satisfying the given Volterra–Stieltjes integral equation and impulse conditions $\Delta^+ x(t_k) = I_k(x(t_k))$ for $k \in \{1, \dots, m\}$. Using this, we achieve the following formulation of the problem:

$$\begin{aligned} x(v) - x(u) &= \int_{t_0}^{v} a(v,s) f(x(s),s)\,\mathrm{d}g(s)\\ &\quad - \int_{t_0}^{u} a(u,s) f(x(s),s)\,\mathrm{d}g(s) \quad \text{for } u, v \in J_k,\ k \in \{0, \dots, m\},\\ \Delta^+ x(t_k) &= I_k(x(t_k)), \qquad k \in \{1, \dots, m\},\\ x(t_0) &= x_0,\end{aligned}$$

where $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k \in \{1, \dots, m-1\}$, and $J_m = (t_m, t_0 + \sigma]$.

The value of the following integrals

$$\int_{t_0}^{v} a(v,s) f(x(s),s)\,\mathrm{d}g(s) \quad \text{and} \quad \int_{t_0}^{u} a(u,s) f(x(s),s)\,\mathrm{d}g(s),$$

where $u, v \in J_k$, are the same if we replace g by a function $\tilde{g}$ such that $g - \tilde{g}$ is a constant function on J_k. This follows from the properties of Henstock–Kurzweil–Stieltjes integral. Also, let us assume g is a left-continuous function which is continuous at t_k, for each $k = 1, \dots, m$. Therefore, it implies that $\Delta^+ g(t_k) = 0$ for every $k \in \{1, \dots, m\}$. Moreover, we assume a is continuous with respect to first variable at $t_1, \dots, t_m$ and also, a satisfies condition (A2) presented in Section 3. Further suppose that f and g satisfy conditions (A1), (A3) and (A4) presented in Section 3. Under these assumptions, our problem can be rewritten as

$$x(t) = x(t_0) + \int_{t_0}^{t} a(t,s) f(x(s), s)\, \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \dots, m\}, \\ t_k < t}} I_k(x(t_k)). \tag{4.1}$$

It is not difficult to see that by the assumptions above, the function

$$t \mapsto \int_{t_0}^{t} a(t,s) f(x(s), s) \mathrm{d}g(s)$$

is continuous at $t_1, \dots, t_m$ (see Remark 4.1 below) and, therefore, $\Delta^+ x(t_k) = I_k(x(t_k))$ for every $k \in \{1, \dots, m\}$.

Remark 4.1. We assume that f, g and a satisfy the assumptions above. Using the same arguments as in the proof of Theorem 3.5, we can prove the following inequality

$$\left| \int_{t_0}^{t} a(t,s) f(x(s), s)\, \mathrm{d}g(s) - \int_{t_0}^{\tau} a(\tau, s) f(x(s), s)\, \mathrm{d}g(s) \right| \leqslant |v(t) - v(\tau)|, \tag{4.2}$$

for all $t, \tau \in [t_0, t_0 + \sigma]$, where v is given by

$$v(t) := \int_{t_0}^{t} cM(s)\, \mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} a(t,s) M(s)\, \mathrm{d}g(s), \qquad t \in [t_0, t_0 + \sigma]. \tag{4.3}$$

Here $c := \sup_{(t,s) \in [t_0, t_0+\sigma]^2} |a(t,s)|$. Notice that every point in $[t_0, t_0 + \sigma]$ at which the function v is continuous, is a continuity point of the function $t \mapsto \int_{t_0}^{t} a(t,s) f(x(s), s) \mathrm{d}g(s)$. Next, let us prove that v given by (4.3) is a continuous function at $t_1, \dots, t_m$. Clearly, $v_1(t) = \int_{t_0}^{t} cM(s)\, \mathrm{d}g(s)$, $t \in [t_0, t_0 + \sigma]$, is continuous at $t_1, \dots, t_m$.

Assertion 1. $v_2(t) = \int_{t_0}^{t_0+\sigma} a(t,s) M(s)\, \mathrm{d}g(s)$, $t \in [t_0, t_0 + \sigma]$, is continuous at $t_1, \dots, t_m$.
Let $i \in \{1, \dots, m\}$ and $(\tau_n)_{n \in \mathbb{N}} \subset [t_0, t_0 + \sigma]$ such that $\tau_n \overset{n \to \infty}{\to} t_i$.

Define the sequence of functions

$$\varphi_n(s) := a(\tau_n, s) M(s), \qquad s \in [t_0, t_0 + \sigma], \tag{4.4}$$

and $\varphi : [t_0, t_0 + \sigma] \to \mathbb{R}$ by $\varphi(s) := a(t_i, s) M(s), s \in [t_0, t_0 + \sigma]$. As $a(\cdot, s)$ is continuous at t_i and $(\tau_n)_{n \in \mathbb{N}} \subset [t_0, t_0 + \sigma]$ is such that $\tau_n \overset{n \to \infty}{\to} t_i$, we have $\lim_{n \to \infty} a(\tau_n, s) = a(t_i, s)$, and therefore,

$$\lim_{n \to \infty} \varphi_n(s) = \lim_{n \to \infty} a(\tau_n, s) M(s) = a(t_i, s) M(s) = \varphi(s).$$

According to condition (A3), $\int_{t_0}^{t_0+\sigma} a(\tau_n, s) M(s)\, \mathrm{d}g(s)$ exists for all $n \in \mathbb{N}$. Using this fact together with (4.4), we get $\int_{t_0}^{t_0+\sigma} \varphi_n(s)\, \mathrm{d}g(s)$ exists for all $n \in \mathbb{N}$.

On the other hand, for all $t \in [t_0, t_0 + \sigma]$, $n \in \mathbb{N}$, we have

$$|\varphi_n(t)| = |a(\tau_n, t) M(t)| \leqslant c\, |M(t)| = cM(t).$$

This implies that

$$\kappa(t) \leqslant \varphi_n(t) \leqslant \omega(t), \qquad t \in [t_0, t_0 + \sigma],$$

where $\omega(t) := cM(t)$ and $\kappa(t) = -cM(t)$. Also, observe that the integrals $\int_{t_0}^{t_0+\sigma} \kappa(s)\,\mathrm{d}g(s)$ and $\int_{t_0}^{t_0+\sigma} \omega(s)\,\mathrm{d}g(s)$ exist, since M is a Henstock–Kurzweil–Stieltjes integrable function. Since all the hypotheses of Theorem 2.7 are satisfied, we obtain

$$\lim_{n\to\infty} \int_{t_0}^{t_0+\sigma} \varphi_n(s)\,\mathrm{d}g(s) = \int_{t_0}^{t_0+\sigma} \varphi(s)\,\mathrm{d}g(s).$$

Hence, the function v_2 is continuous at t_i, for each $i = 1, \dots, m$, proving Assertion 1.

From these facts and by the equality $v(t) = v_1(t) + v_2(t)$, it follow that v is continuous at $t_1, \dots, t_m$.

In the next result, we describe how we can translate the conditions on impulsive Volterra–Stieltjes integral equation to the conditions on Volterra–Stieltjes integral equations. It will be very important in order to prove results for impulsive Volterra–Stieltjes integral equations using known results for Volterra–Stieltjes integral equations.

Lemma 4.2. *Let $m \in \mathbb{N}$, $t_0 \leqslant t_1 < \dots < t_m < t_0 + \sigma$, $D = \{t_0, \dots, t_m\}$, $I_k : \mathbb{R} \to \mathbb{R}$ for $k \in \{1, \dots, m\}$ and let $a : [t_0, t_0+\sigma]^2 \to \mathbb{R}$, $f : \mathbb{R} \times [t_0, t_0+\sigma] \to \mathbb{R}$ and $g : [t_0, t_0+\sigma] \to \mathbb{R}$ satisfy conditions (A1)–(A5). Define*

$$\tilde{a}(t,s) = \begin{cases} a(t,s), & t \in [t_0, t_0+\sigma] \text{ and } s \in [t_0, t_0+\sigma]\backslash D, \\ 1, & t \in [t_0, t_0+\sigma] \text{ and } s \in D, \end{cases} \tag{4.5}$$

$$\tilde{f}(x,s) = \begin{cases} f(x,s), & \text{for } x \in \mathbb{R} \text{ and } s \in [t_0, t_0+\sigma]\backslash D, \\ I_k(x), & \text{for } x \in \mathbb{R} \text{ and } s \in D, \end{cases} \tag{4.6}$$

$$\tilde{g}(s) = \begin{cases} g(\tau), & \text{for } s \in [t_0, t_1], \\ g(s) + k, & \text{for } s \in (t_k, t_{k+1}] \text{ and } k \in \{1, \dots, m-1\}, \\ g(s) + m, & \text{for } s \in (t_m, t_0+\sigma]. \end{cases} \tag{4.7}$$

Also, suppose that $I_1, \dots, I_m : \mathbb{R} \to \mathbb{R}$ satisfy the following condition:

(I) There exists constants $M_2, L_2 > 0$ such that

$$|I_k(x)| \leqslant M_2$$

for every $k \in \{1, \dots, m\}$ and $x \in \mathbb{R}$, and

$$|I_k(x) - I_k(y)| \leqslant L_2\,|x - y|$$

for every $k \in \{1, \dots, m\}$ and $x, y \in \mathbb{R}$.

Then the functions $\tilde{a} : [t_0, t_0+\sigma]^2 \to \mathbb{R}$, $\tilde{f} : \mathbb{R} \times [t_0, t_0+\sigma] \to \mathbb{R}$ and $\tilde{g} : [t_0, t_0+\sigma] \to \mathbb{R}$ also satisfy conditions (A1)–(A5) with $\tilde{a}, \tilde{f}, \tilde{g}$ respectively in the place of a, f, g.

Proof. Since g is nondecreasing and left-continuous, $\tilde{g}$ has the same properties by the definition, proving condition (A1). The condition (A2) is an immediate consequence from the definition of $\tilde{a}$.

Notice that (A3) follows by combining the condition (A1) and the hypotheses from $\tilde{f}$ and $\tilde{a}$ together with Lemma 2.10.

To prove the condition (A4), let $x \in G([t_0, t_0+\sigma], \mathbb{R})$, $c_1, c_2 \in \mathbb{R}$, $[u_1, u_2] \subset [t_0, t_0+\sigma]$ and $b_{u_2,u_1}(s) := c_1 a(u_2, s) + c_2 a(u_1, s)$. From Lemma 2.10, we obtain

$$\begin{aligned}\int_{u_1}^{u_2} b_{u_2,u_1}(s)\tilde{f}(x(s),s)\,\mathrm{d}\tilde{g}(s) &= \int_{u_1}^{u_2} b_{u_2,u_1}(s) f(x(s),s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} b_{u_2,u_1}(t_k)\tilde{f}(x(t_k),t_k)\Delta^+\tilde{g}(t_k)\\ &= \int_{u_1}^{u_2} b_{u_2,u_1}(s) f(x(s),s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} b_{u_2,u_1}(t_k) I_k(x(t_k))\Delta^+\tilde{g}(t_k)\end{aligned}$$

and, therefore,

$$\left|\int_{u_1}^{u_2} b_{u_2,u_1}(s)\tilde{f}(x(s),s)\,\mathrm{d}\tilde{g}(s)\right| \leqslant \int_{u_1}^{u_2} M_1(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} M_2\,|b_{u_2,u_1}(t_k)|\,\Delta^+\tilde{g}(t_k) \tag{4.8}$$

On the other hand, notice that $\tilde{g}(v) - \tilde{g}(u) \geqslant g(v) - g(u)$ whenever $t_0 \leqslant u \leqslant v \leqslant t_0 + \sigma$. It implies together with the definition of the Henstock–Kurzweil–Stieltjes integral and Theorem 2.2 the following

$$\int_{u_1}^{u_2} M_1(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}g(s) \leqslant \int_{u_1}^{u_2} M_1(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}\tilde{g}(s) \leqslant \int_{u_1}^{u_2} \tilde{M}(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}\tilde{g}(s), \tag{4.9}$$

where $\tilde{M}(s) := 1 + M_2 + M_1(s)$ for all $s \in [t_0, t_0+\sigma]$. On the other hand, the function

$$h(t) := \int_{t_0}^{t} \tilde{M}(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}\tilde{g}(s), \qquad t \in [t_0, t_0+\sigma],$$

is nondecreasing and $\Delta^+ h(t_k) = \tilde{M}(t_k)\,|b_{u_2,u_1}(t_k)|\,\Delta^+\tilde{g}(t_k)$ for $k \in \{1, \dots, m\}$ by Theorem 2.5. Hence

$$\sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} M_2\,|b_{u_2,u_1}(t_k)|\,\Delta^+\tilde{g}(t_k) \leqslant \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} \tilde{M}(t_k)\,|b_{u_2,u_1}(t_k)|\,\Delta^+\tilde{g}(t_k) \leqslant h(u_2) - h(u_1).$$

Hence,

$$\sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} M_2\,|b_{u_2,u_1}(t_k)|\,\Delta^+\tilde{g}(t_k) \leqslant \int_{u_1}^{u_2} \tilde{M}(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}\tilde{g}(s) \tag{4.10}$$

Now, by (4.8), (4.9) and (4.10), we get

$$\left|\int_{u_1}^{u_2} b_{u_2,u_1}(s)\tilde{f}(x(s),s)\,\mathrm{d}\tilde{g}(s)\right| \leqslant 2\int_{u_1}^{u_2} \tilde{M}(s)\,|b_{u_2,u_1}(s)|\,\mathrm{d}\tilde{g}(s). \tag{4.11}$$

Now, defining $M(t) = 2\tilde{M}(t)$ for all $t \in [t_0, t_0+\sigma]$, we get the statement (A4).

To prove the condition (A5), consider $x, z \in G([t_0, t_0+\sigma], \mathbb{R})$ and $[u_1, u_2] \subset [t_0, t_0+\sigma]$. Using Lemma 2.10 again, we obtain

$$\begin{aligned}&\int_{u_1}^{u_2} a(u_2,s)\left(\tilde{f}(x(s),s) - \tilde{f}(z(s),s)\right)\mathrm{d}\tilde{g}(s)\\ &\quad= \int_{u_1}^{u_2} a(u_2,s)\left(f(x(s),s) - f(z(s),s)\right)\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} a(u_2,t_k)(I_k(x(t_k)) - I_k(z(t_k)))\Delta^+\tilde{g}(t_k).\end{aligned}$$

Consequently,

$$\left|\int_{u_1}^{u_2} a(u_2,s)\,\big(\tilde f(x(s),s)-\tilde f(z(s),s)\big)\,\mathrm{d}\tilde g(s)\right|$$
$$\leqslant \int\limits_{u_1}^{u_2} L_1(s)\,|a(u_2,s)|\,|x(s)-z(s)|\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} L_2\,|a(u_2,t_k)|\,|x(t_k)-z(t_k)|\Delta^+\tilde g(t_k).$$

Therefore,

$$\int\limits_{u_1}^{u_2} L_1(s)\,|a(u_2,s)|\,|x(s)-z(s)|\mathrm{d}g(s) \leqslant \int\limits_{u_1}^{u_2} \tilde L(s)\,|a(u_2,s)|\,|x(s)-z(s)|\mathrm{d}\tilde g(s),$$

where $\tilde L(s) = 1 + L_2 + L_1(s)$ for all $s\in[t_0,t_0+\sigma]$. Next, we observe that the function

$$\gamma(t) = \int_{t_0}^{t} \tilde L(s)\,|a(u_2,s)|\,|x(s)-z(s)|\,\mathrm{d}\tilde g(s), \qquad t\in[t_0,t_0+\sigma],$$

is nondecreasing and

$$\Delta^+\gamma(t_k) = \tilde L(t_k)\,|a(u_2,t_k)|\,|x(t_k)-z(t_k)|\Delta^+\tilde g(t_k),$$

for $k\in\{1,\dots,m\}$. Hence,

$$\begin{aligned}\sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} L_2\,|a(u_2,t_k)|\,|x(t_k)-z(t_k)|\Delta^+\tilde g(t_k) &\leqslant \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} \tilde L(t_k)\,|a(u_2,t_k)|\,|x(t_k)-z(t_k)|\Delta^+\tilde g(t_k)\\ &\leqslant \gamma(u_2)-\gamma(u_1).\end{aligned}$$

It follows that

$$\left|\int_{u_1}^{u_2} a(u_2,s)\,\big(\tilde f(x(s),s)-\tilde f(z(s),s)\big)\,\mathrm{d}\tilde g(s)\right| \leqslant 2\int_{u_1}^{u_2} \tilde L(s)\,|a(u_2,s)|\,|x(s)-z(s)|\,\mathrm{d}\tilde g(t).$$

Now, defining $L(t) = 2\tilde L(t)$ for all $t\in[t_0,t_0+\sigma]$, we get the desired result. □

The following theorem describes a strong relation between the solutions of impulsive Volterra–Stieltjes integral equations and the solutions of Volterra–Stieltjes integral equations without impulses. We can omit its proof as it follows by arguments analogous to those used in [12] to prove Theorem 3.1.

Theorem 4.3. *Let $m\in\mathbb{N}$, $t_0\leqslant t_1<\dots<t_m<t_0+\sigma$, $D=\{t_0,\dots,t_m\}$, $I_k:\mathbb{R}\to\mathbb{R}$ for $k\in\{1,\dots,m\}$ and $f:\mathbb{R}\times[t_0,t_0+\sigma]\to\mathbb{R}$. Assume that $g:[t_0,t_0+\sigma]\to\mathbb{R}$ satisfies the condition (A1) and $a:[t_0,t_0+\sigma]^2\to\mathbb{R}$ satisfies condition (A2). Furthermore, assume that g and $a(\cdot,s)$, $s\in[t_0,t_0+\sigma]$, are continuous at each $\tau\in D$. Consider the functions $\tilde a:[t_0,t_0+\sigma]^2\to\mathbb{R}$, $\tilde f:\mathbb{R}\times[t_0,t_0+\sigma]\to\mathbb{R}$ and $\tilde g:[t_0,t_0+\sigma]\to\mathbb{R}$ defined in Lemma 4.2, given by* (4.5), (4.6) *and* (4.7) *respectively.*

Then $x:[t_0,t_0+\sigma]\to\mathbb{R}$ is a solution of

$$x(t) = x_0 + \int_{t_0}^{t} a(t,s)f(x(s),s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k(x(t_k)), \tag{4.12}$$

if and only if $x:[t_0,t_0+\sigma]\to\mathbb{R}$ is a solution of

$$x(t) = x_0 + \int_{t_0}^{t} \tilde a(t,s)\tilde f(x(s),s)\,\mathrm{d}\tilde g(s). \tag{4.13}$$

As an immediate consequence, we obtain a result about existence and uniqueness of solutions of impulsive Volterra–Stieltjes integral equation. We omit its proof, since it follows directly from the correspondence and the analogue result for Volterra–Stieltjes integral equation.

Theorem 4.4. *Let $m \in \mathbb{N}$, $t_0 \leqslant t_1 < \cdots < t_m < t_0 + \sigma$, $D = \{t_0, \ldots, t_m\}$, $I_k : \mathbb{R} \to \mathbb{R}$ for $k \in \{1, \ldots, m\}$ and let $a : [t_0, t_0 + \sigma]^2 \to \mathbb{R}$, $f : \mathbb{R} \times [t_0, t_0 + \sigma] \to \mathbb{R}$ and $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfy conditions (A1)–(A5). Furthermore, assume that g and $a(\cdot, s)$, $s \in [t_0, t_0 + \sigma]$, are continuous at each $\tau \in D$. Also, suppose that $I_1, \ldots, I_m : \mathbb{R} \to \mathbb{R}$ satisfies condition (I) from Lemma 4.2.*

Then there exists a unique solution $x : [t_0, t_0 + \sigma] \to \mathbb{R}$ of the impulsive Volterra–Stieltjes integral equation

$$x(t) = x_0 + \int_{t_0}^{t} a(t,s) f(x(s), s)\, \mathrm{d}g(s) + \sum_{\substack{k \in \{1,\ldots,m\},\\ t_k < t}} I_k(x(t_k)). \tag{4.14}$$

Acknowledgements

E. Alvarez was partially supported by DIDI, Project 2016-011. R. Grau was supported by DIDI, Project 2018-28. C. Lizama was partially supported by ANID under FONDECYT grant 1180041. J. G. Mesquita was partially supported by CNPq 407952/2016-0, FAPDF grant 0193.001300/2016 and FEMAT–Fundação de Estudos em Ciências Matemáticas Proc. 042/2018.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 6, 1–56; https://doi.org/10.14232/ejqtde.2021.1.6 www.math.u-szeged.hu/ejqtde/

Geometry and integrability of quadratic systems with invariant hyperbolas

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Received 15 July 2020, appeared 18 January 2021
Communicated by Gabriele Villari

Abstract. Let **QSH** be the family of non-degenerate planar quadratic differential systems possessing an invariant hyperbola. We study this class from the viewpoint of integrability. This is a rich family with a variety of integrable systems with either polynomial, rational, Darboux or more general Liouvillian first integrals as well as non-integrable systems. We are interested in studying the integrable systems in this family from the topological, dynamical and algebraic geometric viewpoints. In this work we perform this study for three of the normal forms of **QSH**, construct their topological bifurcation diagrams as well as the bifurcation diagrams of their configurations of invariant hyperbolas and lines and point out the relationship between them. We show that all systems in one of the three families have a rational first integral. For another one of the three families, we give a global answer to the problem of Poincaré by producing a geometric necessary and sufficient condition for a system in this family to have a rational first integral. Our analysis led us to raise some questions in the last section, relating the geometry of the invariant algebraic curves (lines and hyperbolas) in the systems and the expression of the corresponding integrating factors.

Keywords: quadratic differential systems, invariant algebraic curves, invariant hyperbola, Darboux integrability, Liouvillian integrability.

2020 Mathematics Subject Classification: 34A05, 34C05, 34C45.

1 Introduction

Let $\mathbb{R}[x,y]$ be the set of all real polynomials in the variables x and y. Consider the planar system

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1.1}$$

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where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$ and $P, Q \in \mathbb{R}[x,y]$. We call the degree of system (1.1) the integer $\max\{\deg P, \deg Q\}$. In the case when the polynomial P and Q are relatively prime i. e. they do not have a non-constant common factor, we say that (1.1) is *non-degenerate.*

Consider

$$\chi = P(x,y)\,\frac{\partial}{\partial x} + Q(x,y)\,\frac{\partial}{\partial y} \tag{1.2}$$

the polynomial vector field associated to (1.1).

A real *quadratic differential system* is a polynomial differential system of degree 2, i.e.

$$\begin{aligned} \dot{x} &= p_0 + p_1(\tilde{a},x,y) + p_2(\tilde{a},x,y) \equiv p(\tilde{a},x,y), \\ \dot{y} &= q_0 + q_1(\tilde{a},x,y) + q_2(\tilde{a},x,y) \equiv q(\tilde{a},x,y) \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} &p_0 = a, \quad && p_1(\tilde{a},x,y) = cx + dy, \quad && p_2(\tilde{a},x,y) = gx^2 + 2hxy + ky^2, \\ &q_0 = b, \quad && q_1(\tilde{a},x,y) = ex + fy, \quad && q_2(\tilde{a},x,y) = lx^2 + 2mxy + ny^2. \end{aligned}$$

Here we denote by $\tilde{a} = (a,c,d,g,h,k,b,e,f,l,m,n)$ the 12-tuple of the coefficients of system (1.3). Thus a quadratic system can be identified with a point $\tilde{a}$ in $\mathbb{R}^{12}$.

We denote the class of all real quadratic differential systems with **QS**.

In this work we are interested in polynomial differential equations (1.1) which are endowed with an algebraic geometric structure, i.e. which posses invariant algebraic curves under the flow. We are interested both in their geometry and also in the impact this geometry has on the integrability of the systems.

Definition 1.1 ([11])**.** An algebraic curve $C(x,y) = 0$ with $C(x,y) \in \mathbb{C}[x,y]$ is called an *invariant algebraic curve* of system (1.1) if it satisfies the following identity:

$$C_x P + C_y Q = KC, \tag{1.4}$$

for some $K \in \mathbb{C}[x,y]$ where C_x and C_y are the derivative of C with respect to x and y. K is called the *cofactor* of the curve $C = 0$.

For simplicity we write the curve C instead of the curve $C = 0$ in $\mathbb{C}^2$. Note that if system (1.1) has degree m then the cofactor of an invariant algebraic curve C of the system has degree $m - 1$.

Definition 1.2. Let U be an open subset of $\mathbb{R}^2$. A real function H: $U \to \mathbb{R}$ is a *first integral* of system (1.1) if it is constant on all solution curves $(x(t), y(t))$ of system (1.1), i.e., $H(x(t), y(t)) = k$, where k is a real constant, for all values of t for which the solution $(x(t), y(t))$ is defined on U.

If H is differentiable in U then H is a first integral on U if and only if

$$H_x P + H_y Q = 0. \tag{1.5}$$

Definition 1.3. If a system (1.1) has a first integral of the form

$$H(x,y) = {C_1}^{\lambda_1} \cdots {C_p}^{\lambda_p} \tag{1.6}$$

where C_i are invariant algebraic curves of system (1.1) and $\lambda_i \in \mathbb{C}$ then we say that system (1.1) is *Darboux integrable* and we call the function H a *Darboux function*.

Theorem 1.4 ([11]). *Suppose that a polynomial system* (1.1) *has* m *invariant algebraic curves* $C_i(x,y) = 0$, $i \leq m$, *with* $C_i \in \mathbb{C}[x,y]$ *and with* $m > n(n+1)/2$ *where* n *is the degree of the system. Then there exist complex numbers* $\lambda_1, \ldots, \lambda_m$ *such that* $C_1^{\lambda_1} \ldots C_m^{\lambda_m}$ *is a first integral of the system.*

If a system (1.1) admits a rational first integral we say that (1.1) is *algebraically integrable*. Poincaré was enthustiastic about the work of Darboux [11] which he called "oeuvre magistrale" in [22] and stated *the problem of algebraic integrability* which asks to recognize when a polynomial vector field has a rational first integral. Jouanolou gave a sufficient condition for recognizing that a polynomials system has a rational first integral.

Theorem 1.5 ([15]). *Consider a polynomial system* (1.1) *of degree* n *and suppose that it admits* m *invariant algebraic curves* $C_i(x,y) = 0$ *where* $1 \leq i \leq m$, *then if* $m \geq 2 + \frac{n(n+1)}{2}$, *there exists integers* $N_1, N_2, \ldots, N_m$ *such that* $I(x,y) = \prod_{i=1}^{m} C_i^{N_i}$ *is a first integral of* (1.1).

In connection to this problem Poincaré stated a number of definitions among them the following definitions below.

Let $H = f/g$ be a rational first integral of the polynomial vector field (1.2). We say that H has degree n if n is the maximum of the degrees of f and g. We say that the degree of H is *minimal* among all the degrees of the rational first integrals of χ if any other rational first integral of χ has a degree greater than or equal to n. Let $H = f/g$ be a rational first integral of χ. According to Poincaré [22] we say that $c \in \mathbb{C} \cup \{\infty\}$ is a *remarkable value* of H if $f + cg$ is a reducible polynomial in $\mathbb{C}[x,y]$. Here, if $c = \infty$, then $f + cg$ denotes g. Note that for all $c \in \mathbb{C}$ the algebraic curve $f + cg = 0$ is invariant. The curves in the factorization of $f + cg$, when c is a remarkable value, are called *remarkable curves*.

Now suppose that c is a remarkable value of a rational first integral H and that $u_1^{\alpha_1} \cdots u_r^{\alpha}$ is the factorization of the polynomial $f + cg$ into reducible factors in $\mathbb{C}[x,y]$. If at least one of the α_i is larger than 1 then we say, following again Poincaré (see for instance [14]), that c is a *critical remarkable value* of H, and that $u_i = 0$ having $\alpha_i > 1$ is a *critical remarkable curve* of the vector field (1.2) with exponent α_i.

Since we can think of $c \in \mathbb{C} \cup \{\infty\}$ as the projective line $P_1(\mathbb{R})$ we can also use the following definition.

Definition 1.6. Consider $\mathcal{F}_{(c_1,c_2)} : c_1 f - c_2 g = 0$ where f/g is a rational first integral of (1.2). We say that $[c_1 : c_2]$ is a remarkable value of the curve $\mathcal{F}_{(c_1,c_2)}$ if $\mathcal{F}_{(c_1,c_2)}$ is reducible over $\mathbb{C}$.

It is proved in [4] that there are finitely many remarkable values for a given rational first integral H and if (1.2) has a rational first integral and has no polynomial first integrals, then it has a polynomial inverse integrating factor if and only if the first integral has at most two critical remarkable values.

Given $H = f/g$ a rational first integral, consider $F_{(c_1,c_2)} = c_1 f - c_2 g$ where $\deg F_{(c_1,c_2)} = n$. If $F_{(c_1,c_2)} = f_1 f_2$ where $\deg f_i = n_i < n$ then necessarily the points on the intersection of $f_1 = 0$ and $f_2 = 0$ must be singular points of the curve $F_{(c_1,c_2)}$. So to find the irreducible factors of $F_{(c_1,c_2)}$ we start by finding the singularities of $F_{(c_1,c_2)}$, i.e., the points on the curve which annihilate both first derivatives in x and y.

The following notion was defined by Christopher in [5] where he called it "degenerate invariant algebraic curve".

Definition 1.7. Let $F(x,y) = \exp\left(\frac{G(x,y)}{H(x,y)}\right)$ with $G, H \in \mathbb{C}[x,y]$ coprime. We say that F is an *exponential factor* of system (1.1) if it satisfies the equality

$$F_x P + F_y Q = LF, \tag{1.7}$$

for some $L \in \mathbb{C}[x,y]$. The polynomial L is called the *cofactor* of the exponential factor F.

Definition 1.8. If system (1.1) has a first integral of the form

$$H(x,y) = {C_1}^{\lambda_1} \cdots {C_p}^{\lambda_p} {F_1}^{\mu_1} \cdots {F_q}^{\mu_q} \tag{1.8}$$

where C_i and F_j are the invariant algebraic curves and exponential factors of system (1.1) respectively and $\lambda_i, \mu_j \in \mathbb{C}$, then we say that the system is *generalized Darboux integrable*. We call the function H a *generalized Darboux function*.

Remark 1.9. In [11] Darboux considered functions of the type (1.6), not of type (1.8). In recent works functions of type (1.8) were called Darboux functions. Since in this work we need to pay attention to the distinctions among the various kinds of first integral we call (1.6) a Darboux and (1.8) a generalized Darboux first integral.

Definition 1.10. Let U be an open subset of $\mathbb{R}^2$ and let $R : U \to \mathbb{R}$ be an analytic function which is not identically zero on U. The function R is an integrating factor of a polynomial system (1.1) on U if one of the following two equivalent conditions holds:

$$\operatorname{div}(RP, RQ) = 0, \qquad R_x P + R_y Q = -R \operatorname{div}(P, Q), \tag{1.9}$$

on U.

A first integral H of

$$\dot{x} = RP, \qquad \dot{y} = RQ$$

associated to the integrating factor R is given by

$$H(x,y) = \int R(x,y)P(x,y)dy + h(x),$$

where $H(x,y)$ is a function satisfying $H_x = -RQ$. Then,

$$\dot{x} = H_y, \qquad \dot{y} = -H_x.$$

In order that this function H be well defined the open set U must be simply connected.

Liouvillian functions are functions that are built up from rational functions using exponentiation, integration, and algebraic functions. For more details on Liouvillian functions, see [7].

Theorem 1.11 ([4, 21])**.** *If a planar polynomial vector field* (1.2) *has a generalized Darboux first integral, then it has a rational integrating factor.*

As for a converse, we have the following result which easily follows from [23].

Theorem 1.12 ([8])**.** *If a planar polynomial vector field* (1.2) *has a rational integrating factor, then it has a generalized Darboux first integral.*

An important consequence of Singer's theorem (see [27]) is the following.

Theorem 1.13 ([5, 27])**.** *A planar polynomial differential system* (1.1) *has a Liouvillian first integral if and only if it has a generalized Darboux integrating factor.*

For a proof see [28, p. 134].

We have the following table summing up these results.

First integral		Integrating factor
Generalized Darboux	$\Leftrightarrow$	Rational
Liouvillian	$\Leftrightarrow$	Generalized Darboux

Definition 1.14 ([11])**.** Consider a planar polynomial system (1.1). An algebraic solution $f = 0$ of (1.1) is an algebraic invariant curve which is irreducible over $\mathbb{C}$.

Theorem 1.15 ([6])**.** *Consider a polynomial system* (1.1) *that has k algebraic solutions $C_i = 0$ such that*

(a) all curves $C_i = 0$ are non-singular and have no repeated factor in their highest order terms,

(b) no more than two curves meet at any point in the finite plane and are not tangent at these points,

(c) no two curves have a common factor in their highest order terms,

(d) the sum of the degrees of the curves is $n + 1$, where n is the degree of system (1.1)*.*

Then system (1.1) *has an integrating factor*

$$\mu(x, y) = 1/(C_1 C_2 \cdots C_k).$$

This result of Christopher–Kooij (C–K) is interesting because it relates the geometry of the configuration of invariant algebraic curves of the systems with the expression of the integrating factors involving the polynomials defining the curves. In fact this theorem has a geometric content which is however not completely explicit in the algebraic way their theorem is stated. We restate the above result in geometric terms as follows:

Theorem 1.16. *Consider a polynomial system* (1.1) *that has k algebraic solutions $C_i = 0$ such that*

(a) all curves $C_i = 0$ are non-singular and they intersects transversally the line at infinity $Z = 0$,

(b) no more than two curves meet at any point in the finite plane and are not tangent at these points,

(c) no two curves intersect at a point on the line at infinity $Z = 0$,

(d) the sum of the degrees of the curves is $n + 1$, where n is the degree of system (1.1)*.*

Then system (1.1) *has an integrating factor*

$$\mu(x, y) = 1/(C_1 C_2 \cdots C_k).$$

In the hypotheses of this theorem the way the curves are placed with respect to one another in the totality of the curves, in other words the "geometry of the configuration of invariant algebraic curves" has an impact of the kind of integrating factor we could have. One of our goals is to collect data so as to extend this theorem beyond these limiting geometric conditions.

There are some important invariant polynomials in the study of polynomial vector fields. Considering $C_2(\tilde{a}, x, y) = yp_2(\tilde{a}, x, y) - xq_2(\tilde{a}, x, y)$ as a cubic binary form of x and y we calculate

$$\eta(\tilde{a}) = \text{Discrim}[C_2, \xi], \quad M(\tilde{a}, x, y) = \text{Hessian}[C_2],$$

where $\xi = y/x$ or $\xi = x/y$. It is known that the singular points at infinity of quadratic systems are given by the solutions in x and y of $C_2(\tilde{a}, x, y) = 0$. If $\eta < 0$ then this means we have one real singular point at infinity and two complex.

Remark 1.17. We note that since a system in **QSH** always has an invariant hyperbola then clearly we always have at least 2 real singular points at infinity. So we must have $\eta \geq 0$.

The family **QSH** can be split as follows: $\mathbf{QSH}_{(\eta=0)}$ of systems which possess either exactly two distinct real singularities at infinity or the line at infinity filled up with singularities and $\mathbf{QSH}_{(\eta>0)}$ of systems which possess three distinct real singularities at infinity in $P_2(\mathbb{C})$.

In [18] the authors proved that there are 162 distinct configurations and provided necessary and sufficient conditions for a non-degenerate quadratic differential system to have at least one invariant hyperbola and for the realization of each one of the configurations. These conditions are expressed in terms of the coefficients of the systems. They obtained the normal forms for family **QSH** and in this paper we study the following 3 normal forms:

$$\begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 4a - 3v^2 - \dfrac{4xy}{3} + \dfrac{y^2}{3}, \quad \text{where } a \neq 0. \end{cases} \tag{1.10}$$

$$\begin{cases} \dot{x} = -\dfrac{x^2}{2} - \dfrac{xy}{2} \\ \dot{y} = b - \dfrac{3xy}{2} + \dfrac{y^2}{2}, \quad \text{where } b \neq 0. \end{cases} \tag{1.11}$$

$$\begin{cases} \dot{x} = 2a + gx^2 + xy, \\ \dot{y} = a(2g-1) + (g-1)xy + y^2, \quad \text{where } a(g-1) \neq 0. \end{cases} \tag{1.12}$$

Our first goal in this paper is to do a complete study of these three families of quadratic systems which possess an invariant hyperbola. Our interest is in the geometry of these systems, as expressed in terms of their invariant algebraic curves, in the impact of this geometry on the integrability of these systems, on their phase portraits and in the dynamics of the systems expressed in the bifurcation diagrams of the families we study. Our third goal is to confront our results with the existing results in the literature and bring to light some missing cases in theses other studies which we point out here. Our geometric analysis is done in detail as this is part of a program of collecting data in order to obtain more global results on the family **QSH** and its Darboux theory.

Our paper is organized as follows: in Section 2 we give a number of definitions and propositions useful for the other sections. In Sections 3, 4, 5 we present a complete study of families (1.10), (1.11) and (1.12). The choice of the first two families is motivated by the fact that they do not satisfy all the conditions in the hypothesis of the Christopher–Kooij theorem, here stated in theorem 1.15, but the conclusion of the theorem still holds, while the last family does not always posses a first integral and it will provide a counterpoint. In Section 6 we raise some questions, consider the problem of Poincaré for the family **QSH**, and make some concluding comments.

2 Preliminaries

The notion of *configuration of invariant curves* of a polynomial differential system appears in several works, see for instance [26].

Definition 2.1. Consider a real planar polynomial system (1.1) with a finite number of singular points. By a *configuration of algebraic solutions* of the system we mean a set of algebraic solutions over $\mathbb{C}$ of the system, each one of these curves endowed with its own multiplicity and together with all the real singular points of this system located on these curves, each one of these singularities endowed with its own multiplicity.

Definition 2.2. Suppose we have two systems (S_1), (S_2) in **QSH** with a finite number of singularities, finite or infinite, a finite set of invariant hyperbolas $\mathcal{H}_i^1 : h_i^1(x,y) = 0,\ i = 1,\dots,k$ of (S_1) (respectively $\mathcal{H}_i^2 : h_i^2(x,y) = 0,\ i = 1,\dots,k$ of (S_2)) and a finite set (which could also be empty) of invariant straight lines $\mathsf{L}_j^1 : g_j^1(x,y) = 0,\ j = 1,\dots,k'$ of (S_1) (respectively $\mathsf{L}_j^2 : g_j^2(x,y) = 0,\ j = 1,\dots,k'$ of (S_2)). We say that the two configurations C_1, C_2 of hyperbolas and lines of these systems are *equivalent* if there is a one-to-one correspondence Φ_h between the hyperbolas of C_1 and C_2 and a one-to-one correspondence Φ_l between the lines of C_1 and C_2 such that:

- (i) the correspondences conserve the multiplicities of the hyperbolas and lines (in case there are any) and also send a real invariant curve to a real invariant curve and a complex invariant curve to a complex invariant curve;
- (ii) for each hyperbola $\mathcal{H} : h(x,y) = 0$ of C_1 (respectively each line $\mathsf{L} : g(x,y) = 0$) we have a one-to-one correspondence between the real singular points on $\mathcal{H}$ (respectively on L) and the real singular points on $\Phi_h(\mathcal{H})$ (respectively $\Phi_l(\mathsf{L})$) conserving their multiplicities, their location on branches of hyperbolas and their order on these branches (respectively on the lines);
- (iii) Furthermore, consider the total curves $\mathcal{F}^1 : \prod H_i^1(X,Y,Z) \prod G_j^1(X,Y,Z) Z = 0$ (respectively $\mathcal{F}^2 : \prod H_i^2(X,Y,Z) \prod G_j^2(X,Y,Z) Z = 0$) where $H_i^1(X,Y,Z) = 0,\ G_j^1(X,Y,Z) = 0$ (respectively $H_i^2(X,Y,Z) = 0,\ G_j^2(X,Y,Z) = 0$) are the projective completions of $\mathcal{H}_i^1$, L_j^1 (respectively $\mathcal{H}_i^2$, L_j^2). Then, there is a one-to-one correspondence ψ between the singularities of the curves $\mathcal{F}^1$ and $\mathcal{F}^2$ conserving their multiplicities as singular points of these (total) curves.

It is important to assume that systems (1.3) are non-degenerate because otherwise doing a time rescaling, they can be reduced to linear or constant systems. Under this assumption all the systems in **QSH** have a finite number of finite singular points.

In the family **QSH** we also have cases where we have an infinite number of hyperbolas. In these cases, by a Jouanolou result (see Theorem 1.5 on page 3), we have a rational first integral.

In [18] the authors classified the family **QSH**, according to their geometric properties encoded in the configurations of invariant hyperbolas and invariant straight lines which these systems possess. If a quadratic system has an infinite number of hyperbolas then the system has a finite number of invariant affine straight lines (see [1]). Therefore, we can talk about *equivalence of configurations of the invariant affine lines associated to the system*. Given two such configurations C_{1l} and C_{2l} associated to systems (S_1) and (S_2) of (1.1), we say they are *equivalent* if and only if there is a one-to-one correspondence Φ between the lines of C_{1l} and C_{2l} such that:

(i) the correspondence preserve the multiplicities of the lines and also sends a real (respectively complex) invariant line to a real (respectively complex) invariant line;

(ii) for each line $\mathsf{L} : g(x,y) = 0$ we have a one-to-one correspondence between the real singularities on L and the real singularities on Φ preserving their multiplicities and their order on the lines.

Definition 2.3 ([18]). Consider two systems (S_1) and (S_2) in **QSH** each one with an infinite number of invariant hyperbolas. Consider the configurations C_{1l} and C_{2l} of invariant affine straight lines $\mathsf{L}_j^1 : g_j^1(x,y) = 0$ where $j = 1,2,\ldots,k$ of system (S_1) and respectively $\mathsf{L}_j^2 : g_j^2(x,y) = 0$ where $j = 1,2,\ldots,k$ of system (S_2). We say that the two configurations C_{1l} and C_{2l} are equivalent with respect to the hyperbolas of the systems if and only if:

(i) they are equivalent as configurations of invariant lines, and

(ii) taking any hyperbola $\mathcal{H}_1 : h_1(x,y) = 0$ of (S_1) and any hyperbola $\mathcal{H}_2 : h_2(x,y) = 0$ of (S_2), then we must have a one-to-one correspondence between the real singularities of system (S_1) located on $\mathcal{H}_1$ and of real singularities of system (S_2) located on $\mathcal{H}_2$, preserving their multiplicities, their location and order on branches.

Furthermore, consider the curves $\mathcal{F}_1 : \prod h_1(x,y) \prod g_j^1 = 0$ and $\mathcal{F}_2 : \prod h_2(x,y) \prod g_j^2 = 0$. Then, we have a one-to-one correspondence between the singularities of the curve $\mathcal{F}_1$ with those in the curve $\mathcal{F}_2$ preserving their multiplicities as singularities of these curves.

The definition above is independent of the choice of the two hyperbolas $\mathcal{H}_1 : h_1(x,y) = 0$ of (S_1) and $\mathcal{H}_2 : h_2(x,y) = 0$ of (S_2).

Suppose that a polynomial differential system has an algebraic solution $f(x,y) = 0$ where $f(x,y) \in \mathbb{C}[x,y]$ is of degree n given by

$$f(x,y) = c_0 + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + \cdots + c_{n0}x^n + c_{n-1,1}x^{n-1}y + \cdots + c_{0n}y^n,$$

with $\hat{c} = (c_0, c_{10}, \ldots, c_{0n}) \in \mathbb{C}^N$ where $N = (n+1)(n+2)/2$. We note that the equation

$$\lambda f(x,y) = 0,\ \lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$$

yields the same locus of complex points in the plane as the locus induced by $f(x,y) = 0$. Therefore, a curve of degree n is defined by $\hat{c}$ where

$$[\hat{c}] = [c_0 : c_{10} : \cdots : c_{0n}] \in P_{N-1}(\mathbb{C}).$$

We say that a sequence of curves $f_i(x,y) = 0$, each one of degree n, *converges* to a curve $f(x,y) = 0$ if and only if the sequence of points $[c_i] = [c_{i0} : c_{i10} : \cdots : c_{i0n}]$ converges to $[\hat{c}] = [c_0 : c_{10} : \cdots : c_{0n}]$ in the topology of $P_{N-1}(\mathbb{C})$.

We observe that if we rescale the time $t' = \lambda t$ by a positive constant λ the geometry of the systems (1.1) (phase curves) does not change. So for our purposes we can identify a system (1.1) of degree n with a point

$$[a_0 : a_{10} : \cdots : a_{0n} : b_0 : b_{10} : \cdots : b_{0n}] \in \mathbb{S}^{N-1}(\mathbb{R})$$

where $N = (n+1)(n+2)$.

Definition 2.4.

(1) We say that an invariant curve

$$\mathcal{L} : f(x,y) = 0, \qquad f \in \mathbb{C}[x,y]$$

for a polynomial system (S) of degree n has geometric multiplicity m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}(\mathbb{R})$ where $N = (n+1)(n+2)$ such that each (S_k) has m distinct invariant curves

$$\mathcal{L}_{1,k} : f_{1,k}(x,y) = 0, \ldots, \mathcal{L}_{m,k} : f_{m,k}(x,y) = 0$$

over $\mathbb{C}$, $deg(f) = deg(f_{i,k}) = r$, converging to $\mathcal{L}$ as $k \to \infty$, in the topology of $P_{R-1}(\mathbb{C})$, with $R = (r+1)(r+2)/2$ and this does not occur for $m+1$.

(2) We say that the line at infinity

$$\mathcal{L}_\infty : Z = 0$$

of a polynomial system (S) of degree n has geometric multiplicity m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}(\mathbb{R})$ where $N = (n+1)(n+2)$ such that each (S_k) has $m-1$ distinct invariant lines

$$\mathcal{L}_{1,k} : f_{1,k}(x,y) = 0, \ldots, \mathcal{L}_{m-1,k} : f_{m-1,k}(x,y) = 0$$

over $\mathbb{C}$, converging to the line at infinity $\mathcal{L}_\infty$ as $k \to \infty$, in the topology of $P_2(\mathbb{C})$ and this does not occur for m.

Definition 2.5 ([9])**.** Let $\mathbb{C}_m[x,y]$ be the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x,y]$ of degree at most m and of dimension $R = \binom{2+m}{2}$. Let $\{v_1, v_2, \ldots, v_R\}$ be a base of $\mathbb{C}_m[x,y]$. We denote by $M_R(m)$ the $R \times R$ matrix

$$M_R(m) = \begin{pmatrix} v_1 & v_2 & \cdots & v_R \\ \chi(v_1) & \chi(v_2) & \cdots & \chi(v_R) \\ \vdots & \vdots & \ddots & \vdots \\ \chi^{R-1}(v_1) & \chi^{R-1}(v_2) & \cdots & \chi^{R-1}(v_R) \end{pmatrix}, \tag{2.1}$$

where $\chi^{k+1}(v_i) = \chi(\chi^k(v_i))$. The mth extactic curve of χ, $\mathcal{E}_m(\chi)$, is given by the equation $\det M_R(m) = 0$. We also call $\mathcal{E}_m(\chi)$ the mth extactic polynomial.

From the properties of the determinant we note that the extactic curve is independent of the choice of the base of $\mathbb{C}_m[x,y]$.

Theorem 2.6 ([20])**.** *Consider a planar vector field* (1.2)*. We have* $\mathcal{E}_m(\chi) = 0$ *and* $\mathcal{E}_{m-1}(\chi) \neq 0$ *if and only if* χ *admits a rational first integral of exact degree* m*.*

Observe that if $f = 0$ is an invariant algebraic curve of degree m of χ , then f divides $\mathcal{E}_m(\chi)$. This is due to the fact that if f is a member of a base of $\mathbb{C}_m[x,y]$, then f divides the whole column in which f is located.

Definition 2.7 ([9])**.** We say that an invariant algebraic curve $f = 0$ of degree $m \geq 1$ has algebraic multiplicity k if $\det M_R(m) \neq 0$ and k is the maximum positive integer such that f^k divides $\det M_R(m)$; and it has no defined algebraic multiplicity if $\det M_R(m) \equiv 0$.

Definition 2.8 ([9])**.** We say that an invariant algebraic curve $f = 0$ of degree $m \geq 1$ has integrable multiplicity k with respect to χ if k is the largest integer for which the following is true: there are $k-1$ exponential factors $\exp(g_j/f^j)$, $j = 1, \ldots, k-1$, with $\deg g_j \leq jm$, such that each g_j is not a multiple of f.

In the next result we see that the algebraic and integrable multiplicity coincide if $f = 0$ is an irreducible invariant algebraic curve.

Theorem 2.9 ([16])**.** *Consider an irreducible invariant algebraic curve $f = 0$ of degree $m \geq 1$ of χ. Then f has algebraic multiplicity k if and only if the vector field* (1.2) *has $k-1$ exponential factors $\exp(g_j/f^j)$, where $(g_j, f) = 1$ and g_j is a polynomial of degree at most jm, for $j = 1, \ldots, k-1$.*

In [9] the authors showed that the definitions of geometric, algebraic and integrable multiplicity are equivalent when $f = 0$ is an irreducible invariant algebraic curve of vector field (1.2).

In order to use the infinity of $\mathbb{R}^2$ as an additional invariant curve for studying the integrability of the vector field χ, we need the Poincaré compactification of the vector field χ. For $Z \neq 0$ consider the change of variables

$$x = \frac{1}{Z}, \qquad y = \frac{Y}{Z}$$

the vector field χ is transformed to

$$\overline{\chi} = -Z\,\overline{P}(Z,Y)\frac{\partial}{\partial Z} + \left(\overline{Q}(Z,Y) - Y\,\overline{P}(Z,Y)\right)\frac{\partial}{\partial Y}$$

where $\overline{P}(Z,Y) = Z^2 P\left(\frac{1}{Z}, \frac{Y}{Z}\right)$ and $\overline{Q}(Z,Y) = Z^2 Q\left(\frac{1}{Z}, \frac{Y}{Z}\right)$.

We note that $Z = 0$ is an invariant line of the vector field $\overline{\chi}$ and that the infinity of $\mathbb{R}^2$ corresponds to $Z = 0$ of the vector field $\overline{\chi}$. So we can define the algebraic multiplicity of $Z = 0$ for the vector field $\overline{\chi}$.

Definition 2.10. We say that the infinity of χ has algebraic multiplicity k if $Z = 0$ has algebraic multiplicity k for the vector field $\overline{\chi}$; and that it has no defined algebraic multiplicity if $Z = 0$ has no defined algebraic multiplicity for $\overline{\chi}$.

Let's recall the algebraic-geometric definition of an r-cycle on an irreducible algebraic variety of dimension n.

Definition 2.11. Let V be an irreducible algebraic variety of dimension n over a field $\mathbb{K}$. A cycle of dimension r or r-cycle on V is a formal sum

$$\sum_W n_W W$$

where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in \mathbb{Z}$, and only a finite number of n_W's are non-zero. We call degree of an r-cycle the sum

$$\sum_W n_W.$$

An $(n-1)$-cycle is called a divisor.

Definition 2.12. For a non-degenerate polynomial differential systems (S) possessing a finite number of algebraic solutions

$$\mathcal{F} = \{f_i\}_{i=1}^{m},\ f_i(x,y) = 0,\ f_i(x,y) \in \mathbb{C},$$

each with multiplicity n_i and a finite number of singularities at infinity, we define the algebraic solutions divisor (also called the invariant curves divisor) on the projective plane,

$$ICD_{\mathcal{F}} = \sum_{n_i} n_i \mathcal{C}_i + n_\infty \mathcal{L}_\infty$$

where $C_i : F_i(X,Y,Z) = 0$ are the projective completions of $f_i(x,y) = 0$, n_i is the multiplicity of the curve $C_i = 0$ and n_∞ is the multiplicity of the line at infinity $\mathcal{L}_\infty : Z = 0$.

It is well known (see [1]) that the maximum number of invariant straight lines, including the line at infinity, for polynomial systems of degree $n \geq 2$ is $3n$.

Proposition 2.13 ([1]). *Every quadratic differential system has at most six invariant straight lines, including the line at infinity.*

In the case we consider here, we have a particular instance of the divisor ICD because the invariant curves will be invariant hyperbolas and invariant lines of a quadratic differential system, in case these are in finite number. In case we have an infinite number of hyperbolas we can construct the divisor of the invariant straight lines which are always in finite number.

Another ingredient of the configuration of algebraic solutions are the real singularities situated on these curves. We also need to use here the notion of multiplicity divisor of real singularities of a system, located on the algebraic solutions of the system.

Definition 2.14.

1. Suppose a real quadratic system (1.3) has a non-zero finite number of invariant hyperbolas

   $$\mathcal{H}_i : h_i(x,y) = 0,\ i = 1,2,\dots,k$$

   and a finite number of affine invariant lines

   $$\mathcal{L}_j : f_j(x,y) = 0,\ j = 1,2,\dots,l.$$

   We denote the line at infinity $\mathcal{L}_\infty : Z = 0$. Let us assume that on the line at infinity we have a finite number of singularities. The divisor of invariant hyperbolas and invariant lines on the complex projective plane of the system is the following

   $$ICD = n_1\mathcal{H}_1 + \cdots + n_k\mathcal{H}_k + m_1\mathcal{L}_1 + \cdots + m_l\mathcal{L}_l + m_\infty\mathcal{L}_\infty$$

   where n_i (respectively m_j) is the multiplicity of the hyperbola $\mathcal{H}_i$ (respectively m_j of the line $\mathcal{L}_j$), and m_∞ is the multiplicity of $\mathcal{L}_\infty$. We also mark the complex (non-real) invariant hyperbolas (respectively lines) denoting them by $\mathcal{H}_i^C$ (respectively $\mathcal{L}_i^C$). We define the total multiplicity TM of the divisor as the sum $\sum_i n_i + \sum_j m_j + m_\infty$.

2. The zero-cycle on the real projective plane, of singularities of a quadratic system (1.3) located on the configuration of invariant lines and invariant hyperbolas, is given by

   $$M_{0CS} = r_1P_1 + \cdots + r_lP_l + v_1P_1^\infty + \cdots + v_nP_n^\infty$$

where P_i (respectively P_j^∞) are all the finite (respectively infinite) such singularities of the system and r_i (respectively v_j) are their corresponding multiplicities. We mark the complex singular points denoting them by P_i^C. We define the total multiplicity TM of zero-cycles as the sum $\sum_i r_i + \sum_j v_j$.

In the family QSH we have configurations which have an infinite number of hyperbolas. These are of two kinds: those with a finite number of singular points at infinity, and those with the line at infinity filled up with singularities. To distinguish these two cases we define $|\mathrm{Sing}_\infty|$ to be the cardinality of the set of singular points at infinity of the systems. In the first case we have $|\mathrm{Sing}_\infty| = 2$ or 3, and in the second case $|\mathrm{Sing}_\infty|$ is the continuum and we simply write $|\mathrm{Sing}_\infty| = \infty$. Since in both cases the systems admit a finite number of affine invariant straight lines we can use them to distinguish the configurations.

Definition 2.15.

(1) In case we have an infinite number of hyperbolas and just two or three singular points at infinity but we have a finite number of invariant straight lines we define

$$ILD = m_1\mathcal{L}_1 + \cdots + m_l\mathcal{L}_l + m_\infty\mathcal{L}_\infty.$$

(2) In case we have an infinite number of hyperbolas, the line at infinity is filled up with singularities and we have a finite number of affine lines, we define

$$ILD = m_1\mathcal{L}_1 + \cdots + m_l\mathcal{L}_l.$$

Suppose we have a finite number of invariant hyperbolas and invariant straight lines of a system (S) and that they are given by equations

$$f_i(x,y) = 0,\ i \in \{1,2,\dots,k\},\ f_i \in \mathbb{C}[x,y].$$

Let us denote by $F_i(X,Y,Z) = 0$ the projection completion of the invariant curves $f_i = 0$ in $P_2(\mathbb{C})$.

Definition 2.16. The total invariant curve of the system (S) in QSH, on $P_2(\mathbb{R})$, is the curve

$$T(S) = \prod_i F_i(X,Y,Z)Z = 0.$$

In case one of the curves is multiple then it will appear with its multiplicity.

For example, if a system (S) admits an invariant hyperbola $h(x,y)$ with multiplicity two and the line at infinity $Z = 0$ has multiplicity one, then the total invariant curve of this system is

$$T(S) = H(X,Y,Z)^2 Z = 0$$

where $H(X,Y,Z) = 0$ is the projection completion of $h = 0$. The degree of $T(S)$ is 5.

The singular points of the system (S) situated on $T(S)$ are of two kinds: those which are simple (or smooth) points of $T(S)$ and those which are multiple points of $T(S)$.

Remark 2.17. To each singular point of the system we have its associated multiplicity as a singular point of the system. In addition, when these singular points are situated on the total curve, we also have the multiplicity of these points as points on the total curve $T(S)$. Through a singular point of the systems there may pass several of the curves $F_i = 0$ and $Z = 0$. Also we may have the case when this point is a singular point of one or even of several of the curves in case we work with invariant curves with singularities. This leads to the multiplicity of the point as point of the curve $T(S)$. The simple points of the curve $T(S)$ are those of multiplicity one. They are also the smooth points of this curve.

Definition 2.18. The zero-cycle of the total curve $T(S)$ of system (S) is given by

$$M_{0CT} = r_1 P_1 + \cdots + r_l P_l + v_1 P_1^\infty + \cdots + v_n P_n^\infty$$

where P_i (respectively P_j^∞) are all the finite (respectively infinite) singularities situated on $T(S)$ and r_i (respectively v_j) are their corresponding multiplicities as points on the total curve $T(S)$. We define the total multiplicity TM of zero-cycles of the total invariant curve as the sum $\sum_i r_i + \sum_j v_j$.

Remark 2.19. If two curves intersects transversally, this point will be a simple point of intersection. If they are tangent, we would have an intersection multiplicity higher than or equal to two.

Definition 2.20 ([24])**.** Two polynomial differential systems S_1 and S_2 are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of S_1 to the oriented phase curves of S_2 and preserving the orientation.

To cut the number of non equivalent phase portraits in half we use here another equivalence relation.

Definition 2.21. Two polynomial differential systems S_1 and S_2 are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of S_1 to the oriented phase curves of S_2, preserving or reversing the orientation.

We use the notation for singularities as introduced in [2] and [3]. We say that a singular point is *elemental* if it possess two eigenvalues not zero; *semi-elemental* if it possess exactly one eigenvalue equal to zero and *nilpotent* if it posses two eigenvalues zero. We call *intricate* a singular point with its Jacobian matrix identically zero.

We will place first the finite singular points which will be denoted with lower case letters and secondly we will place the infinite singular points which will be denoted by capital letters, separating them by a semicolon ';'.

In our study we will have real and complex finite singular points and from the topological viewpoint only the real ones are interesting. When we have a simple (respectively double) complex finite singular point we use the notation © (respectively $©_{(2)}$).

For the elemental singular points we use the notation 's', 'S' for saddles, 'n', 'N' for nodes, 'f' for foci and 'c' for centers.

Non-elemental singular points are multiple points. Here we introduce a special notation for the infinity non-elemental singular point. We denote by $\binom{a}{b}$ the maximum number a (respectively b) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example, when we have a non-elemental point at infinity obtained by the coalescence from a node at infinite with a saddle at infinite we will denoted it by $\binom{0}{2}SN$.

The semi-elemental singular points can either be nodes, saddles or saddle-nodes (finite or infinite). If they are finite singular points we will denote them by '$n_{(2)}$', '$s_{(2)}$' and '$sn_{(2)}$', respectively and if they are infinite singular points by '$\binom{a}{b}N$', '$\binom{a}{b}S$' and '$\binom{a}{b}SN$', where $\binom{a}{b}$ indicates their multiplicity. We note that semi-elemental nodes and saddles are respectively topologically equivalent with elemental nodes and saddles.

The nilpotent singular points can either be saddles, nodes, saddle-nodes, elliptic-saddles, cusps, foci or centers. The only finite nilpotent points for which we need to introduce notation are the elliptic-saddles and cusps which we denote respectively by 'es' and 'cp'.

The intricate singular points are degenerate singular points. It is known that the neighbourhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic (p), hyperbolic (h) and elliptic (e) (see [12]). In this work we have the following finite intricate singular points of multiplicity four described according their sectoral decomposition:

- $hpphpp_{(4)}$
- $phph_{(4)}$
- $epep_{(4)}$

The degenerate systems are systems with a common factor in the polynomials defining the system. We will denote this case with the symbol $\ominus$. The degeneracy can be produced by a common factor of degree one which defines a straight line or a common quadratic factor which defines a conic. In this paper we have just the first case happening. Following [2] we use the symbol $\ominus[|]$ for a real straight line.

Moreover, we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If some singular points remain on this curve we will use the corresponding notation of their various kinds. In this situation, the geometrical properties of the singularity that remain after the removal of the degeneracy, may produce topologically different phenomena, even if they are topologically equivalent singularities. So, we will need to keep the geometrical information associated to that singularity.

In this study we use the notation $(\ominus[|]; n^d)$ which denotes the presence of a real straight line filled up with singular points in the system such that the reduced system has a node n^d on this line where n^d is a one-direction node, that is, a node with two identical eigenvalues whose Jacobian matrix cannot be diagonal.

The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points.

We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non-isolated singularity. There is a detailed description of this notation in [2]. In case that after the removal of the finite degeneracy, a singular point at infinity remains at the same place, we must denote it with all its geometrical properties since they may influence the local topological phase portrait. In this study we use the notation $\binom{0}{2}SN, (\ominus[|]; \emptyset)$ that means that the system has at infinity a saddle-node, and one non-isolated singular point which is part of a real straight line filled up with singularities (other that the line at infinity), and that the reduced linear system has no infinite singular point in that position. See [2] and [3] for more details.

In order to distinguish topologically the phase portraits of the systems we obtained, we also use some invariants introduced in [25]. Let SC be the total number of separatrix connections, i.e. of phase curves connecting two singularities which are local separatrices of the two singular points. We denote by

- SC_f^f the total number of SC connecting two finite singularities,
- SC_f^∞ the total number of SC connecting a finite with an infinite singularity,
- SC_∞^∞ the total number of SC connecting two infinite.

A *graphic* as defined in [13] is formed by a finite sequence of singular points $r_1, r_2, \dots, r_n$ (with possible repetitions) and non-trivial connecting orbits γ_i for $i = 1, \dots, n$ such that γ_i has r_i as α-limit set and r_{i+1} as ω-limit set for $i < n$ and γ_n has r_n as α-limit set and r_1 as ω-limit set. Also normal orientations n_j of the non-trivial orbits must be coherent in the sense that if γ_{j-1} has left-hand orientation then so does γ_j. A *polycycle* is a graphic which has a Poincaré return map.

A *degenerate graphic* is a graphic where it is also allowed that one or several (even all) connecting orbits γ_i can be formed by an infinite number of singular points. For more details, see [13].

3 Geometric analysis of family (1.10)

Consider the family

$$(1.10)\quad \begin{cases} \dot{x} = a - \dfrac{x^2}{3} - \dfrac{2xy}{3} \\ \dot{y} = 4a - 3v^2 - \dfrac{4xy}{3} + \dfrac{y^2}{3}, \quad \text{where } a \neq 0. \end{cases}$$

This is a two parameter family depending on $(a, v) \in (\mathbb{R}\backslash\{0\}) \times \mathbb{R}$. We display below the full geometric analysis of the systems in this family, which is endowed with at least three invariant algebraic curves. In the generic situation

$$av(a - v^2)(a - 3v^2/4)(a + 3v^2)(a - 8v^2/9) \neq 0 \tag{3.1}$$

the systems have only two invariant lines J_1 and J_2 and only two invariant hyperbolas J_3 and J_4 with respective cofactors α_i, $1 \leq i \leq 4$ where

$$\begin{aligned} J_1 &= -3\sqrt{-a + v^2} - x + y, & \alpha_1 &= \sqrt{-a + v^2} - \tfrac{x}{3} + \tfrac{y}{3}, \\ J_2 &= 3\sqrt{-a + v^2} - x + y, & \alpha_2 &= -\sqrt{-a + v^2} - \tfrac{x}{3} + \tfrac{y}{3}, \\ J_3 &= -3a + 3vx - x^2 + xy, & \alpha_3 &= -v - \tfrac{2x}{3} - \tfrac{y}{3}, \\ J_4 &= -3a - 3vx - x^2 + xy, & \alpha_4 &= v - \tfrac{2x}{3} - \tfrac{y}{3}. \end{aligned}$$

We see that since the number of invariant curve is four, these systems are Darboux integrable. We note that if $v = 0$ then the two hyperbolas coincide and we get a double hyperbola. Also if $a = v^2$ the two lines coincide and we get a double line. So to have four distinct curves we need to put $v(a - v^2) \neq 0$. We inquire when we could have an additional line. Calculations yield that this happens when $a - 3v^2/4 = 0$. We also inquire when we could have an additional hyperbola. Calculations yield that this happens when $(a + 3v^2)(a - 8v^2/9) = 0$.

Straightforward calculations lead us to the tables listed below. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using for lines the 1st and for hyperbola the 2nd extactic polynomial, respectively.

(i) $av(a - v^2)(a - 3v^2/4)(a + 3v^2)(a - 8v^2/9) \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -3\sqrt{-a+v^2} - x + y$ $J_2 = 3\sqrt{-a+v^2} - x + y$ $J_3 = -3a + 3vx - x^2 + xy$ $J_4 = -3a - 3vx - x^2 + xy$ $\alpha_1 = \sqrt{-a+v^2} - \frac{x}{3} + \frac{y}{3}$ $\alpha_2 = \sqrt{-a+v^2} - \frac{x}{3} + \frac{y}{3}$ $\alpha_3 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_4 = v - \frac{2x}{3} - \frac{y}{3}$	$P_1 = (-v - \sqrt{v^2-a}, -v + 2\sqrt{v^2-a})$ $P_2 = (v - \sqrt{v^2-a}, v + 2\sqrt{v^2-a})$ $P_3 = (-v + \sqrt{v^2-a}, -v - 2\sqrt{v^2-a})$ $P_4 = (v + \sqrt{v^2-a}, v - 2\sqrt{v^2-a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $v^2 > a$ we have $n, s, s, n; N, N, S$ if $v > 0$ $s, n, n, s; N, N, S$ if $v < 0$ For $v^2 < a$ we have $©, ©, ©, ©; N, N, S$	$\overline{J}_1 \cap \overline{J}_2 = P_2^\infty$ simple $\overline{J}_1 \cap \overline{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \overline{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\overline{J}_2 \cap \overline{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\overline{J}_2 \cap \overline{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\overline{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\overline{J}_3 \cap \overline{J}_4 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\overline{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\overline{J}_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty \text{ if } v^2 > a \\ J_1^C + J_2^C + J_3 + J_4 + \mathcal{L}_\infty \text{ if } v^2 < a \end{cases}$	5 5
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } v^2 > a \\ P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } v^2 < a \end{cases}$	7 7
$T = Z\overline{J}_1\overline{J}_2\overline{J}_3\overline{J}_4 = 0$	7
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 5P_2^\infty + P_3^\infty \text{ if } v^2 > a \\ 3P_1^\infty + 5P_2^\infty + P_3^\infty \text{ if } v^2 < a \end{cases}$	17 9

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double and

2) five distinct tangents at P_2^∞.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{\frac{\lambda_1\sqrt{v^2-a}}{v}} J_4^{-\frac{\lambda_1\sqrt{v^2-a}}{v}}$	$R = J_1^{\lambda_1} J_2^{-\lambda_1-2} J_3^{\frac{(\lambda_1+1)\sqrt{v^2-a}}{v}-1} J_4^{-\frac{(\lambda_1+1)\sqrt{v^2-a}}{v}-1}$
Simple example	$\mathcal{I} = \frac{J_1}{J_2}\left(\frac{J_3}{J_4}\right)^{\frac{\sqrt{v^2-a}}{v}}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_3 J_4}$

(ii) $av(a-v^2)(a-3v^2/4)(a+3v^2)(a-8v^2/9)=0$.

(ii.1) $v=0$ and $a\neq 0$.

Here the two hyperbolas coalesce yielding a double hyperbola so we compute the exponential factor E_4.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1=-3i\sqrt{a}+x-y$ $J_2=3i\sqrt{a}+x-y$ $J_3=-3a+x(y-x)$ $E_4=e^{\frac{g_1x}{-3a+x(y-x)}}$ $\alpha_1=-i\sqrt{a}-\frac{x}{3}+\frac{y}{3}$ $\alpha_2=i\sqrt{a}-\frac{x}{3}+\frac{y}{3}$ $\alpha_3=-\frac{2x}{3}-\frac{y}{3}$ $\alpha_4=-\frac{g_1}{3}$	$P_1=(-i\sqrt{a},2i\sqrt{a})$ $P_2=(i\sqrt{a},-2i\sqrt{a})$ $P_1^\infty=[0:1:0]$ $P_2^\infty=[1:1:0]$ $P_3^\infty=[1:0:0]$ For $a<0$ we have $sn_{(2)},sn_{(2)};N,N,S$ For $a>0$ we have $©_{(2)},©_{(2)};N,N,S$	$\bar{J}_1\cap\bar{J}_2=P_2^\infty$ simple $\bar{J}_1\cap\bar{J}_3=\begin{cases}P_2^\infty \text{ simple}\\ P_2 \text{ simple}\end{cases}$ $\bar{J}_1\cap\mathcal{L}_\infty=P_2^\infty$ simple $\bar{J}_2\cap\bar{J}_3=\begin{cases}P_2^\infty \text{ simple}\\ P_1 \text{ simple}\end{cases}$ $\bar{J}_2\cap\mathcal{L}_\infty=P_2^\infty$ simple $\bar{J}_3\cap\mathcal{L}_\infty=\begin{cases}P_1^\infty \text{ simple}\\ P_2^\infty \text{ simple}\end{cases}$

Divisor and zero-cycles	Degree
$ICD=\begin{cases}J_1+J_2+2J_3+\mathcal{L}_\infty \text{ if } a<0\\ J_1^C+J_2^C+2J_3+\mathcal{L}_\infty \text{ if } a>0\end{cases}$	5 5
$M_{0CS}=\begin{cases}2P_1+2P_2+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } a<0\\ 2P_1^C+2P_2^C+P_1^\infty+P_2^\infty+P_3^\infty \text{ if } a>0\end{cases}$	7 7
$T=Z\bar{J}_1\bar{J}_2\bar{J}_3^2=0.$	7
$M_{0CT}=\begin{cases}3P_1+3P_2+3P_1^\infty+5P_2^\infty+P_3^\infty \text{ if } a<0\\ 3P_1^\infty+5P_2^\infty+P_3^\infty \text{ if } a>0\end{cases}$	15 9

where the total curve T has
1) only two distinct tangents at P_1 (and at P_2), but one of them is double;
2) only two distinct tangents at P_1^∞, but one of them is double and
3) only four distinct tangents at P_2^∞, but one of them is double.

	First integral	Integrating Factor
General	$I=J_1^{\lambda_1}J_2^{-\lambda_1}J_3^0E_4^{-\frac{6i\sqrt{a}\lambda_1}{g_1}}$	$R=J_1^{\lambda_1}J_2^{-2-\lambda_1}J_3^{-2}E_4^{-\frac{6i\sqrt{a}(1+\lambda_1)}{g_1}}$
Simple example	$\mathcal{I}=\frac{J_1}{J_2{E_4}^{6i\sqrt{a}}}$	$\mathcal{R}=\frac{1}{J_1J_2J_3^2}$

(ii.2) $a = v^2$.

Here the two lines coalesce yielding a double line so we compute the exponential factor E_4.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = -3v^2 + 3vx - x^2 + xy$ $J_3 = -3v^2 - 3vx - x^2 + xy$ $E_4 = e^{\frac{g_0 + g_1(x-y)}{x-y}}$ $\alpha_1 = -\frac{x}{3} + \frac{y}{3}$ $\alpha_2 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_3 = v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_4 = \frac{g_0}{3}$	$P_1 = (-v, -v)$ $P_2 = (v, v)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $sn_{(2)}, sn_{(2)}; N, N, S$	$\overline{J}_1 \cap \overline{J}_2 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \overline{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\overline{J}_2 \cap \overline{J}_3 = \begin{cases} P_2^\infty \text{ triple} \\ P_1 \text{ simple} \end{cases}$ $\overline{J}_2 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\overline{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = 2J_1 + J_2 + J_3 + \mathcal{L}_\infty$	5
$M_{0CS} = 2P_1 + 2P_2 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\overline{J}_1^2\overline{J}_2\overline{J}_3 = 0$	7
$M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$	15

where the total curve T has

1) only two distinct tangents at P_1 and at P_2, but one of them is double;

2) only two distinct tangents at P_1^∞, but one of them is double and

3) only four distinct tangents at P_2^∞, but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^{-\lambda_2} E_4^{\frac{6v\lambda_2}{g_0}}$	$R = J_1^{-2} J_2^{\lambda_2} J_3^{-2-\lambda_2} E_4^{\frac{6v(1+\lambda_2)}{g_0}}$
Simple example	$\mathcal{I} = \frac{J_2 E_4{}^{6v}}{J_3}$	$\mathcal{R} = \frac{1}{J_1^2 J_2 J_3}$

(ii.3) $a = 3v^2/4$.

Here we have, apart from two lines and two hyperbolas, a third invariant line. Then, we have five invariant algebraic curves and hence according to Jouanolou's theorem the corresponding system has a rational first integral.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -\frac{3v}{2} + x - y$ $J_2 = \frac{3v}{2} + x - y$ $J_3 = y$ $J_4 = -\frac{x^2}{3v} + \frac{xy}{3v} - \frac{3v}{4} + x$ $J_5 = \frac{x^2}{3v} - \frac{xy}{3v} + \frac{3v}{4} + x$ $\alpha_1 = \frac{1}{6}(-3v - 2x + 2y)$ $\alpha_2 = \frac{1}{6}(3v - 2x + 2y)$ $\alpha_3 = \frac{y}{3} - \frac{4x}{3}$ $\alpha_4 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_5 = v - \frac{2x}{3} - \frac{y}{3}$	$P_1 = \left(-\frac{3v}{2}, 0\right)$ $P_2 = \left(-\frac{v}{2}, -2v\right)$ $P_3 = \left(\frac{v}{2}, 2v\right)$ $P_4 = \left(\frac{3v}{2}, 0\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $n, s, s, n; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_4$ double $\bar{J}_3 \cap \bar{J}_5 = P_1$ double $\bar{J}_3 \cap \mathcal{L}_\infty = P_3^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$	6
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 5P_2^\infty + 2P_3^\infty$	20

where the total curve T has
1) only two distinct tangents at P_1 (and at P_4), but one of them is double,
2) only two distinct tangents at P_1^∞, but one of them is double and
3) five distinct tangents at P_2^∞.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{2} - \frac{\lambda_2}{2}} J_4^{\frac{\lambda_2}{2}} J_5^{\frac{\lambda_1}{2}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-1-\frac{\lambda_1}{2} - \frac{\lambda_2}{2}} J_4^{-\frac{1}{2}+\frac{\lambda_2}{2}} J_5^{-\frac{1}{2}+\frac{\lambda_1}{2}}$
Simple example	$\mathcal{I}_1 = \frac{J_1^2 J_5}{J_3}, \quad \mathcal{I}_2 = \frac{J_2^2 J_4}{J_3}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_4 J_5}$

Remark 3.1. Consider $\mathcal{F}^1_{(c_1,c_2)} = c_1 J_1^2 J_5 - c_2 J_3 = 0$, $\deg \mathcal{F}^1_{(c_1,c_2)} = 4$. The remarkable

values of $\mathcal{F}^1_{(c_1,c_2)}$ are $[1:9v^2/2]$ and $[1:0]$ for which we have

$$\mathcal{F}^1_{(1,9v^2/2)} = -J_2^2 J_4, \qquad \mathcal{F}^1_{(1,0)} = J_1^2 J_5.$$

Therefore, J_1, J_2, J_4, J_5 are remarkable curves of $\mathcal{I}_1$, $[1:9v^2/2]$ and $[1:0]$ are the only two critical remarkable values of $\mathcal{I}_1$ and J_1, J_2 are critical remarkable curves of $\mathcal{I}_1$. The singular points are P_1, P_3 for $\mathcal{F}^1_{(1,9v^2/2)}$ and P_2, P_4 for $\mathcal{F}^1_{(1,0)}$.

Considering the first integral $\mathcal{I}_2$ with its associated curve $\mathcal{F}^2_{(c_1,c_2)} = c_1 J_2^2 J_4 - c_2 J_3$ we have the same remarkable values $[1:9v^2/2]$ and $[1:0]$ and the same remarkable curves J_1, J_2, J_4, J_5. However, the singular point are P_1, P_3 for $\mathcal{F}^2_{(1,0)}$ and P_2, P_4 for $\mathcal{F}^2_{(1,9v^2/2)}$.

(ii.4) $a = -3v^2$.

Here we have, apart from two lines and two hyperbolas, a third invariant hyperbola. Then, we have five invariant algebraic curves and hence according to Jouanolou's theorem the corresponding system has a rational first integral.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -6v + x - y$ $J_2 = 6v + x - y$ $J_3 = 9v^2 + xy$ $J_4 = 9v^2 + 3vx - x^2 + xy$ $J_5 = 9v^2 - 3vx - x^2 + xy$ $\alpha_1 = -2v - \frac{x}{3} + \frac{y}{3}$ $\alpha_2 = 2v - \frac{x}{3} + \frac{y}{3}$ $\alpha_3 = -\frac{5x}{3} - \frac{y}{3}$ $\alpha_4 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_5 = v - \frac{2x}{3} - \frac{y}{3}$	$P_1 = (-3v, 3v)$ $P_2 = (-v, 5v)$ $P_3 = (v, -5v)$ $P_4 = (3v, -3v)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $n, s, s, n; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ double $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ double $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ triple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ double} \\ P_2^\infty \text{ double} \end{cases}$ $\bar{J}_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$	6
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	9
$M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 4P_1^\infty + 5P_2^\infty + 2P_3^\infty$	21

where the total curve T has
1) only two distinct tangents at P_1 (and at P_4), but one of them is double,
2) only two distinct tangents at P_1^∞, but one of them is triple,
3) only four tangents at P_2^∞, but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\lambda_1-\lambda_2} J_4^{2\lambda_2} J_5^{2\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-2-\lambda_1-\lambda_2} J_4^{1+2\lambda_2} J_5^{1+2\lambda_1}$
Simple example	$\mathcal{I}_1 = \frac{J_1 J_5^2}{J_3}, \quad \mathcal{I}_2 = \frac{J_2 J_4^2}{J_3}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_4 J_5}$

Remark 3.2. Consider $\mathcal{F}^1_{(c_1,c_2)} = c_1 J_1 J_5^2 - c_2 J_3 = 0$, $\deg \mathcal{F}^1_{(c_1,c_2)} = 5$. The remarkable values of $\mathcal{F}^1_{(c_1,c_2)}$ are $[1 : -108v^3]$ and $[1 : 0]$ for which we have

$$\mathcal{F}^1_{(1,-108v^3)} = J_2 J_4^2, \qquad \mathcal{F}^1_{(1,0)} = J_1 J_5^2.$$

Therefore, J_1, J_2, J_4, J_5 are remarkable curves of $\mathcal{I}_1$, $[1 : -108v^3]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathcal{I}_1$ and J_4, J_5 are critical remarkable curves of $\mathcal{I}_1$. The singular points are P_2, P_4 for $\mathcal{F}^1_{(1,-108v^3)}$ and P_1, P_3 for $\mathcal{F}^1_{(1,0)}$.

Considering the first integral $\mathcal{I}_2$ with its associated curve $\mathcal{F}^2_{(c_1,c_2)} = c_1 J_2 J_4^2 - c_2 J_3$ we have the remarkable values $[1 : 108v^3]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_4, J_5. The singular point are P_1, P_3 for $\mathcal{F}^2_{(1,108v^3)}$ and P_2, P_4 for $\mathcal{F}^2_{(1,0)}$.

(ii.5) $a = 8v^2/9$.

Here we have, apart from two lines and two hyperbolas, a third invariant hyperbola. Then, we have five invariant algebraic curves and hence according to Jouanolou's theorem the corresponding system has a rational first integral.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -v + x - y$ $J_2 = v + x - y$ $J_3 = y(x-y) - \frac{v^2}{3}$ $J_4 = -\frac{8v^2}{3} + 3vx + x(y-x)$ $J_5 = -\frac{8v^2}{3} - 3vx + x(y-x)$ $\alpha_1 = \frac{1}{3}(-v - x + y)$ $\alpha_2 = \frac{1}{3}(v - x + y)$ $\alpha_3 = \frac{2y}{3} - \frac{5x}{3}$ $\alpha_4 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_5 = v - \frac{2x}{3} - \frac{y}{3}$	$P_1 = \left(-\frac{4v}{3}, -\frac{v}{3}\right)$ $P_2 = \left(-\frac{2v}{3}, -\frac{5v}{3}\right)$ $P_3 = \left(\frac{2v}{3}, \frac{5v}{3}\right)$ $P_4 = \left(\frac{4v}{3}, \frac{v}{3}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $n, s, s, n; N, N, S$	$\overline{J}_1 \cap \overline{J}_2 = P_2^\infty$ simple $\overline{J}_1 \cap \overline{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \overline{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \overline{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\overline{J}_2 \cap \overline{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\overline{J}_2 \cap \overline{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\overline{J}_2 \cap \overline{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\overline{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\overline{J}_3 \cap \overline{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ triple} \end{cases}$ $\overline{J}_3 \cap \overline{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ triple} \end{cases}$ $\overline{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\overline{J}_4 \cap \overline{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\overline{J}_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\overline{J}_5 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$	6
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\overline{J}_1\overline{J}_2\overline{J}_3\overline{J}_4\overline{J}_5 = 0$	9
$M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 6P_2^\infty + 2P_3^\infty$	21

where the total curve T has
1) only two distinct tangents at P_1 (and at P_4), but one of them is double,
2) only two distinct tangents at P_1^∞, but one of them is double and
3) six distinct tangents at P_2^∞.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{3}-\frac{\lambda_2}{3}} J_4^{\frac{\lambda_2}{3}} J_5^{\frac{\lambda_1}{3}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{3}-\frac{\lambda_2}{3}-\frac{2}{3}} J_4^{\frac{\lambda_2}{3}-\frac{2}{3}} J_5^{\frac{\lambda_1}{3}-\frac{2}{3}}$
Simple example	$\mathcal{I}_1 = \frac{J_1^3 J_5}{J_3}, \quad \mathcal{I}_2 = \frac{J_2^3 J_4}{J_3}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_4 J_5}$

Remark 3.3. Consider $\mathcal{F}^1_{(c_1,c_2)} = c_1 J_1^3 J_5 - c_2 J_3 = 0$, $\deg \mathcal{F}^1_{(c_1,c_2)} = 5$. The remarkable values of $\mathcal{F}^1_{(c_1,c_2)}$ are $[1 : -16v^3]$ and $[1 : 0]$ for which we have

$$\mathcal{F}^1_{(1,-16v^3)} = J_2^3 J_4, \qquad \mathcal{F}^1_{(1,0)} = J_1^3 J_5.$$

Therefore, J_1, J_2, J_4, J_5 are remarkable curves of $\mathcal{I}_1$, $[1 : -16v^3]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathcal{I}_1$ and J_1, J_2 are critical remarkable curves of $\mathcal{I}_1$. The singular points are P_1, P_3 for $\mathcal{F}^1_{(1,-16v^3)}$ and P_2, P_4 for $\mathcal{F}^1_{(1,0)}$.

Considering the first integral $\mathcal{I}_2$ with its associated curves $\mathcal{F}^2_{(c_1,c_2)} = c_1 J_2^3 J_4 - c_2 J_3$ we have the remarkable values $[1 : 16v^3]$ and $[1 : 0]$ and the same remarkable curves J_1, J_2, J_4, J_5. The singular point are P_1, P_3 for $\mathcal{F}^2_{(1,0)}$ and P_2, P_4 for $\mathcal{F}^2_{(1,16v^3)}$.

(ii.6) $a = 0$ and $v \neq 0$.

Under this condition, systems (1.10) do not belong to **QSH**, but we study them seeking a complete understanding of the bifurcation diagram of the systems in the full family (1.10). All the invariant lines are $x = 0$ and $\pm 3v - x + y = 0$ that are simple. Perturbing this system in the family (1.10) we can obtain two distinct configurations of lines and hyperbolas. By perturbing the reducible conics $x(-3v - x + y) = 0$ and $x(3v - x + y) = 0$ we can produce two distinct hyperbolas $-3a - 3vx - x^2 + xy = 0$ and $-3a + 3vx - x^2 + xy = 0$, respectively. Furthermore, the cubic $x(3v - x + y)(-3v - x + y) = 0$ has integrable multiplicity two.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = -3v - x + y$ $J_2 = 3v - x + y$ $J_3 = x$ $E_4 = e^{-\frac{6g_0\left(6v^2+x(y-x)\right)+g_1 x\left((x-y)^2-9v^2\right)}{2x(-3v+x-y)(3v+x-y)}}$ $\alpha_1 = v - \frac{x}{3} + \frac{y}{3}$ $\alpha_2 = -v - \frac{x}{3} + \frac{y}{3}$ $\alpha_3 = -\frac{x}{3} - \frac{2y}{3}$ $\alpha_4 = g_0$	$P_1 = (0, -3v)$ $P_2 = (2v, -v)$ $P_3 = (-2v, v)$ $P_4 = (0, 3v)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$ For $v \neq 0$ we have $s, n, n, s; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = P_1^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ if $v \neq 0$	4
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$ if $v \neq 0$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0.$	4
$M_{0CT} = 2P_1 + P_2 + P_3 + 2P_4 + 2P_1^\infty + 3P_2^\infty + P_3^\infty$ if $v \neq 0$	12

where the total curve T has three distinct tangents at P_2^∞.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^0 E_4^{-\frac{2\lambda_1 v}{g_0}}$	$R = J_1^{\lambda_1} J_2^{-4-\lambda_1} J_3^{-2} E_4^{-\frac{2v(\lambda_1+2)}{g_0}}$
Simple example	$\mathcal{I} = \frac{J_1}{J_2 E_4^{2v}}$	$\mathcal{R} = \frac{1}{J_1^2 J_2^2 J_3^2}$

(ii.7) $a = v = 0.$

Under this condition, systems (1.10) do not belong to **QSH**, but we study them seeking a complete understanding of the bifurcation diagram of the systems in the full family (1.10). Here we have a single system which has a rational first integral that foliates the plane into quartic curves. All the invariant affine lines are $x = 0$, $y = 0$ that are simple and $x - y = 0$ that is double. The lines $x = 0$ and $x - y = 0$ are remarkable curves. Perturbing this system in the full family (1.10) we can obtain up to ten distinct configurations of lines and hyperbolas. By perturbing the reducible conic $x(x - y) = 0$ we can produce 2 distinct hyperbolas $-3a + 3vx - x^2 + xy = 0$ and $-3a - 3vx - x^2 + xy = 0$. Perturbing the reducible conic $y(x - y) = 0$ we can produce a third hyperbola $y(x - y) - \frac{v^2}{3} = 0$ and by perturbing $xy = 0$ we can produce the hyperbola $9v^2 + xy = 0$. We get a double hyperbola $-3a + x(y - x) = 0$ by perturbing the double reducible conic $x^2(x - y)^2 = 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $E_4 = e^{\frac{g_0 + g_1(x-y)}{x-y}}$ $\alpha_1 = \frac{y}{3} - \frac{4x}{3}$ $\alpha_2 = -\frac{x}{3} - \frac{2y}{3}$ $\alpha_3 = \frac{y}{3} - \frac{x}{3}$ $\alpha_4 = \frac{g_0}{3}$	$P_1 = (0, 0)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$ $hpphpp_{(4)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = P_2^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$	10

where the total curve T has
1) only three distinct tangents at P_1, but one of them is double;
2) only two distinct tangentes at P_2^∞, but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{-3\lambda_1} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-4-3\lambda_1} E_4^0$
Simple example	$\mathcal{I}_1 = \frac{J_1}{J_2 J_3^3}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_3}$

Remark 3.4. Consider $\mathcal{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_2 J_3^3 = 0$, $\deg \mathcal{F}^1_{(c_1,c_2)} = 4$. The remarkable value of $\mathcal{F}^1_{(c_1,c_2)}$ is $[0:1]$ for which we have

$$\mathcal{F}^1_{(0,1)} = -J_2 J_3^3.$$

Therefore, J_2, J_3 are remarkable curves of $\mathcal{I}_1$, $[0:1]$ is the only critical remarkable values of $\mathcal{I}_1$ and J_3 is critical remarkable curve of $\mathcal{I}_1$. The singular point is P_1 for $\mathcal{F}^1_{(0,1)}$.

We sum up the topological, dynamical and algebraic geometric features of family (1.10) and we also confront our results with previous results in the literature in the following proposition. We show that there exists two more configurations of invariant hyperbolas and lines than in [18], there are four more phase portraits than the ones appearing in [17] and there is one more phase portrait than the ones appearing in [10].

Proposition 3.5.

(a) *For the family (1.10) we have nine distinct configurations $C_1^{(1.10)} - C_9^{(1.10)}$ of invariant hyperbolas and lines (see Figure 3.1 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space is $av(a-v^2)(a+3v^2)(a-3v^2/4)(a-8v^2/9) = 0$. On $v(a-v^2) = 0$ one of the algebraic solutions is double. On $(a+3v^2)(a-3v^2/4)(a-8v^2/8) = 0$ we have an additional line or an additional hyperbola. The configurations $C_8^{(1.10)}$ and $C_9^{(1.10)}$ are not equivalent with anyone of the configurations for systems (3.97) (here family (1.10)) in [18].*

(b) *All systems in family (1.10) have an inverse integrating factor which is polynomial. All systems in family (1.10) satisfying the genericity condition (3.1) have a Darboux first integral. If $a = v^2$ then the systems have a double invariant line. If $v = 0$ then the systems have a double invariant hyperbola. In both cases, the systems have a generalized Darboux first integral. In* *all the following three cases, we have a rational first integral. If $a = 3v^2/4$ then the systems have an additional invariant line and the plane is foliated into quartic algebraic curves. If $a = -3v^2$ the plane is foliated by quintic algebraic curves. If $a = 8v^2/9$ then the systems have an additional invariant hyperbola and the plane is foliated in quintic algebraic curves. The remarkable curves are J_1, J_2, J_4, J_5 for these three algebraically integrable cases of family (1.10) for each case correspondingly.*

(c) For the family (1.10) we have five topologically distinct phase portraits $P_1^{(1.10)} - P_5^{(1.10)}$. The topological bifurcation diagram of family (1.10) is done in Figure 3.2. The bifurcation set of singularities is the half line $v = 0$ and $a < 0$, the parabola $a = v^2$ and the line $a = 0$. The phase portraits $P_1^{(1.10)}$, $P_3^{(1.10)}$, $P_4^{(1.10)}$ and $P_5^{(1.10)}$ are not topologically equivalent with anyone of the phase portraits in [17]. The phase portrait $P_1^{(1.10)}$ is not topologically equivalent with anyone of the phase portraits in [10].

Proof. (a) We have the following types of divisors and zero-cycles of the total invariant curve T for the configurations of family (1.10) :

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(1.10)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_2^{(1.10)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_3^{(1.10)}$	$ICD = J_1^C + J_2^C + J_3 + J_4 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_4^{(1.10)}$	$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_5^{(1.10)}$	$ICD = J_1^C + J_2^C + 2J_3 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_6^{(1.10)}$	$ICD = 2J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_7^{(1.10)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 5P_2^\infty + 2P_3^\infty$
$C_8^{(1.10)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 4P_1^\infty + 5P_2^\infty + 2P_3^\infty$
$C_9^{(1.10)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 6P_2^\infty + 2P_3^\infty$

Although $C_1^{(1.10)}$ and $C_2^{(1.10)}$ admit the same type of divisors and zero-cycles we can see they are different because in $C_1^{(1.10)}$ each branch of the hyperbolas intersects one line while $C_2^{(1.10)}$ have two branches intersecting both lines and two branches intersecting no line. Therefore, the configurations $C_1^{(1.10)}$ up to $C_9^{(1.10)}$ are all distinct. For the limit cases of family (1.10) we have the following configurations:

Configurations	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + P_2 + P_3 + 2P_4 + 2P_1^\infty + 3P_2^\infty + P_3^\infty$
c_2	$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$

The other statements in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves were done in Remarks 3.1, 3.2 and 3.3 .

(c) We have that:

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(1.10)}$	(N,N,S)	(n,s,s,n)	$2SC_f^f\ \ 8SC_f^\infty\ \ 0SC_\infty^\infty$
$P_2^{(1.10)}$	(N,N,S)	(n,s,s,n)	$4SC_f^f\ \ 6SC_f^\infty\ \ 0SC_\infty^\infty$
$P_3^{(1.10)}$	(N,N,S)	$(©,©,©,©)$ $(©_{(2)},©_{(2)})$	$0SC_f^f\ \ 0SC_f^\infty\ \ 2SC_\infty^\infty$
$P_4^{(1.10)}$	(N,N,S)	$(sn_{(2)},sn_{(2)})$	$1SC_f^f\ \ 6SC_f^\infty\ \ 0SC_\infty^\infty$
$P_5^{(1.10)}$	(N,N,S)	$(sn_{(2)},sn_{(2)})$	$0SC_f^f\ \ 8SC_f^\infty\ \ 0SC_\infty^\infty$

Therefore, we have five distinct phase portraits for systems (1.10). For the limit cases of family (1.10) we have the following phase portraits:

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
p_1	(N,N,S)	(n,s,s,n)	$3SC_f^f\ \ 6SC_f^\infty\ \ 0SC_\infty^\infty$
p_2	(N,N,S)	$hpphpp_{(4)}$	$0SC_f^f\ \ 6SC_f^\infty\ \ 0SC_\infty^\infty$

On the table below we list the phase portraits of Llibre–Yu in [17] that satisfy the following conditions: the phase portraits admit 3 singular points at infinity with the type (N,N,S), and it has either 0, 1, 2 or 4 real singular points in the finite region.

Phase Portraits	Sing. at ∞	Real finite sing.	Separatrix connections
R_{01},Ω_6	(N,S,N)	$\emptyset$	$0SC_f^f\ \ 0SC_f^\infty\ \ 1SC_\infty^\infty$
L_{11},L_{12}	(N,S,N)	cp	$0SC_f^f\ \ 2SC_f^\infty\ \ 1SC_\infty^\infty$
P_2	(N,S,N)	$pphpph$	$0SC_f^f\ \ 6SC_f^\infty\ \ 0SC_\infty^\infty$
L_{31},L_{32}	(N,S,N)	(s,es)	$2SC_f^f\ \ 6SC_f^\infty\ \ 2SC_\infty^\infty$
L_{33}	(N,S,N)	(c,es)	$1SC_f^f\ \ 4SC_f^\infty\ \ 1SC_\infty^\infty$
R_1,R_2	(N,S,N)	(s,c)	$1SC_f^f\ \ 2SC_f^\infty\ \ 1SC_\infty^\infty$
R_3,Ω_5	(N,S,N)	(c,c)	$2SC_f^f\ \ 0SC_f^\infty\ \ 3SC_\infty^\infty$
R_5	(N,S,N)	(s,n,n,s)	$4SC_f^f\ \ 6SC_f^\infty\ \ 0SC_\infty^\infty$
R_8,Ω_1	(N,S,N)	(s,n,n,s)	$4SC_f^f\ \ 6SC_f^\infty\ \ 0SC_\infty^\infty$

Therefore, we can see from the two above tables that the phase portraits $P_1^{(1.10)}$, $P_3^{(1.10)}$, $P_4^{(1.10)}$ and $P_5^{(1.10)}$ are not topologically equivalent with anyone of the phase portraits in [17]. They are however phase portraits of systems possessing and invariant hyperbola and an invariant line.

On the table below we list the phase portraits of Coll–Ferragut–Llibre in [17] that admit 3 singular points at infinity with the type (N, N, S), and it has either 0, 1, 2 or 4 real singular points in the finite region:

Phase Portrait	Sing. at ∞	Real finite sing.	Separatrix connections
(20)	(N, N, S)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
(42)	(N, N, S)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
(59)	(N, N, S)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
(21)	(N, N, S)	cp	$0SC_f^f$ $2SC_f^\infty$ $2SC_\infty^\infty$
(43)	(N, S, N)	cp	$0SC_f^f$ $2SC_f^\infty$ $1SC_\infty^\infty$
(57)	(N, N, S)	$pphpph$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
(22)	(N, N, S)	(s, c)	$1SC_f^f$ $2SC_f^\infty$ $2SC_\infty^\infty$
(23)	(N, N, S)	(s, c)	$0SC_f^f$ $4SC_f^\infty$ $1SC_\infty^\infty$
(28)	(N, N, S)	(s, c)	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$
(44)	(N, N, S)	(s, c)	$1SC_f^f$ $2SC_f^\infty$ $1SC_\infty^\infty$
(45)	(N, N, S)	(es, s)	$2SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$
(58)	(N, N, S)	(sn, sn)	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
(77)	(N, N, S)	(sn, sn)	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
(102)	(N, N, S)	(s, es)	$2SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
(35)	(N, N, S)	(n, s, s, n)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
(115)	(N, N, S)	(n, s, s, n)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Therefore, the phase portrait $P_1^{(1.10)}$ is not topologically equivalent with anyone of the phase portraits in [10]. It is however a phase portrait of a systems possessing a polynomial inverse integrating factor. $\square$

3.1 The solution of the Poincaré problem for the family (1.10)

We can recognize when a system in this family has a rational first integral. The following is the answer to Poincaré's problem for the family (1.10):

Theorem 3.6.

i) A necessary and sufficient condition for a system in family (1.10) to have a rational first integral is that $v^2 - a > 0$ and that (a, v) be situated on a parabola of the form $a = (1 - r^2)v^2$ with $r \in \mathbb{Q}$.

ii) The set of all points (a, v)'s satisfying these two conditions is dense in the set $v^2 - a > 0$ with $v \neq 0$.

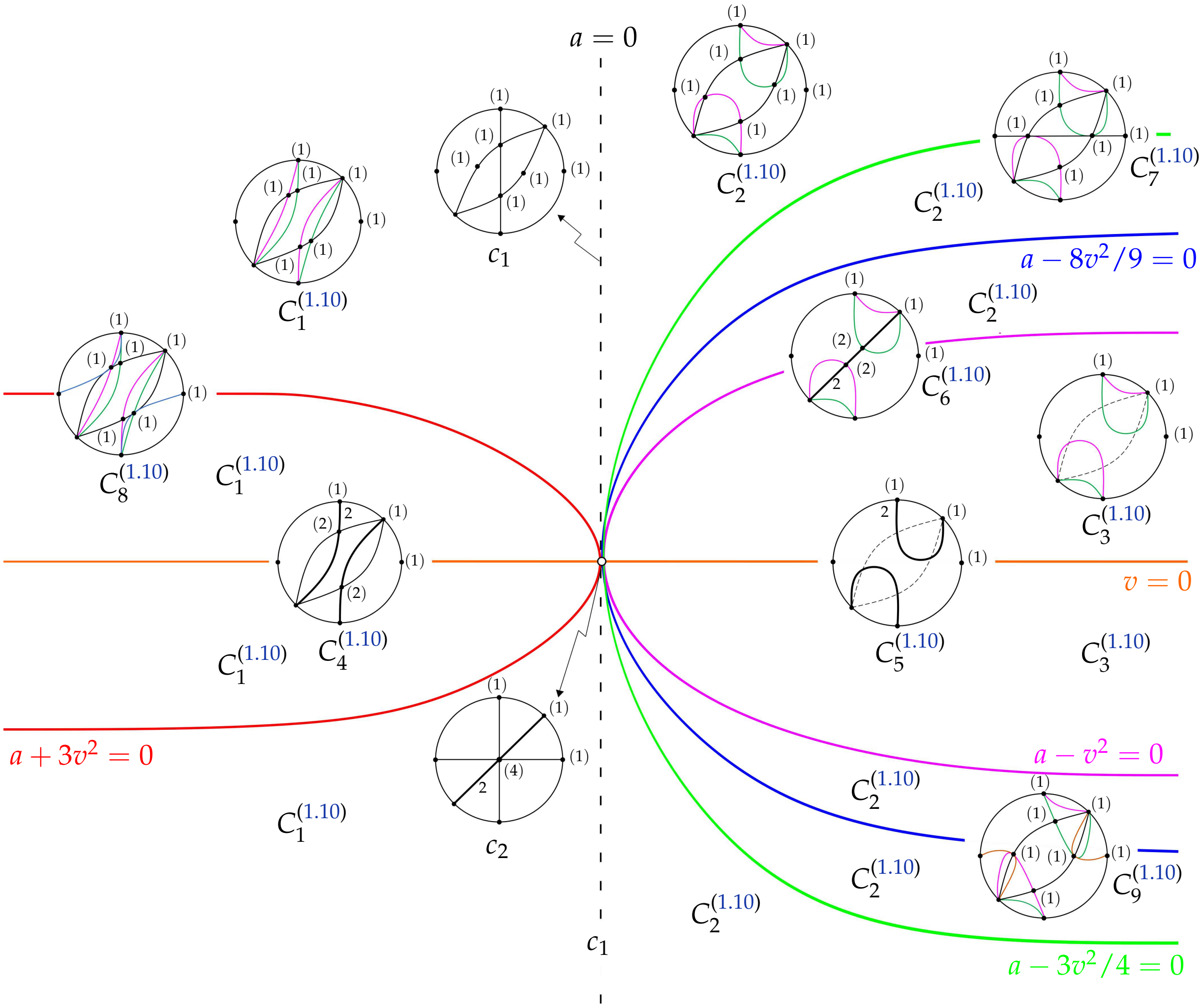

Figure 3.1: Bifurcation diagram of configurations for family (1.10): In this figure on the dashed line $a = 0$ both hyperbolas become reducible into two lines one of them $x = 0$. On the bifurcation curves we either have an additional line or additional hyperbola or coalescing lines or coalescing hyperbolas or real lines becoming complex. The dashed lines represent complex lines.

Proof. i) We first prove that the condition is necessary. So assume that we have a system of parameters (a, v) that has a rational first integral. Assume now that (a, v) is in the generic situations $v(a - v^2)(a + 3v^2)(a - 8v^2/9)(a - 3v^2/4) \neq 0$. Any first integral of the system is then of the following general form:

$$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{\frac{\lambda_1\sqrt{v^2-a}}{v}} J_4^{-\frac{\lambda_1\sqrt{v^2-a}}{v}}.$$

This is a rational first integral if and only if $\lambda_1 \in \mathbb{Z}$ and $\frac{\lambda_1\sqrt{v^2-a}}{v} \in \mathbb{Z}$ in which case we must have that $r = \sqrt{v^2 - a}/v$ must be a rational number. In view of our generic hypothesis $r \neq 0$. Since $r = \sqrt{v^2 - a}/v$ is rational we have $v^2 - a \geq 0$ and by hypothesis $v^2 - a \neq 0$. Therefore $v^2 - a > 0$. We also have $a = (1 - r^2)v^2$ and therefore the condition is necessary in this case. Consider now the case when $v(a - v^2)(a + 3v^2)(a - 8v^2/9)(a - 3v^2/4) = 0$. Since on $v(a - v^2) = 0$ we cannot have a rational first integral because as we see in the tables for

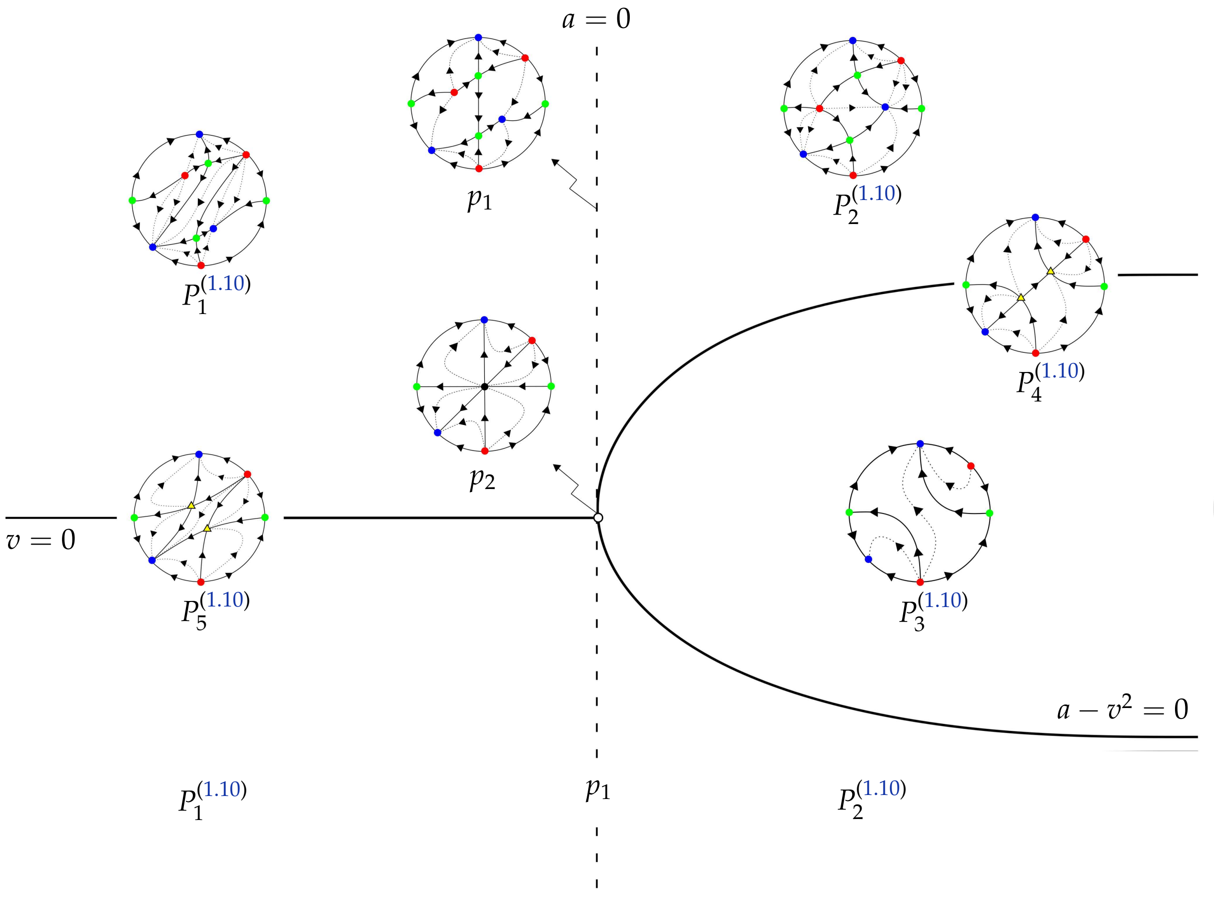

Figure 3.2: Topological bifurcation diagram for family (1.10).

these two cases, we have exponential factors in the first integrals and hence we must have $v(a-v^2) \neq 0$. Therefore our previous assumption is reduced to $(a+3v^2)(a-8v^2/9)(a-3v^2/4) = 0$. Suppose first that the point (a,v) is located on the parabola $a = -3v^2$. Then this parabola can be written as $a = (1-r^2)v^2$ where $r = 2$. We then have $v^2 - a = r^2v^2 = 4v^2 > 0$. If the point (a,v) is on the parabola $a - 8y^2/9 = 0$ then this parabola can be written as $a = (1-r^2)v^2$ for $r = 1/3$. Here again we have that $v^2 - a = r^2v^2 = v^2/9 > 0$. So the system situated on the parabola $a - 8y^2/9 = 0$ satisfies $v^2 - a > 0$ and for $r = 1/3$ the point is located on the parabola $a = (1-r^2)v^2$. So also in this case these conditions are necessary. There remains only the case when (a,v) is on the parabola $a - 3v^2/4 = 0$. In this case we can write this parabola as $a = (1-r^2)v^2$ by taking $r = 1/2$. Also here $v^2 - a = r^2v^2 = v^2/4 > 0$, i.e. $v^2 - a > 0$. So the necessity of the conditions is proved in this case too.

We now prove the sufficiency of the conditions. Let us assume that $v^2 - a > 0$, $v \neq 0$ and (a,v) is located on a parabola $a = (1-r^2)v^2$ with $r \in \mathbb{Q}$. Then clearly $r \neq 0$, otherwise $v^2 - a = r^2v^2 = 0$ contrary to our assumption. In case $r = 2, 1/3, 1/2$ we are on one of the three parabolas obtained from the condition $(a+3v^2)(a-8v^2/9)(a-3v^2/4) = 0$ and for these parabolas the tables give us rational first integrals. If the generic condition is satisfied, i.e. $v(a-v^2)(a+3v^2)(a-8v^2/9)(a-3v^2/4) \neq 0$, then we know that we have the corresponding first integral indicated in the Tables for this case where the exponents for the curves J_i are λ_1 and $\lambda_1\sqrt{v^2-a}/v$. But we know by our assumption that (a,v) is located on a parabola $a = (1-r^2)v^2$ for some rational number r. From this equation we have that $r^2 = (a-v^2)/v^2$.

Hence $r = \sqrt{v^2 - a}/v$ is rational. We may suppose $r = m/n$ with $m, n \in \mathbb{Z}$ and m, n coprime. Then by taking in the general expression of the first integral $\lambda_1 = n$ and $r = \sqrt{v^2 - a}/v$ we obtained a rational first integral in this case.

ii) Let us denote by $\mathcal{P}_r$ the parabola corresponding to a rational number r, i.e.

$$\mathcal{P}_r := \{(a, v) \in \mathbb{R}^2 : (1 - r^2)v^2 = a\}.$$

Thus a system in the family (1.10) has a rational first integral if and only if it corresponds to a point (a, v) such that $v^2 - a > 0$ with $v \neq 0$ and the point is situated on a parabola $\mathcal{P}_r$ for some rational number r. In the parameter plane $\mathbb{R}^2$ let the a-axis be the horizontal line and the v-axis be the vertical one. The parabolas $a = (1 - r^2)v^2$ are symmetric with respect to the a-axis. Because of this it would suffice to prove the density of points (a, v) on parabolas $\mathcal{P}_r$ and inside $v^2 - a > 0$ and $v > 0$.

Claim: The set of all points in $A =: \cup_{r \in \mathbb{Q}} \mathcal{P}_r$ with $v > 0$ is dense in the set $S^+ = \{(a, v) : v^2 - a > 0,\ v > 0\}$.

Take an arbitrary point point $p_0 = (a_0, v_0) \in S^+$. So we have $v_0^2 - a_0 > 0$ and $v_0 > 0$. We only need to consider p_0 inside the first or second quadrant. Indeed the line $a = 0$ is outside the parameter space of our family. So $a_0 \neq 0$. In view of our assumption we have that $(v_0^2 - a_0)/v_0^2 > 0$. So let $r_0 = \sqrt{(v_0^2 - a_0)/v_0^2} > 0$. Hence we have $a_0 = (1 - r_0^2)v_0^2$. Here r_0 is not necessarily a rational number. But it can be approximated with rational numbers. So take a sequence of rational numbers r_n such that $r_n \to r_0$. At this point let us assume that the point (a_0, v_0) is in the second quadrant, i.e. $a_0 < 0$. In this case $r_0 > 1$ and since $r_n \to r_0$ there exists a number N such that for $n > N$ $r_n > 1$ and hence $r_n^2 > 1$ for all $n > N$. So $\sqrt{a_0/(1 - r_n^2)} > 0$. Denote by $v_n = \sqrt{a_0/(1 - r_n^2)}$. Then $v_n \to v_0$ and hence $(a_0, v_n) \to (a_0, v_0)$. But each point (a_0, v_n) is located on the corresponding parabola $a_0 = (1 - r_n^2)v_n^2$ and hence p_0 is an accumulation point of points situated on such parabolas with r_n rational. Assume now that the point p_0 is in the first quadrant. Then $a_0 > 0$ and since $(v_0^2 - a_0)/v_0^2 > 0$ we have that $0 < r_0 = \sqrt{1 - a_0/v_0^2} < 1$ which means that there exists a natural number N such that for $n > N$ we have $0 < r_n < 1$ and hence $r_n^2 < 1$ and hence we can take again $v_n = \sqrt{a_0/(1 - r_n^2)}$. Then clearly $v_n \to v_0$ and we obtain a sequence of points (a_0, v_n) sitting on parabolas $a_0 = (1 - r_n^2)v_n^2$ with r_n rational. And $v_0^2 - a_0 = r_n^2 > 0$. Since $v_0 > 0$ then there is a natural number M such that for all $n > M$ $v_n > 0$. $\square$

Considering $r = m_1/m_2$ where $m_1, m_2 \in \mathbb{Z}$ we can say that

$$I = \left(\frac{J_1}{J_2}\right)^{m_2} \left(\frac{J_3}{J_4}\right)^{m_1}$$

is a rational first integral of (1.10) when $a = (1 - (m_1/m_2)^2)v^2$. Consider

$$\mathcal{F}_{(c_1,c_2)} = c_1 J_1^{m_2} J_3^{m_1} - c_2 J_2^{m_2} J_4^{m_1} = 0.$$

We have that $[1 : 0]$ and $[0 : 1]$ are remarkable values for $\mathcal{I}$, since

$$\mathcal{F}_{(1,0)} = J_1^{m_2} J_3^{m_1}, \qquad \mathcal{F}_{(0,1)} = -J_2^{m_2} J_4^{m_1}.$$

The case $m_1 = m_2 = 1$ is when $a = 0$ and this case was done previously. Suppose $m_1 \neq 1$ or $m_2 \neq 1$. If $m_1 > 1$ then $[1 : 0]$ and $[0 : 1]$ are the only two critical remarkable values for $\mathcal{I}$ and J_3, J_4 are critical remarkable curves. If we also have $m_2 > 1$ then J_1, J_2 also are critical remarkable curves.

There are some additional remarkable curves when $a = (1 - (m_1/m_2)^2)v^2$ for especial values of m_1 and m_2, see examples in the Appendix. We could find among these examples curves of degree $5, 6, 7, 8, 10, 12$ etc.

4 Geometric analysis of family (1.11)

Consider the family

$$(1.11)\quad \begin{cases} \dot{x} = -\dfrac{x^2}{2} - \dfrac{xy}{2} \\ \dot{y} = b - \dfrac{3xy}{2} + \dfrac{y^2}{2}, \quad \text{where } b \neq 0. \end{cases}$$

This is a one parameter family depending on $b \in \mathbb{R}\backslash\{0\}$. Every system in the family (1.11) is endowed with five invariant algebraic curves: three lines J_1, J_2, J_3 and two hyperbolas J_4, J_5 with respective cofactors α_i, $1 \leq i \leq 5$ where

$$\begin{aligned} J_1 &= -i\sqrt{2}\sqrt{b} - x + y, & \alpha_1 &= \tfrac{i\sqrt{b}}{\sqrt{2}} - \tfrac{x}{2} + \tfrac{y}{2}, \\ J_2 &= i\sqrt{2}\sqrt{b} - x + y, & \alpha_2 &= -\tfrac{i\sqrt{b}}{\sqrt{2}} - \tfrac{x}{2} + \tfrac{y}{2}, \\ J_3 &= x, & \alpha_3 &= -\tfrac{x}{2} - \tfrac{y}{2}, \\ J_4 &= x(y-x) - b, & \alpha_4 &= -x, \\ J_5 &= xy - \tfrac{b}{2}, & \alpha_5 &= -2x. \end{aligned}$$

Straightforward calculations lead us to the tables listed below. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using for lines the 1st and for hyperbola the 2nd extactic polynomial, respectively.

(i) $b \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -i\sqrt{2}\sqrt{b} - x + y$ $J_2 = i\sqrt{2}\sqrt{b} - x + y$ $J_3 = x$ $J_4 = x(y-x) - b$ $J_5 = xy - \frac{b}{2}$ $\alpha_1 = \frac{i\sqrt{b}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}$ $\alpha_2 = -\frac{i\sqrt{b}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}$ $\alpha_3 = -\frac{x}{2} - \frac{y}{2}$ $\alpha_4 = -x$ $\alpha_5 = -2x$	$P_1 = \left(\frac{i\sqrt{b}}{\sqrt{2}}, -\frac{i\sqrt{b}}{\sqrt{2}}\right)$ $P_2 = \left(-\frac{i\sqrt{b}}{\sqrt{2}}, \frac{i\sqrt{b}}{\sqrt{2}}\right)$ $P_3 = \left(0, -i\sqrt{2}\sqrt{b}\right)$ $P_4 = \left(0, i\sqrt{2}\sqrt{b}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ For $b < 0$ we have $n, n, s, s; N, N, S$ For $b > 0$ we have ©, ©, ©, ©; N, N, S	$J_1 \cap J_2 = P_2^\infty$ simple $J_1 \cap J_3 = P_4$ simple $J_1 \cap J_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $J_1 \cap J_5 = P_2$ double $J_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $J_2 \cap J_3 = P_3$ simple $J_2 \cap J_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $J_2 \cap J_5 = P_1$ double $J_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $J_3 \cap J_4 = P_1^\infty$ double $J_3 \cap J_5 = P_1^\infty$ double $J_3 \cap \mathcal{L}_\infty = P_1^\infty$ simple $J_4 \cap J_5 = \begin{cases} P_1, P_2 \text{ simple} \\ P_1^\infty \text{ double} \end{cases}$ $J_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $J_5 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty \text{ if } b < 0 \\ J_1^C + J_2^C + J_3 + J_4 + J_5 + \mathcal{L}_\infty \text{ if } b > 0 \end{cases}$	6 6
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } b < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty \text{ if } b > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = \begin{cases} 3P_1 + 3P_2 + 2P_3 + 2P_4 + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } b < 0 \\ 4P_1^\infty + 4P_2^\infty + 2P_3^\infty \text{ if } b < 0 \end{cases}$	20 10

where the total curve T has

1) only two distinct tangents at P_1 (and at P_2), but one of them is double,

2) only three distinct tangents at P_1^∞, but one of them is double and

3) four distinct tangents at P_2^∞.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{2\lambda_1} J_4^{\lambda_4} J_5^{-\lambda_1-\frac{\lambda_4}{2}}$	$R = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{1+2\lambda_1} J_4^{\lambda_4} J_5^{-\lambda_1-\frac{\lambda_4}{2}-\frac{3}{2}}$
Simple example	$\mathcal{I}_1 = \frac{J_4^2}{J_5}, \quad \mathcal{I}_2 = \frac{J_1 J_2 J_3^2}{J_5}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_3 J_4}$

Remark 4.1.

- Consider $\mathcal{F}^1_{(c_1,c_2)} = c_1 J_4^2 - c_2 J_5 = 0$, $\deg \mathcal{F}^1_{(c_1,c_2)} = 4$. The remarkable values of $\mathcal{F}^1_{(c_1,c_2)}$ are $[1 : -2b]$ and $[1 : 0]$ for which we have

$$\mathcal{F}^1_{(1,-2b)} = J_1 J_2 J_3^2, \qquad \mathcal{F}^1_{(1,0)} = J_4^2.$$

  Therefore, J_1, J_2, J_3, J_4 are remarkable curves of $\mathcal{I}_1$, $[1 : -2b]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathcal{I}_1$ and J_3, J_4 are critical remarkable curves of $\mathcal{I}_1$. The singular points are P_3, P_4 for $\mathcal{F}^1_{(1,-2b)}$ and P_1, P_2 for $\mathcal{F}^1_{(1,0)}$.

- Consider $\mathcal{F}^2_{(c_1,c_2)} = c_1 J_1 J_2 J_3^2 - c_2 J_5 = 0$, $\deg \mathcal{F}^2_{(c_1,c_2)} = 4$. The remarkable values of $\mathcal{F}^2_{(c_1,c_2)}$ are $[1 : 2b]$ and $[1 : 0]$ for which we have

$$\mathcal{F}^2_{(1,2b)} = J_4^2, \qquad \mathcal{F}^2_{(1,0)} = J_1 J_2 J_3^2.$$

  Therefore, J_1, J_2, J_3, J_4 are remarkable curves of $\mathcal{I}_2$, $[1 : 2b]$ and $[1 : 0]$ are the only two critical remarkable values of $\mathcal{I}_2$ and J_3, J_4 are critical remarkable curves of $\mathcal{I}_2$. The singular points are P_1, P_2 for $\mathcal{F}^2_{(1,2b)}$ and P_3, P_4 for $\mathcal{F}^2_{(1,0)}$.

(ii) $b = 0$.

Under this condition, the system (1.11) does not belong to **QSH**, but we study it seeking a complete understanding of the bifurcation diagram of the system in the full family

(1.11). Here we have a single system which has a rational first integral that foliates the plane into cubic curves. All the affine invariant lines are $x = 0$, $y = 0$ that are simple and $x - y = 0$ that is double. The lines $x = 0$ and $x - y = 0$ are remarkable curves. Perturbing this system in the family (1.11) we can obtain two distinct configurations of lines and hyperbolas. By perturbing the reducible conics $x(x - y) = 0$ and $xy = 0$ we obtain the hyperbolas $x(y - x) - b = 0$ and $xy - \frac{b}{2} = 0$, respectively.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $E_4 = e^{\frac{g_0+g_1(x-y)}{x-y}}$ $\alpha_1 = \frac{y}{2} - \frac{3x}{2}$ $\alpha_2 = -\frac{x}{2} - \frac{y}{2}$ $\alpha_3 = \frac{y}{2} - \frac{x}{2}$ $\alpha_4 = \frac{g_0}{2}$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:1:0]$ $P_3^\infty = [1:0:0]$ $hpphpp_{(4)}; N, N, S$	$\overline{J}_1 \cap \overline{J}_2 = P_1$ simple $\overline{J}_1 \cap \overline{J}_3 = P_1$ simple $\overline{J}_1 \cap \mathcal{L}_\infty = P_3^\infty$ simple $\overline{J}_2 \cap \overline{J}_3 = P_1$ simple $\overline{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\overline{J}_3 \cap \mathcal{L}_\infty = P_2^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\overline{J}_1\overline{J}_2\overline{J}_3^2 = 0.$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$	11

where the total curve T has

1) only three distinct tangents at P_1, but one of them is double;

2) only two distinct tangents at P_2^∞, but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{-2\lambda_1} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-3-2\lambda_1} E_4^0$
Simple example	$\mathcal{I}_1 = \frac{J_1}{J_2 J_3^2}, \quad \mathcal{I}_2 = \frac{J_2 J_3^2}{J_1}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_3}$

Remark 4.2. Consider $\mathcal{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_2 J_3^2 = 0$, $\deg \mathcal{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathcal{F}^1_{(c_1,c_2)}$ is $[0:1]$ for which we have

$$\mathcal{F}^1_{(0,1)} = -J_2 J_3^2.$$

Therefore, J_2, J_3 are remarkable curves of $\mathcal{I}_1$, $[1:0]$ is the only critical remarkable values of $\mathcal{I}_1$ and J_3 is critical remarkable curve of $\mathcal{I}_1$. The singular point is P_1 for $\mathcal{F}^1_{(0,1)}$. Considering the first integral $\mathcal{I}_2$ with its associated curves $\mathcal{F}^2_{(c_1,c_2)} = c_1 J_2 J_3^2 - c_2 J_1$ we have the remarkable value $[1:0]$ and the same remarkable curves J_2, J_3. The singular point is P_1 for $\mathcal{F}^2_{(1,0)}$.

We sum up the topological, dynamical and algebraic geometric features of family (1.11) and we also confront our results with previous results in the literature in the following proposition. We show that there are two more phase portraits than the ones appearing in [17] and there is one more phase portrait than the ones appearing in [10].

Proposition 4.3.

(a) For the family (1.11) we have two distinct configurations $C_1^{(1.11)}$ and $C_2^{(1.11)}$ of invariant hyperbolas and lines (see Figure 4.1 for the complete bifurcation diagram of configurations of such family). The bifurcation set in the full parameter space contains only the point $b = 0$.

(b) All systems in family (1.11) have an inverse integrating factor which is polynomial. All systems in family (1.11) have a rational first integral and the plane is foliated into quartic algebraic curves. The remarkable curves are J_1, J_2, J_3, J_4 for family (1.11).

(c) For the family (1.11) we have two topologically distinct phase portraits $P_1^{(1.11)}$ and $P_2^{(1.11)}$. The topological bifurcation diagram in the full parameter space is done in Figure 4.2. The bifurcation set of singularities is the point $b = 0$. The phase portraits $P_1^{(1.11)}$ and $P_2^{(1.11)}$ are not topologically equivalent with anyone of the phase portraits in [17].

Proof.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (1.11):

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(1.11)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 2P_3 + 2P_4 + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty$
$C_2^{(1.11)}$	$ICD = J_1^C + J_2^C + J_3 + J_4 + J_5 + \mathcal{L}_\infty$ $M_{0CT} = 4P_1^\infty + 4P_2^\infty + 2P_3^\infty$

Therefore, the configurations $C_1^{(1.11)}$ and $C_2^{(1.11)}$ are distinct. For the limit case of family (1.11) we have the following configuration:

Configuration	Divisors and zero-cycles of the total inv. curve T
c_2	$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$

(b) It follows directly from Jouanolou's theorem that we always have a rational first integral for family (1.11) since we have five invariant algebraic curves. The computations for the remarkable curves were done in Remark 4.1.

(c) We have that:

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(1.11)}$	(N,N,S)	(n,s,s,n)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(1.11)}$	(N,N,S)	$(©,©,©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$

Therefore, we have two distinct phase portraits for systems (1.11). For the limit case of family (1.11) we have the following phase portrait:

Phase Portrait	Sing. at ∞	Finite sing.	Separatrix connections
p_2	(N,N,S)	$hpphpp_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Note that $P_1^{(1.11)} \cong_{top} p_1$ and $P_2^{(1.11)} \cong_{top} P_3^{(1.10)}$. We saw in the study of the previous family that $P_3^{(1.10)}$ is not topologically equivalent with anyone if the phase portraits in [17].

On the table below we list the phase portraits of Llibre–Yu in [17] that admit 3 singular points at infinity with the type (N,N,S) and with 4 real singular points in the finite region.

Phase Portrait	Sing. at ∞	Real finite sing.	Separatrix connections
R_5	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
R_8, Ω_1	(N,S,N)	(s,n,n,s)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Therefore, the phase portraits $P_1^{(1.11)}$ is not topologically equivalent with anyone of the phase portraits in [17]. It is however a phase portrait of systems possessing an invariant line and an invariant hyperbola. □

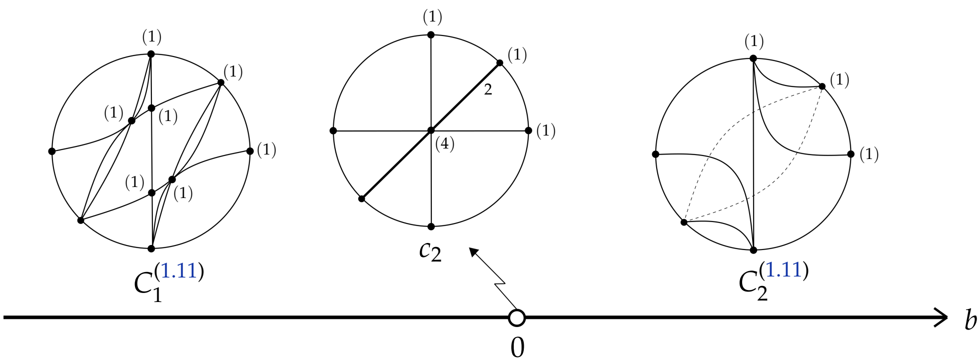

Figure 4.1: Bifurcation diagram of configurations for family (1.11). At $b = 0$ the two hyperbolas become reducible into the lines $x = 0$, $x - y = 0$ and $x = 0$, $y = 0$.

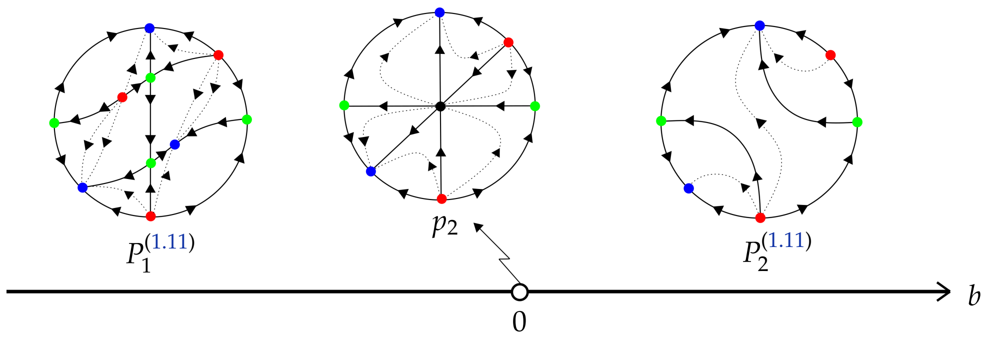

Figure 4.2: Topological bifurcation diagram for family (1.11). The only bifurcation point is at $b = 0$ where all 4 singularities (real on the left or complex on the right) coalesce with $(0, 0)$.

5 Geometric analysis of family (1.12)

Consider the family

$$(1.12)\quad \begin{cases} \dot{x} = 2a + gx^2 + xy \\ \dot{y} = a(2g-1) + (g-1)xy + y^2, \quad \text{where } a(g-1) \neq 0. \end{cases}$$

This is a two parameter family depending on $(a, g) \in \mathbb{R}^2 \setminus \{(0, 1)\}$. Every system in the family (1.12) is endowed with at least one invariant hyperbola J_1 with cofactor α_1 given by

$$J_1 = a + xy, \qquad \alpha_1 = (-1 + 2g)x + 2y.$$

Except for a denumerable set of lines in the parameter space, i.e. except for

$$L_k : 2g - k = 0, \qquad k \in \mathbb{N} = \{0, 1, 2, \dots\} \quad \text{and} \quad L : 4g - 1 = 0,$$

systems in (1.12) are not Liouvillian integrable (see [19]). It thus remains to be shown what happens on these lines and we consider here the case L_1 and L.

Straightforward calculations lead us to the tables listed below. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using for lines the 1st and for hyperbola the 2nd extactic polynomial, respectively.

(i) $ag(g-1)(2g-1)(4g-1) \neq 0$.

In [19] it is proved that except for the denumerable set of lines $\cup_{k\in\mathbb{N}} L_k \cup L$,

$$L_k = \{(a, g) \in \mathbb{R}^2 \setminus \{(0, 1)\} : 2g - k = 0\}, \qquad k \in \mathbb{N},$$

$$L = \{(a, g) \in \mathbb{R}^2 \setminus \{(0, 1)\} : 4g - 1 = 0\}$$

systems (1.12) are neither Darboux nor Liouvillian integrable. We prove below that when $(a, g) \in L_1$ systems (1.12) are generalized Darboux integrable and when $(a, g) \in L$ systems (1.12) are Liouvillian integrable. The cases where $(a, g) \in \cup_{k\in\mathbb{N}} L_k - L_1$ are still open. For these cases we were not able to prove the non-integrability and we also could not find other invariant algebraic curves, which we managed to search up to degree four. Although we are unable to guarantee the existence of a first integral in $\cup_{k\in\mathbb{N}} L_k - L_1$, it is still possible to obtain the complete topological bifurcation diagram of this family.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = a + xy$ $\alpha_1 = (-1+2g)x + 2y$	$P_1 = \left(-2i\sqrt{a}, i\left(2\sqrt{a}g - \sqrt{a}\right)\right)$ $P_2 = \left(2i\sqrt{a}, -i\left(2\sqrt{a}g - \sqrt{a}\right)\right)$ $P_3 = \left(-\frac{i\sqrt{a}}{\sqrt{g}}, -i\sqrt{a}\sqrt{g}\right)$ $P_4 = \left(\frac{i\sqrt{a}}{\sqrt{g}}, i\sqrt{a}\sqrt{g}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $a<0$ we have $f, f, ©, ©; \binom{0}{2}SN, S$ if $g<0$ $f, f, s, s; \binom{0}{2}SN, N$ if $0<g<\frac{7}{32}$ $n, n, s, s; \binom{0}{2}SN, N$ if $\frac{7}{32} \leq g < \frac{1}{4}$ $s, s, n, n; \binom{0}{2}SN, N$ if $g > \frac{1}{4}$ For $a>0$ we have $©, ©, n, n; \binom{0}{2}SN, S$ if $g<0$ $©, ©, ©, ©; \binom{0}{2}SN, N$ if $g>0$	$\bar{J}_1 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + \mathcal{L}_\infty$	2
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty \text{ if } a<0 \text{ and } g<0 \\ P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty \text{ if } a<0 \text{ and } g>0 \\ P_1^C + P_2^C + P_3 + P_4 + 2P_1^\infty + P_2^\infty \text{ if } a>0 \text{ and } g<0 \\ P_1^C + P_2^C + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty \text{ if } a>0 \text{ and } g>0 \end{cases}$	7 7 7 7
$T = Z\bar{J}_1 = 0$	3
$M_{0CT} = \begin{cases} 2P_1^\infty + 2P_2^\infty \text{ if } ag>0 \\ P_3 + P_4 + 2P_1^\infty + 2P_2^\infty \text{ if } ag<0 \end{cases}$	4 6

(ii) $ag(g-1)(2g-1)(4g-1) = 0$.

(ii.1) $g = 0$ and $a \neq 0$.

Under this condition, $(a,g) \in L_0$ which corresponds to an open case regarding the integrability.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = a + xy$ $\alpha_1 = -x + 2y$	$P_1 = \left(2i\sqrt{a}, -i\sqrt{a}\right)$ $P_2 = \left(2i\sqrt{a}, i\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $f, f; \binom{0}{2}SN, \binom{1}{2}S$ if $a < 0$ ©, ©; $\binom{0}{2}SN, \binom{1}{2}N$ if $a > 0$	$\bar{J}_1 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + \mathcal{L}_\infty$	2
$M_{0CS} = \begin{cases} P_1 + P_2 + 2P_1^\infty + 3P_2^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + 2P_1^\infty + 3P_2^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\overline{J_1} = 0$	3
$M_{0CT} = \begin{cases} P_1 + P_2 + 2P_1^\infty + 2P_2^\infty \text{ if } a < 0 \\ 2P_1^\infty + 2P_2^\infty \text{ if } a > 0 \end{cases}$	6 4

(ii.2) $g = \frac{1}{2}$ and $a \neq 0$.

Here we have an additional invariant line which is simple and the invariant hyperbola becomes double so we compute the exponential factor E_3.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = a + xy$ $E_3 = e^{-\frac{a(g_1-g_0)+g_1xy-2g_0y^2}{2(a+xy)}}$ $\alpha_1 = \frac{x}{2} + y$ $\alpha_2 = 2y$ $\alpha_3 = -g_0y$	$P_1 = \left(-2i\sqrt{a}, 0\right)$ $P_2 = \left(2i\sqrt{a}, 0\right)$ $P_3 = \left(-i\sqrt{2}\sqrt{a}, -\frac{i\sqrt{a}}{\sqrt{2}}\right)$ $P_4 = \left(i\sqrt{2}\sqrt{a}, \frac{i\sqrt{a}}{\sqrt{2}}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, s, n, n; \binom{0}{2}SN, N$ if $a < 0$ ©, ©, ©, ©; $\binom{0}{2}SN, N$ if $a > 0$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ double $\bar{J}_1 \cap \mathcal{L}_\infty =$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + 2J_2 + \mathcal{L}_\infty$	4
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2^2 = 0$	6
$M_{0CT} = \begin{cases} P_1 + P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty \text{ if } a < 0 \\ 3P_1^\infty + 4P_2^\infty \text{ if } a > 0 \end{cases}$	13 7

where the total curve T has

1) only two distinct tangents at P_1^∞, but one of them is double,

2) only two distinct tangents at P_2^∞, but one of them is triple.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} E_3^{\frac{2\lambda_2}{g_0}}$	$R = J_1^1 J_2^{\lambda_2} E_3^{\frac{2(2+\lambda_2)}{g_0}}$
Simple example	$\mathcal{I} = J_2 E_3^2$	$\mathcal{R} = \frac{J_1}{J_2^2}$

(ii.3) $g = \frac{1}{4}$ and $a \neq 0$.

Here the hyperbola becomes double so we compute the exponential factor E_2.

Inv. cur./exp. fac. and cofac.	Singularities	Intersection points
$J_1 = a + xy$ $E_2 = e^{\frac{ag_0 + g_0 xy + g_1 y^2}{(a+xy)}}$ $\alpha_1 = -\frac{x}{2} + 2y$ $\alpha_2 = -g_1 y$	$P_1 = \left(2i\sqrt{a}, -\frac{i\sqrt{a}}{2}\right)$ $P_2 = \left(2i\sqrt{a}, \frac{i\sqrt{a}}{2}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $sn_{(2)}, sn_{(2)}; \binom{0}{2}SN, N$ if $a < 0$ $©_{(2)}, ©_{(2)}; \binom{0}{2}SN, N$ if $a > 0$	$\bar{J}_1 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = 2J_1 + \mathcal{L}_\infty$	3
$M_{0CS} = \begin{cases} 2P_1 + 2P_2 + 2P_1^\infty + P_2^\infty \text{ if } a < 0 \\ 2P_1^C + 2P_2^C + 2P_1^\infty + P_2^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1^2 = 0$	5
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + 3P_1^\infty + 3P_2^\infty \text{ if } a < 0 \\ 3P_1^\infty + 3P_2^\infty \text{ if } a > 0 \end{cases}$	10 6

where the total curve T has only two distinct tangents at P_1^∞ (and P_2^∞), but one of them is double.

	First integral	Integrating Factor
General	$I = \frac{1}{2}\left(2\sqrt{2}aDawsonF\left(\frac{\sqrt{2}y}{\sqrt{a+xy}}\right) + x\sqrt{a+xy}\right)\left(e^{\frac{ag_0+g_0xy+g_1y^2}{a+xy}}\right)^{\frac{2}{g_1}}$	$R = J_1^{-\frac{1}{2}} E_2^{\frac{2}{g_1}}$
Simple example	$\mathcal{I} = \frac{1}{2}\left(2\sqrt{2}aDawsonF\left(\frac{\sqrt{2}y}{\sqrt{a+xy}}\right) + x\sqrt{a+xy}\right)\left(e^{\frac{y^2}{a+xy}}\right)^2$	$\mathcal{R} = J_1^{-\frac{1}{2}} E_2^2$

(ii.4) $g = 1$ and $a \neq 0$.

Here we have, apart from a simple hyperbola, two additional invariant lines (real or complex, depending on the sign of the parameter a).

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = 1 - \frac{iy}{\sqrt{a}}$ $J_2 = 1 + \frac{iy}{\sqrt{a}}$ $J_3 = a + xy$ $\alpha_1 = y - i\sqrt{a}$ $\alpha_2 = y + i\sqrt{a}$ $\alpha_3 = x + 2y$	$P_1 = \left(-i\sqrt{a}, -i\sqrt{a}\right)$ $P_2 = \left(i\sqrt{a}, i\sqrt{a}\right)$ $P_3 = \left(-2i\sqrt{a}, i\sqrt{a}\right)$ $P_4 = \left(2i\sqrt{a}, -i\sqrt{a}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $n, n, s, s; \binom{0}{2}SN, N$ if $a < 0$ ©, ©, ©, ©; $\binom{0}{2}SN, N$ if $a > 0$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_1 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + \mathcal{L}_\infty \text{ if } a < 0 \\ J_1^C + J_2^C + J_3 + \mathcal{L}_\infty \text{ if } a > 0 \end{cases}$	4 4
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty \text{ if } a < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty \text{ if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	5
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + P_3 + P_4 + 2P_1^\infty + 4P_2^\infty \text{ if } a < 0 \\ 2P_1^\infty + 4P_2^\infty \text{ if } a > 0 \end{cases}$	12 6

where the total curve T has four distinct tangents at P_2^∞.

	First integral	Integrating Factor
General	$I = \left(\sqrt{a+y^2}+y\right)^{-\frac{\sqrt{\frac{a+y^2}{a}}}{\sqrt{a+y^2}}} e^{\frac{\sqrt{\frac{a+y^2}{a}}(x-y)}{a+xy}}$	$R = J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} J_3^{-2}$
Simple example	$\mathcal{I} = \left(\sqrt{a+y^2}+y\right)^{-\frac{\sqrt{\frac{a+y^2}{a}}}{\sqrt{a+y^2}}} e^{\frac{\sqrt{\frac{a+y^2}{a}}(x-y)}{a+xy}}$	$\mathcal{R} = J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} J_3^{-2}$

(ii.5) $a = 0$ and $g \neq 0, 1$.

Under this condition, systems (1.12) do not belong to **QSH**, but we study them seeking a complete understanding of the bifurcation diagram of the systems in the full family (1.12). All the affine invariant lines are $y = 0$ that is simple and $x = 0$ that is double so we compute the exponential factor E_3. By perturbing the reducible conic $xy = 0$ we produce the hyperbola $a + xy = 0$.

Inv. cur./exp. fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $E_3 = e^{\frac{g_0 x + g_1 y}{x}}$ $\alpha_1 = (-1+g)x + y$ $\alpha_2 = gx + y$ $\alpha_3 = -g_1 y$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $epep_{(4)}; \binom{0}{2}SN, S$ if $g < 0$ $phph_{(4)}; \binom{0}{2}SN, N$ if $g > 0$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = J_1 + 2J_2 + \mathcal{L}_\infty$ if $g \neq 1$	4
$M_{0CS} = 4P_1 + 2P_1^\infty + P_2^\infty$ if $g \neq 0$	7
$T = Z\overline{J}_1\overline{J}_2^2 = 0$ if $g \neq 1$	3
$M_{0CT} = 3P_1 + 3P_1^\infty + 2P_2^\infty$ if $g \neq 0$	8

where the total curve T has only two distinct tangents at P_1^∞, but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\frac{(g-1)\lambda_1}{g}} E_3^{\frac{\lambda_1}{g_1 g}}$	$R = J_1^{\lambda_1} J_2^{-\frac{(g-1)\lambda_1}{g} - \frac{3g-1}{g}} E_3^{\frac{1+\lambda_1}{g_1 g}}$
Simple example	$\mathcal{I} = J_1^g J_2^{(1-g)} E_3$	$\mathcal{R} = \frac{1}{J_1 J_2^2}$

(ii.6) $a = 0$ and $g = 1$.

Under this condition, systems (1.12) do not belong to **QSH**, but we study them seeking a complete understanding of the bifurcation diagram of the systems in the full family (1.12). All the affine invariant lines are $y = 0$ and $x = 0$ that are double so we compute the exponential factor E_3 and E_4. By perturbing the reducible conic $xy = 0$ we produce the hyperbola $a + xy = 0$.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $E_3 = e^{\frac{g_0 x + g_1 y}{x}}$ $E_4 = e^{\frac{h_0 + h_1 y}{y}}$ $\alpha_1 = y$ $\alpha_2 = x + y$ $\alpha_3 = -g_1 y$ $\alpha_4 = -h_0$	$P_1 = (0, 0)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 0 : 0]$ $phph_{(4)}; \binom{0}{2}SN, N$	$\overline{J}_1 \cap \overline{J}_2 = P_1$ simple $\overline{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\overline{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = 2J_1 + 2J_2 + \mathcal{L}_\infty$	4
$M_{0CS} = 4P_1 + 2P_1^\infty + P_2^\infty$	7
$T = Z\overline{J}_1^2\overline{J}_2^2 = 0$	5
$M_{0CT} = 4P_1 + 3P_1^\infty + 3P_2^\infty$	10

where the total curve T has only two distinct tangents at P_1^∞ (and P_2^∞), but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^0 E_3^{\frac{\lambda_1}{g_1}} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2} E_3^{\frac{1+\lambda_1}{g_1}} E_4^0$
Simple example	$\mathcal{I} = J_1 E_3$	$\mathcal{R} = \frac{1}{J_1 J_2^2}$

(ii.7) $a = g = 0$.

Under this condition, systems (1.12) do not belong to **QSH**, but we study them seeking a complete understanding of the bifurcation diagram of the systems in the full family (1.12). The line $y = 0$ is filled up with singularities, therefore this is a degenerate system. The following study is done with the reduced system. For this system the line $x = 0$ is double so we compute the exponential factor E_2.

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $E_2 = e^{\frac{g_0 x + g_1 y}{x}}$ $\alpha_1 = 1$ $\alpha_2 = -g_1$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $(\ominus[\|]; n^d); \binom{0}{2}SN, (\ominus[\|]; \emptyset)$	$\bar{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = 2J_1 + \mathcal{L}_\infty$	3
$M_{0CS} = P_1 + 2P_1^\infty$	3
$T = Z\bar{J}_1^2 = 0$	3
$M_{0CT} = 2P_1 + 3P_1^\infty$	5

where the total curve T has only two distinct tangents at P_1^∞, but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{g_1 \lambda_2} E_2^{\lambda_2}$	$R = J_1^{-2+g_1\lambda_2} E_2^{\lambda_2}$
Simple example	$\mathcal{I} = J_1 E_2$	$\mathcal{R} = \frac{1}{J_1^2}$

We sum up the topological, dynamical and algebraic geometric features of family (1.12) and also confront our results with previous results in literature in the following proposition. We show that there are two more phase portraits than the ones appearing in [17].

Proposition 5.1.

(a) *For the family* (1.12) *we obtained seven distinct configurations* $C_1^{(1.12)} - C_7^{(1.12)}$ *of invariant hyperbolas and lines (see Figure 5.1 for the complete bifurcation diagram of configurations of this family). The bifurcation set of configurations in the full parameter space is* $ag(g-1)(g-1/2)(g-1/4) = 0$. *On* $(g-1/2)(g-1/4) = 0$ *the invariant hyperbola is double. On* $g = 1/2$ *we have an additional invariant line and on* $g = 1$ *we have two additional invariant lines. On* $g = 0$ *we just have a simple invariant hyperbola. On* $a = 0$ *the hyperbola becomes reducible into two lines and when* $a = g = 0$ *one of the lines is filled up with singularities.*

(b) *The family* (1.12) *is generalized Darboux integrable when* $g = 1/2$ *and it is Liouvillian integrable when* $g = 1/4$.

(c) *For the family* (1.12) *we have seven topologically distinct phase portraits* $P_1^{(1.12)} - P_7^{(1.12)}$. *The topological bifurcation diagram of family* (1.12) *is done in Figure 5.2. The bifurcation set are the half lines* $g = 1/4$ *and* $g = 1/2$ *with* $a < 0$ *and the lines* $g = 0$ *and* $a = 0$. *The half line* $g = 1/4$ *with* $a < 0$ *and the lines* $g = 0$, $a = 0$ *are bifurcation sets of singularities and the half line* $g = 1/2$ *with* $a < 0$ *is a bifurcation of saddle to saddle connection. The phase portraits* $P_4^{(1.12)}$ *and* $P_6^{(1.12)}$ *are not topologically equivalent with anyone of the phase portraits in [17].*

Proof.

(a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (1.12):

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(1.12)}$	$ICD = J_1 + \mathcal{L}_\infty$ $M_{0CT} = P_3 + P_4 + 2P_1^\infty + P_2^\infty$
$C_2^{(1.12)}$	$ICD = J_1 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1^\infty + P_2^\infty$
$C_3^{(1.12)}$	$ICD = J_1 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1^\infty + P_2^\infty$
$C_4^{(1.12)}$	$ICD = J_1 + 2J_2 + \mathcal{L}_\infty$ $M_{0CT} = P_1 + P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty$
$C_5^{(1.12)}$	$ICD = J_1 + 2J_2 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1^\infty + 4P_2^\infty$
$C_6^{(1.12)}$	$ICD = 2J_1 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 3P_1^\infty + 3P_2^\infty$
$C_7^{(1.12)}$	$ICD = 2J_1 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1^\infty + 3P_2^\infty$

Therefore, the configurations $C_1^{(1.12)}$ up to $C_7^{(1.12)}$ are all distinct. For the limit cases of family (1.12) we have the following configurations:

Configurations	Divisors and zero-cycles of the total inv. curve T
c_3	$ICD = J_1 + 2J_2 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 3P_1^\infty + 2P_2^\infty$
c_4	$ICD = 2J_1 + 2J_2 + \mathcal{L}_\infty$ $M_{0CT} = 4P_1 + 3P_1^\infty + 3P_2^\infty$
c_5	$ICD = 2J_1 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 3P_1^\infty$
c_6	$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + P_3 + P_4 + 2P_1^\infty + 4P_2^\infty$
c_7	$ICD = J_1^C + J_2^C + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1^\infty + 4P_2^\infty$

(b) This is shown in the previously exhibited tables.

(c) We have that:

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(1.12)}$	$(\binom{0}{2}SN, N)$	(n,n,s,s)	$2SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(1.12)}$	$(\binom{0}{2}SN, N)$	(s,s,n,n)	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_3^{(1.12)}$	$(\binom{0}{2}SN, S)$ $(\binom{0}{2}SN, \binom{1}{2}S)$	$(f,f,©,©)$ (f,f)	$0SC_f^f$ $2SC_f^\infty$ $2SC_\infty^\infty$
$P_4^{(1.12)}$	$(\binom{0}{2}SN, N)$ $(\binom{0}{2}SN, N)$ $(\binom{0}{2}SN, \binom{1}{2}N)$	$(©,©,©,©)$ $(©_{(2)},©_{(2)})$ $(©,©)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_5^{(1.12)}$	$(\binom{0}{2}SN, S)$	$(©,©,n,n)$	$0SC_f^f$ $2SC_f^\infty$ $0SC_\infty^\infty$
$P_6^{(1.12)}$	$(\binom{0}{2}SN, N)$	(s,s,n,n)	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_7^{(1.12)}$	$(\binom{0}{2}SN, N)$	$(sn_{(2)}, sn_{(2)})$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Therefore, we have seven distinct phase portraits for systems (1.12). For the limit cases of family (1.12) we have the following phase portraits:

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
p_3	$(\binom{0}{2}SN, N)$	$phph_{(4)}$	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$
p_4	$\binom{0}{2}SN, (\ominus[\|];\emptyset)$	$(\ominus[\|]; n^d)$	$0SC_f^f$ $2SC_f^\infty$ $0SC_\infty^\infty$
p_5	$(\binom{0}{2}SN, S)$	$epep_{(4)}$	$0SC_f^f$ $4SC_f^\infty$ $0SC_\infty^\infty$

On the table below we list all the phase portraits of Llibre-Yu in [17] that admit 2 singular points at infinity with the type (SN, N):

Phase Portrait	Sing. at ∞	Real finite sing.	Separatrix connections
L_{01}	(SN, N)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $3SC_\infty^\infty$
L_{03}	(SN, N)	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $3SC_\infty^\infty$
ω_1	(SN, N)	(s, n)	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

The phase portraits $P_4^{(1.12)}$ and $P_6^{(1.12)}$ are not topologically equivalent with anyone of the phase portraits in [17]. They are however phase portraits of systems possessing an invariant line and an invariant hyperbola (when $g = 1/2$). □

Remark 5.2. The family (1.12) does not have any case where the inverse integrating factor is polynomial. We just have a polynomial inverse integrating factor on the limit case $a = 0$ of family (1.12).

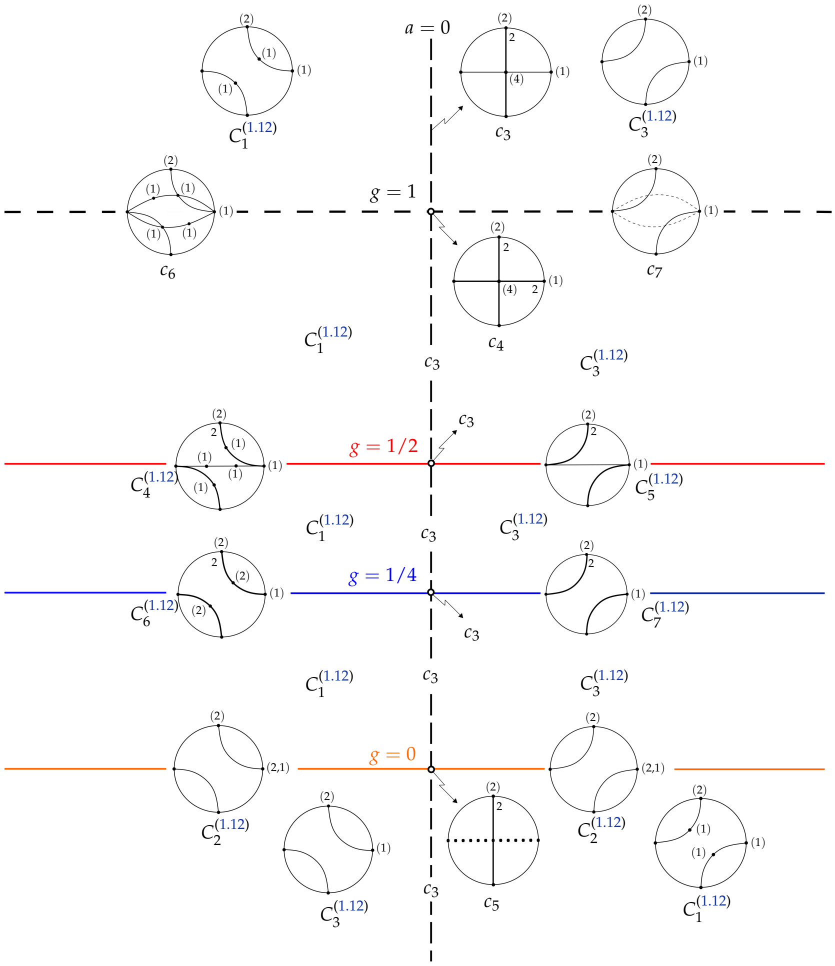

Figure 5.1: Bifurcation diagram of configurations for family (1.12): In this figure on the dashed line $a = 0$ the hyperbola becomes reducible into two lines $x = 0$ and $y = 0$. When $a = g = 0$ the line $y = 0$ is filled up with singularities. For the bifurcation curves we either have an additional line or coalescing hyperbolas or a change in the multiplicity of a infinity singularity. On the dashed line $g = 1$ we have two additional lines. The dashed lines represent complex lines.

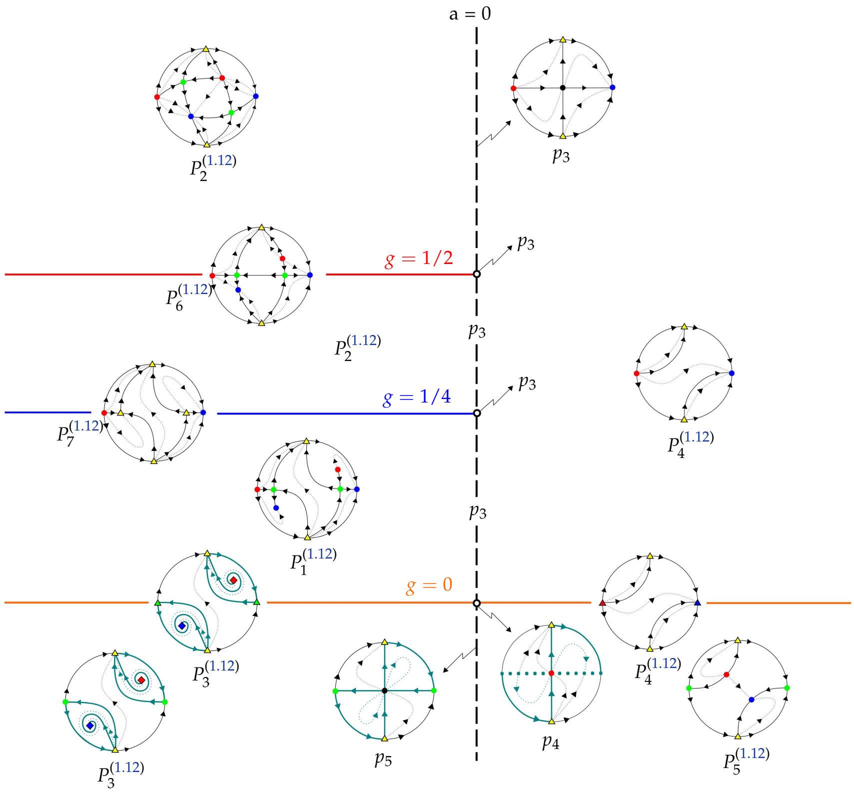

Figure 5.2: Topological bifurcation diagram for family (1.12). Note that the phase portraits p_4, p_5 and $P_3^{(1.12)}$ possess graphics in their first and third quadrant.

We have the following number of distinct configurations and phase portraits in the normals forms (1.10), (1.11) and (1.12), denoted here by NF studied, as well as their limit points:

Config. in the NF studied	All config.	Phase port. in the NF studied	All Phase port.
18	25	12	17

6 Questions, the problem of Poincaré and concluding comments

We are interested both in the algebraic-geometric properties of the systems in the family **QSH**, as expressed in their global geometric configurations of algebraic solutions, and on their impact on the integrability of the systems. We are also interested in the topological phase portraits of systems in **QSH**. This family is 3-dimensional modulo the action of the affine group of transformations and time rescaling (see [18]). As we have seen in the three families we discussed in this work, the class **QSH** forms a rich testing ground for exploring integrability in terms of the global algebraic geometric features of the systems occurring in these normal forms. The geometric analysis of the systems we studied bring out a number of questions. We expect to find answers to these questions, once the full study of all the normal forms of **QSH** will be completed.

6.1 The problem of Poincaré

For two out of the three families discussed in this work we have an answer to Poincaré's problem of algebraic integrability for each one of the systems in these families. The answer is given entirely in geometric terms (see Theorem 3.6 in Subsection 3.1) and all systems in the family (1.11) are algebraically integrable. This raises the following question: For how many of the remaining normal forms could we solve the problem of Poincaré for all the systems in the families defined by their respective normal forms? Could this problem be solved in geometrical terms as it was possible for the normal form (A)?

6.2 The problem of generalizing the Christopher–Kooij Theorem 1.15

We saw that under the "generic" conditions of Christopher and Kooij (C–K), formulated algebraically on the algebraic invariant curves $f_1(x,y), \dots, f_k(x,y)$ of a polynomial differential system, we are assured to have a polynomial inverse integrating factor of the special form

$$f_1(x,y)\dots f_k(x,y).$$

In this article we see cases where these "generic conditions" of (C–K) are not satisfied and yet we still have an integrating factor which is polynomial. Furthermore, in some cases, this polynomial inverse integrating factor is of the same form as the one in the (C–K) theorem. Here are some examples occurring in the families we considered.

(I) For the family (1.10).

(1) All the systems in family (1.10) have an inverse integrating factor which is polynomial, they are Darboux integrable and have in the generic case only two invariant lines J_1, J_2 and two invariant hyperbolas J_3, J_4. An inverse polynomial factor is $J_1J_2J_3J_4$ just like in C–K theorem. The condition (a) of the C–K theorem 1.15 is satisfied since our curves are lines and hyperbolas and they are, of course, non-singular and irreducible. The condition (b) is also satisfied since the coefficients in M_{0ST} are all equal to 2. The condition (c) is not satisfied because both of the hyperbolas J_3 and J_4 intersect the line at infinity at P_1^∞ and they are tangent at this point. The condition (d) is not satisfied because the sum of the degrees of the curves is 6 and not 3. However, the conclusion is the same as in theorem 1.15.

(2) In the non-generic cases $(a - 3v^2/4)(a + 3v^2)(a - 8v^2/9) = 0$ we have a similar situation and an inverse polynomial integrating factor. We have, as in the generic case, the two invariant lines J_1, J_2 and we have, apart from the two invariant hyperbolas J_4, J_5 and additional invariant curve J_3. We again have (a) and (b) satisfied but not (c) and (d) of (C–K) theorem 1.15. However, if we restrict our attention only to the remarkable curves J_1, J_2, J_4, J_5 then we still have an inverse integrating factor of the form $J_1J_2J_4J_5$ as in the (C–K) theorem.

(II) Consider now the family (1.11).

(1) The systems in the family (1.11) have in the generic case three invariant lines J_1, J_2, J_3 and two invariant hyperbolas J_4, J_5. Let us now consider for our discussion only the remarkable curves, the three lines J_1, J_2, J_3 and the hyperbola J_4. These of course satisfy the conditions (a) and (b). However they do not satisfy (c) because for example J_1, J_2, J_4 intersect at P_2. They also do not satisfy (d). If we limit our attention to the four curves J_1, J_2, J_3, J_4 we see that we have as an inverse integrating factor the polynomial $J_1J_2J_3J_4$ which we get by taking in the general expression of the integrating factor $\lambda_1 = \lambda_2 = \lambda_4 = -1$.

So we can ask the following questions:

Question 1: How should the geometry of the configuration of algebraic solutions $J_1, \ldots, J_k$ of a polynomial system be so as to have an inverse integrating factor which is polynomial? In particular, how should this geometry be in order to have an inverse integrating factor $J_1 \cdots J_k$? How could we relax, generalize, the hypotheses in the (C–K) theorem such that the same conclusion holds?

Question 2: If a system has a rational first integral do we always have an inverse integrating factor involving only remarkable curves?

Consider now the non-generic case $v(a - v^2) = 0$ of the family (1.10). We have that one of the invariant curves becomes double. In the case $v = 0$ we have two simple invariant lines J_1, J_2 and one double invariant hyperbola J_3. A polynomial inverse integrating factor in this case is $J_1 J_2 J_3^2$. In the case $a - v^2 = 0$ we have a double line J_1 and two simple hyperbolas J_2, J_3. We have a polynomial integrating factor $J_1^2 J_2 J_3$. In this case we still have a polynomial inverse integrating factor.

Question 3: Can we generalize the (C–K) Theorem 1.15 so as to include multiplicity? In what cases there is a relation between the multiplicity s of an algebraic solution J_i and the exponent of J_i appearing in the polynomial inverse integrating factor?

6.3 On the bifurcation diagrams

We have two kinds of bifurcation diagrams: topological and geometrical, i.e., of geometric configurations of algebraic solutions (lines and hyperbolas).

Question 1: What is the relation between these two kinds of bifurcation diagrams?

In all three families the topological bifurcation set of the phase portraits is a subset of the bifurcation set of configurations of algebraic solutions. This inclusion is strict for the first and last families.

The bifurcation set Bif_A for topological phase portraits in the family (A) is formed by the half-line of $v = 0$, $a < 0$ ($(\mathrm{Bif}_A)^{(1)}$); the non-zero points on the parabola $a = v^2$ ($(\mathrm{Bif}_A)^{(2)}$).

On $(\mathrm{Bif}_A)^{(1)}$ and on $(\mathrm{Bif}_A)^{(2)}$ 4 real finite singular points coalesce into 2 real finite double points. In the first case, after crossing the half-line they split again into 4 real singular points, while in the second case they split into 4 complex finite singular points which are finite points of intersection of the complexifications of each one of two real hyperbolas with the two complex invariant lines, respectively.

It is interesting to observe that these topological bifurcation points have an impact on the bifurcation set of geometrical configurations. Indeed, first we mention that above and below the half-line $v = 0$ and $a < 0$ we have two couples of real singularities, the points in each couple are located on distinct branches of one hyperbola. When two singular points on different hyperbolas coalesce this yields the coalescence of the respective branches and also of the two hyperbolas, producing a double hyperbola.

On the non-zero points of the parabola $a = v^2$ the coalescence of the 4 real finite singular points into two couples of double real singular points yields the coalescence of the two lines into a double real line which afterwards splits into two complex lines. In this case again we see that the topological bifurcation points produce also bifurcations in the configurations.

We note that we have a saddle to saddle connection on the parabola $a = v^2$ for $(a, v) \neq (0, 0)$.

On the bifurcation points situated on the remaining three parabolas we either have the occurrence of an additional hyperbola (on $a - 8v^2/9 = 0$ or on $a + 3v^2 = 0$) or the appearance of an additional invariant line (on $a - 3v^2/4 = 0$). The presence of these additional invariant curves does not affect in any way the bifurcation diagram of the systems.

In conclusion we have:

(i) Impact of the topological bifurcations on the bifurcations of configurations: The bifurcation points of singularities of the systems located on the algebraic solutions, when real singular points become multiple, become also bifurcation points for the multiplicity of the algebraic solutions, inducing coalescence of the respective curves and hence of their geometric configuration.

(ii) The bifurcation points of configurations due to the appearance of additional invariant curves (three hyperbolas instead of two or three lines instead of two lines) have no consequence for the topological bifurcation diagram of this family.

(iii) Inside the parabola $a = v^2$ i.e. for points (a, v) such that $v^2 - a < 0$ where we have complex singularities, we have no bifurcation points of phase portraits but we have, on the half-line $v = 0$, $a > 0$ bifurcation points of configurations, the two hyperbolas coalescing into a double hyperbola. Here we need to stress the fact that on this half-line we have two *double complex singularities* and while this fact has no impact on the topological bifurcation it is important for the bifurcations of the configurations. Indeed, when the four complex singularities become two double complex singularities on this half-line, the two hyperbolas on which they are lying coalesce becoming a double hyperbola.

Limit points of the bifurcation diagram for (A) Let us discuss the bifurcation phenomena which occur at the limiting points of our parameter space for systems in the family (A), i.e. the points on $a = 0$. The topological bifurcation on this line is easy to understand. Indeed, except for the the point $(0, 0)$ where all four singularities collide, all the other points on $a = 0$ are bifurcation points of saddle to saddle connections. All the points on the line $a = 0$ are also points of bifurcation of configurations of algebraic solutions. However this bifurcation is a bit harder to understand. Indeed, at these points say on $v > 0$ we have a configuration with three *simple* affine invariant lines, the vertical line intersecting the two parallel lines at two points and forming a saddle-to saddle connection. It is clear that this configuration splits into the configuration $C_1^{(1.10)}$ on the left which has two hyperbolas and two invariant lines. So in some sense the configuration on $a = 0$ should be considered as a *multiple configuration* since it yields new algebraic solutions. Analyzing the bifurcation phenomenon we see that each one of the two hyperbolas splits into two lines on $a = 0$ and $v > 0$. Indeed, the hyperbola J_4 splits into the line $x = 0$ and the line J_1 and the hyperbola J_3 splits into J_2 and $x = 0$. So that although for $a = 0$ each one of the lines is simple, each line contributes to the multiplicity of the configuration. Considering the composite cubic curve $xJ_1J_2 = 0$ we may say that this configuration has (geometric) multiplicity two in this family as it splits into two cubic curves J_1J_4 and J_2J_3. On the other hand we see that we have on $a = 0$ an exponential factor involving in its exponent at the denominator of the rational function, the polynomial xJ_1J_2 which turns out to be of integrable multiplicity two. The notions of integrable multiplicity and geometric multiplicity in [9] are not restricted to algebraic solutions. But the authors say there clearly that the equivalence between integrable and geometric multiplicities occurs only for integrable solutions. In the above family these two multiplicities coincide. So we have the following

Question: Under what condition on (finite) configurations of algebraic solutions do the two multiplicities, integrable and geometric coincide?

Finally we note that the point $(0,0)$ produces in perturbations all nine configurations which we encounter in the extension of the family (A), apart from the fact that we are interested in producing all the phase portraits of family **QSH** as well as fully understanding the integrability of this family, the questions raised above are additional motivation for completing the study of this family.

7 Appendix

Considering $r = m_1/m_2$ where $m_1, m_2 \in \mathbb{Z}$ we can say that

$$I = \left(\frac{J_1}{J_2}\right)^{m_2} \left(\frac{J_3}{J_4}\right)^{m_1}$$

is a rational first integral of (1.10) when $a = (1-(m_1/m_2)^2)v^2$. Consider

$$\mathcal{F}_{(c_1,c_2)} = c_1 J_1^{m_2} J_3^{m_1} - c_2 J_2^{m_2} J_4^{m_1} = 0.$$

We have the following:

- Taking $m_1 = 2$ and $m_2 = 4$ (i.e. $a = 3v^2/4$) we have that

  $$\mathcal{F}_{(1,1)} = -\tfrac{27}{16} v^3 y \left(81v^4 + 36v^2\left(-2x^2 + xy + y^2\right) + 16x(x-y)^3\right).$$

  Therefore, we have a line and a quartic as remarkable curves.

- Taking $m_1 = 2$ and $m_2 = 6$ (i.e. $a = 8v^2/9$) we have that

  $$\mathcal{F}_{(1,1)} = \tfrac{32}{9} v^3 \left(v^2 + 3y(y-x)\right)\left(3v^4(5x-8y) - 2v^2(x-y)^2(5x+4y) + 3x(x-y)^4\right).$$

  Therefore, we have a hyperbola and a quintic as remarkable curves.

- Taking $m_1 = 2$ and $m_2 = 8$ (i.e. $a = 15v^2/16$) we have that

  $$\begin{aligned}\mathcal{F}_{(1,1)} = {} & \tfrac{27}{262144} v^3 (36v^2x - 45v^2y - 80x^2y + 160xy^2 - 80y^3) \\ & (3645v^6 + 19440v^4x^2 - 58320v^4xy + 38880v^4y^2 - 11520v^2x^4 + 23040v^2x^3y \\ & - 23040v^2xy^3 + 11520v^2y^4 + 4096x^6 - 20480x^5y + 40960x^4y^2 - 40960x^3y^3 \\ & + 20480x^2y^4 - 4096xy^5).\end{aligned}$$

  Therefore, we have a cubic and a polynomial of degree 6 as remarkable curves.

- Taking $m_1 = 3$ and $m_2 = 6$ (i.e. $a = 3v^2/4$) we have that

  $$\begin{aligned}\mathcal{F}_{(1,1)} = {} & \tfrac{81}{512} v^3 y (6561v^8 - 1944v^6\left(6x^2 - 3xy - 5y^2\right) \\ & + 1296v^4(x-y)^2\left(6x^2 + 2xy + y^2\right) - 1152v^2x(x-y)^4(2x+y) + 256x^2(x-y)^6).\end{aligned}$$

  Therefore, we have a line and a polynomial of degree 8 as remarkable curves.

- Taking $m_1 = 3$ and $m_2 = 9$ (i.e. $a = 8v^2/9$) we have that

$$\begin{aligned}\mathcal{F}_{(1,1)} = {} & \tfrac{16}{27}v^3\left(v^2 - 3xy + 3y^2\right)\left(64v^{10} + 675v^8x^2 - 2544v^8xy + 2112v^8y^2\right.\\ & - 900v^6x^4 + 2520v^6x^3y - 612v^6x^2y^2 - 2736v^6xy^3 + 1728v^6y^4\\ & + 570v^4x^6 - 2232v^4x^5y + 3420v^4x^4y^2 - 2760v^4x^3y^3 + 1530v^4x^2y^4\\ & - 720v^4xy^5 + 192v^4y^6 - 180v^2x^8 + 936v^2x^7y - 1836v^2x^6y^2 + 1440v^2x^5y^3\\ & + 180v^2x^4y^4 - 1080v^2x^3y^5 + 684v^2x^2y^6 - 144v^2xy^7 + 27x^{10} - 216x^9y\\ & \left.+ 756x^8y^2 - 1512x^7y^3 + 1890x^6y^4 - 1512x^5y^5 + 756x^4y^6 - 216x^3y^7 + 27x^2y^8\right).\end{aligned}$$

  Therefore, we have a hyperbola and a polynomial of degree 10 as remarkable curves.

- Taking $m_1 = 4$ and $m_2 = 2$ (i.e. $a = -3v^2$) we have that

$$\begin{aligned}\mathcal{F}_{(1,1)} = {} & 216v^3(9v^2 + xy)(405v^4x - 81v^4y - 45v^2x^3 + 63v^2x^2y - 18v^2xy^2\\ & + x^5 - 3x^4y + 3x^3y^2 - x^2y^3).\end{aligned}$$

  Therefore, we have a hyperbola and a quintic as remarkable curves.

- Taking $m_1 = 4$ and $m_2 = 6$ (i.e. $a = 5v^2/9$) we have that

$$\begin{aligned}\mathcal{F}_{(1,1)} = {} & -\tfrac{8}{81}v^3(100v^4 - 21v^2x^2 + 270v^2xy + 75v^2y^2 - 45x^3y + 90x^2y^2 - 45xy^3)\\ & (420v^6x + 300v^6y - 385v^4x^3 + 255v^4x^2y + 105v^4xy^2 + 25v^4y^3 + 105v^2x^5\\ & + 90x^4y^3 - 45x^3y^4 + 9x^2y^5).\end{aligned}$$

  Therefore, we have a quartic and a polynomial of degree 7 as remarkable curves.

- Taking $m_1 = 4$ and $m_2 = 8$ (i.e. $a = 3v^2/4$) we have that

$$\begin{aligned}\mathcal{F}_{(1,1)} = {} & -\tfrac{27}{2048}v^3y(81v^4 - 72v^2x^2 + 36v^2xy + 36v^2y^2 + 16x^4 - 48x^3y\\ & + 48x^2y^2 - 16xy^3)(6561v^8 - 11664v^6x^2 + 5832v^6xy + 17496v^6y^2\\ & + 7776v^4x^4 - 12960v^4x^3y + 3888v^4x^2y^2 + 1296v^4y^4 - 2304v^2x^6\\ & + 8064v^2x^5y - 9216v^2x^4y^2 + 2304v^2x^3y^3 + 2304v^2x^2y^4 - 1152v^2xy^5\\ & + 256x^8 - 1536x^7y + 3840x^6y^2 - 5120x^5y^3 + 3840x^4y^4 - 1536x^3y^5 + 256x^2y^6).\end{aligned}$$

  Therefore, we have a line, a quartic and a polynomial of degree 8 as remarkable curves.

- Taking $m_1 = 4$ and $m_2 = 12$ (i.e. $a = 8v^2/9$) we have that

$$\begin{aligned}\mathcal{F}_{(1,1)} = {} & \tfrac{64}{81}v^3(v^2 - 3xy + 3y^2)(15v^4x - 24v^4y - 10v^2x^3 + 12v^2x^2y + 6v^2xy^2\\ & - 8v^2y^3 + 3x^5 - 12x^4y + 18x^3y^2 - 12x^2y^3 + 3xy^4)(64v^{10} + 225v^8x^2\\ & - 1104v^8xy + 960v^8y^2 - 300v^6x^4 + 840v^6x^3y + 180v^6x^2y^2 - 1680v^6xy^3\\ & + 960v^6y^4 + 190v^4x^6 - 744v^4x^5y + 1140v^4x^4y^2 - 920v^4x^3y^3 + 510v^4x^2y^4\\ & - 240v^4xy^5 + 64v^4y^6 - 60v^2x^8 + 312v^2x^7y - 612v^2x^6y^2 + 480v^2x^5y^3\\ & + 60v^2x^4y^4 - 360v^2x^3y^5 + 228v^2x^2y^6 - 48v^2xy^7 + 9x^{10} - 72x^9y + 252x^8y^2\\ & - 504x^7y^3 + 630x^6y^4 - 504x^5y^5 + 252x^4y^6 - 72x^3y^7 + 9x^2y^8).\end{aligned}$$

  Therefore, we have a hyperbola, a quintic and a polynomial of degree 10 as remarkable curves.

- Taking $m_1 = 4$ and $m_2 = 16$ (i.e. $a = 15v^2/16$) we have that

$$\begin{aligned}\mathcal{F}_{(1,1)} = {} & \tfrac{27}{2199023255552} v^3(36v^2x - 45v^2y - 80x^2y + 160xy^2 - 80y^3) \\ & (3645v^6 + 19440v^4x^2 - 58320v^4xy + 38880v^4y^2 - 11520v^2x^4 + 23040v^2x^3y \\ & - 23040v^2xy^3 + 11520v^2y^4 + 4096x^6 - 20480x^5y + 40960x^4y^2 - 40960x^3y^3 \\ & + 20480x^2y^4 - 4096xy^5)(13286025v^{12} + 383582304v^{10}x^2 - 1029814560v^{10}xy \\ & + 661348800v^{10}y^2 + 293932800v^8x^4 - 3174474240v^8x^3y + 8406478080v^8x^2y^2 \\ & - 8465264640v^8xy^3 + 2939328000v^8y^4 - 418037760v^6x^6 + 2090188800v^6x^5y \\ & - 2090188800v^6x^4y^2 - 4180377600v^6x^3y^3 + 10450944000v^6x^2y^4 \\ & - 7942717440v^6xy^5 + 2090188800v^6y^6 + 291962880v^4x^8 - 1804861440v^4x^7y \\ & + 4830658560v^4x^6y^2 - 7431782400v^4x^5y^3 + 7431782400v^4x^4y^4 \\ & - 5202247680v^4x^3y^5 + 2601123840v^4x^2y^6 - 849346560v^4xy^7 + 132710400v^4y^8 \\ & - 94371840v^2x^{10} + 660602880v^2x^9y - 1887436800v^2x^8y^2 + 2642411520v^2x^7y^3 \\ & - 1321205760v^2x^6y^4 - 1321205760v^2x^5y^5 + 2642411520v^2x^4y^6 \\ & - 1887436800v^2x^3y^7 + 660602880v^2x^2y^8 - 94371840v^2xy^9 + 16777216x^{12} \\ & - 167772160x^{11}y + 754974720x^{10}y^2 - 2013265920x^9y^3 + 3523215360x^8y^4 \\ & - 4227858432x^7y^5 + 3523215360x^6y^6 - 2013265920x^5y^7 + 754974720x^4y^8 \\ & - 167772160x^3y^9 + 16777216x^2y^{10}).\end{aligned}$$

Therefore, we have a cubic, a polynomial of degree 6 and a polynomial of degree 12 as remarkable curves.

These computations suggest that the remarkable curves of algebraically integrable systems in the family (A) have an unbounded degree.

Acknowledgments

The first author is partially supported by FAPESP grant number 2019/21181–0 and by CNPq grant number 304766/2019–4. The second author is partially supported by the grant NSERC Grant RN000355. The third author was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

The authors are thankful to the referee for the corrections and comments he/she made.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 7, 1–24; https://doi.org/10.14232/ejqtde.2021.1.7 www.math.u-szeged.hu/ejqtde/

On the stochastic Allen–Cahn equation on networks with multiplicative noise

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Received 12 August 2020, appeared 25 January 2021
Communicated by Péter L. Simon

Abstract. We consider a system of stochastic Allen–Cahn equations on a finite network represented by a finite graph. On each edge in the graph a multiplicative Gaussian noise driven stochastic Allen–Cahn equation is given with possibly different potential barrier heights supplemented by a continuity condition and a Kirchhoff-type law in the vertices. Using the semigroup approach for stochastic evolution equations in Banach spaces we obtain existence and uniqueness of solutions with sample paths in the space of continuous functions on the graph. We also prove more precise space-time regularity of the solution.

Keywords: stochastic evolution equations, stochastic reaction-diffusion equations on networks, analytic semigroups, stochastic Allen–Cahn equation.

2020 Mathematics Subject Classification: 60H15, 35R60 (Primary); 35R02, 47D06 (Secondary).

1 Introduction

We consider a finite connected network, represented by a finite graph G with m edges $\mathsf{e}_1, \dots, \mathsf{e}_m$ and n vertices $\mathsf{v}_1, \dots, \mathsf{v}_n$. We normalize and parametrize the edges on the interval $[0, 1]$. We denote by $\Gamma(\mathsf{v}_i)$ the set of all the indices of the edges having an endpoint at v_i, i.e.,

$$\Gamma(\mathsf{v}_i) := \left\{ j \in \{1, \dots, m\} : \mathsf{e}_j(0) = \mathsf{v}_i \text{ or } \mathsf{e}_j(1) = \mathsf{v}_i \right\}.$$

Denoting by $\Phi := (\phi_{ij})_{n\times m}$ the so-called incidence matrix of the graph G, see Subsection 2.1 for more details, we aim to analyse the existence, uniqueness and regularity of solutions of

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the problem

$$
\begin{cases}
\dot{u}_j(t,x) = (c_j u_j')'(t,x) - p_j(x)u_j(x,t) \\
\qquad\qquad + \beta_j^2 u(x,t) - u(x,t)^3 \\
\qquad\qquad + g_j(t,x,u_j(t,x))\dfrac{\partial w_j}{\partial t}(t,x), & t \in (0,T],\ x \in (0,1),\ j = 1,\ldots,m, \\
u_j(t,\mathsf{v}_i) = u_\ell(t,\mathsf{v}_i) =: q_i(t), & t \in (0,T],\ \forall j,\ell \in \Gamma(\mathsf{v}_i),\ i = 1,\ldots,n, \\
[Mq(t)]_i = -\displaystyle\sum_{j=1}^{m} \phi_{ij}\mu_j c_j(\mathsf{v}_i)u_j'(t,\mathsf{v}_i), & t \in (0,T],\ i = 1,\ldots,n, \\
u_j(0,x) = \mathsf{u}_j(x), & x \in [0,1],\ j = 1,\ldots,m,
\end{cases}
\tag{1.1}
$$

where $\frac{\partial w_j}{\partial t}$ are independent space-time white noises. The reaction terms in (1.1) are classical Allen–Cahn nonlinearities $h_j(\eta) = -\eta^3 + \beta_j^2\eta$ with $\beta_j > 0$, $j = 1,\ldots,m$. Note that $h_j = -H_j'$ where $H_j(\eta) = \frac{1}{4}(\eta^2 - \beta_j^2)^2$ is a double well potential for each j with potential barrier height $\beta_j^4/4$. The diffusion coefficients g_j are assumed to be locally Lipschitz continuous and of linear growth. The coefficients of the linear operator satisfy standard smoothness assumptions, see Subsection 2.1, the matrix M satisfies Assumptions 2.7 and μ_j, $j = 1,\ldots,m$, are positive constants. The classical Allen–Cahn equation belongs to the class of phase field models and is a classical tool to model processes involving thin interface layers between almost homogeneous regions, see [3]. It is a particular case of a reaction-diffusion equation of bistable type and it can be used to study front propagations as in [7] . Effects due to, for example, thermal fluctuations of the system can be accounted for by adding a Wiener type noise in the equation, see [20].

While deterministic evolution equations on networks are well studied, see, [1,2,5,6,8–11,17, 18,25,29–31,34–38] which is, admittedly, a rather incomplete list, the study of their stochastic counterparts is surprisingly scarce despite their strong link to applications, see e.g. [12,13,44] and the references therein. In [12] additive Lévy noise is considered that is square integrable with drift being a cubic polynomial. In [14] multiplicative square integrable Lévy noise is considered but with globally Lipschitz drifts f_j and diffusion coefficients and with a small time dependent perturbation of the linear operator. Paper [13] treats the case when the noise is an additive fractional Brownian motion and the drift is zero. In [22] multiplicative Wiener perturbation is considered both on the edges and vertices with globally Lipschitz diffusion coefficient and zero drift and time-delayed boundary condition. Finally, in [21], the case of multiplicative Wiener noise is treated with bounded and globally Lipschitz continuous drift and diffusion coefficients and noise both on the edges and vertices.

In all these papers the semigroup approach is utilized in a Hilbert space setting and the only work that treats non-globally Lipschitz continuous drifts on the edges, similar to the ones considered here, is [12] but the noise is there additive and square-integrable. In this case, energy arguments are possible using the additive nature of the equation which does not carry over to the multiplicative case. Therefore, we use an entirely different tool set based on the semigroup approach for stochastic evolution equations in Banach spaces [39], or for the classical stochastic reaction-diffusion setting [32,33], see also, [15,16,19,41]. We are able to rewrite (1.1) in a form that fits into this framework. After establishing various embedding and isomorphy results of function spaces and interpolation spaces, we may use [33, Theorem 4.9] to prove our main existence and uniqueness result, Theorem 3.15, which guarantees existence and uniqueness of solutions with sample paths in the space of continuous functions on the graph, denoted by B in the paper (see Definition 3.4); that is, in the space of continuous functions that are continuous on the edges and also across the vertices. When the initial data is sufficiently regular, then Theorem 3.15 also yields certain space-time regularity of the solution.

The paper is organized as follows. In Section 2 we collect partially known semigroup results for the linear deterministic version of (1.1). In Subsection 3.1 we first recall an abstract result from [32, 33] regarding abstract stochastic Cauchy problems in Banach spaces. In order to utilize the abstract framework in our setting we prove various preparatory results in Subsection 3.2: embedding and isometry results are contained in Lemma 3.5, Lemma 3.6 and Corollary 3.7, and a semigroup result in Proposition 3.8. Subsection 3.3 contains our main results where we first consider the abstract stochastic Itô equation corresponding to a slightly more general version of (1.1). An existence and uniqueness result for the abstract stochastic Itô problem is contained in Theorem 3.13 followed by a space-time regularity result in Theorem 3.14. These are then applied to the Itô equation corresponding (1.1) to yield the main result of the paper, Theorem 3.15, concerning the existence, uniqueness and space-time regularity of the solution of (1.1).

2 Heat equation on a network

2.1 The system of equations

We consider a finite connected network, represented by a finite graph G with m edges $\mathsf{e}_1, \dots, \mathsf{e}_m$ and n vertices $\mathsf{v}_1, \dots, \mathsf{v}_n$. We normalize and parametrize the edges on the interval $[0,1]$.

The structure of the network is given by the $n \times m$ matrices $\Phi^+ := (\phi^+_{ij})$ and $\Phi^- := (\phi^-_{ij})$ defined by

$$\phi^+_{ij} := \begin{cases} 1, & \text{if } \mathsf{e}_j(0) = \mathsf{v}_i, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \phi^-_{ij} := \begin{cases} 1, & \text{if } \mathsf{e}_j(1) = \mathsf{v}_i, \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 1, \dots, n$ and $j = 1, \dots m$. We denote by $\mathsf{e}_j(0)$ and $\mathsf{e}_j(1)$ the 0 and the 1 endpoint of the edge e_j, respectively. We refer to [30] for terminology. The $n \times m$ matrix $\Phi := (\phi_{ij})$ defined by

$$\Phi := \Phi^+ - \Phi^-$$

is known in graph theory as *incidence matrix* of the graph G. Further, let $\Gamma(\mathsf{v}_i)$ be the set of all the indices of the edges having an endpoint at v_i, i.e.,

$$\Gamma(\mathsf{v}_i) := \left\{ j \in \{1, \dots, m\} : \mathsf{e}_j(0) = \mathsf{v}_i \text{ or } \mathsf{e}_j(1) = \mathsf{v}_i \right\}.$$

For the sake of simplicity, we will denote the values of a continuous function defined on the (parameterized) edges of the graph, that is of

$$f = (f_1, \dots, f_m)^\top \in (C[0,1])^m \cong C\left([0,1], \mathbb{R}^m\right)$$

at 0 or 1 by $f_j(\mathsf{v}_i)$ if $\mathsf{e}_j(0) = \mathsf{v}_i$ or $\mathsf{e}_j(1) = \mathsf{v}_i$, respectively, and $f_j(\mathsf{v}_i) := 0$ otherwise, for $j = 1, \dots, m$.

We start with the problem

$$\begin{cases} \dot{u}_j(t,x) = (c_j u_j')'(t,x) - p_j(x) u_j(t,x), & t > 0,\ x \in (0,1),\ j = 1, \dots, m, & (a) \\ u_j(t, \mathsf{v}_i) = u_\ell(t, \mathsf{v}_i) =: q_i(t), & t > 0,\ \forall j, \ell \in \Gamma(\mathsf{v}_i),\ i = 1, \dots, n, & (b) \\ [Mq(t)]_i = -\sum_{j=1}^m \phi_{ij} \mu_j c_j(\mathsf{v}_i) u_j'(t, \mathsf{v}_i), & t > 0,\ i = 1, \dots, n, & (c) \\ u_j(0,x) = \mathsf{u}_j(x), & x \in [0,1],\ j = 1, \dots, m & (d) \end{cases} \tag{2.1}$$

on the network. Note that $c_j(\cdot)$, $p_j(\cdot)$ and $u_j(t,\cdot)$ are functions on the edge e_j of the network, so that the right-hand side of $(2.1a)$ reads in fact as

$$(c_j u_j')'(t,\cdot) = \frac{\partial}{\partial x}\left(c_j \frac{\partial}{\partial x} u_j\right)(t,\cdot) - p_j(\cdot)u_j(t,\cdot), \qquad t \geq 0,\ j = 1,\dots,m.$$

The functions $c_1,\dots,c_m$ are (variable) diffusion coefficients or conductances, and we assume that

$$0 < c_j \in C^1[0,1], \qquad j = 1,\dots,m.$$

The functions $p_1,\dots,p_m$ are nonnegative, continuous functions, hence

$$0 \leq p_j \in C[0,1], \qquad j = 1,\dots,m.$$

Equation $(2.1b)$ represents the continuity of the values attained by the system at the vertices in each time instant, and we denote by $q_i(t)$ the common functions values in the vertice i, for $i = 1,\dots,n$ and $t > 0$.

In $(2.1c)$, $M := \left(b_{ij}\right)_{n\times n}$ is a matrix satisfying the following

Assumption 2.1. *The matrix $M = \left(b_{ij}\right)_{n\times n}$ is real, symmetric and negative semidefinite, $M \not\equiv 0$.*

On the left-hand-side, $[Mq(t)]_i$ denotes the ith coordinate of the vector $Mq(t)$. On the right-hand-side, the coefficients

$$0 < \mu_j, \quad j = 1,\dots,m$$

are strictly positive constants that influence the distribution of impulse happening in the ramification nodes according to the Kirchhoff-type law $(2.1c)$.

We now introduce the $n \times m$ *weighted incidence matrices*

$$\Phi_w^+ := (\omega_{ij}^+) \quad \text{and} \quad \Phi_w^- := (\omega_{ij}^-)$$

with entries

$$\omega_{ij}^+ := \begin{cases} \mu_j c_j(\mathsf{v}_i), & \text{if } \mathsf{e}_j(0) = \mathsf{v}_i, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \omega_{ij}^- := \begin{cases} \mu_j c_j(\mathsf{v}_i), & \text{if } \mathsf{e}_j(1) = \mathsf{v}_i, \\ 0, & \text{otherwise.} \end{cases}$$

With these notations, the Kirchhoff law $(2.1c)$ becomes

$$Mq(t) = -\Phi_w^+ u'(t,0) + \Phi_w^- u'(t,1), \qquad t \geq 0. \tag{2.2}$$

In equation $(2.1d)$ we pose the initial conditions on the edges.

2.2 Spaces and operators

We are now in the position to rewrite our system in form of an abstract Cauchy problem, following the concept of [31]. First we consider the (real) Hilbert space

$$E_2 := \prod_{j=1}^{m} L^2(0,1;\mu_j dx) \tag{2.3}$$

as the *state space* of the edges, endowed with the natural inner product

$$\langle u, v\rangle_{E_2} := \sum_{j=1}^{m} \int_0^1 u_j(x)v_j(x)\mu_j dx, \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \in E_2.$$

Observe that E_2 is isomorphic to $\left(L^2(0,1)\right)^m$ with equivalence of norms.

We further need the *boundary space* $\mathbb{R}^n$ of the vertices. According to (2.1b) we will consider such functions on the edges of the graph those values coincide in each vertex. Therefore we introduce the *boundary value operator*

$$L\colon\ (C[0,1])^m \subset E_2 \to \mathbb{R}^n$$

with

$$\begin{aligned} D(L) &= \left\{u \in (C[0,1])^m : u_j(\mathsf{v}_i) = u_\ell(\mathsf{v}_i),\ \forall j,\ell \in \Gamma(\mathsf{v}_i),\ i = 1,\dots,n\right\};\\ Lu &:= (q_1,\dots,q_n)^\top \in \mathbb{R}^n, \qquad q_i = u_j(\mathsf{v}_i) \quad \text{for some } j \in \Gamma(\mathsf{v}_i),\ i = 1,\dots,n. \end{aligned} \tag{2.4}$$

The condition $u(t,\cdot) \in D(L)$ for each $t > 0$ means that (2.1b) is for the function $u(\cdot,\cdot)$ satisfied.

On E_2 we define the operator

$$A_{\max} := \begin{pmatrix} \frac{d}{dx}\left(c_1 \frac{d}{dx}\right) - p_1 & & 0 \\ & \ddots & \\ 0 & & \frac{d}{dx}\left(c_m \frac{d}{dx}\right) - p_m \end{pmatrix} \tag{2.5}$$

with domain

$$D(A_{\max}) := \left(H^2(0,1)\right)^m \cap D(L). \tag{2.6}$$

This operator can be regarded as *maximal* since no other boundary condition except continuity is supposed for the functions in its domain.

We further define the so called *feedback operator* acting on $D(A_{\max})$ and having values in the boundary space $\mathbb{R}^n$ as

$$\begin{aligned} D(C) &= D(A_{\max});\\ Cu &:= -\Phi_w^+ u'(0) + \Phi_w^- u'(1), \end{aligned}$$

compare with (2.2).

With these notations, we can finally rewrite (2.1) in form of an abstract Cauchy problem. Define

$$\begin{aligned} A &:= A_{\max} \\ D(A) &:= \{u \in E_2 : u \in D(A_{\max}) \text{ and } MLu = Cu\}, \end{aligned} \tag{2.7}$$

see the definitions above. Using this, (2.1) becomes

$$\begin{cases} \dot{u}(t) = Au(t), & t > 0,\\ u(0) = \mathsf{u}, \end{cases} \tag{2.8}$$

with $\mathsf{u} = (\mathsf{u}_1,\dots,\mathsf{u}_m)^\top$.

2.3 Well-posedness of the abstract Cauchy problem

To prove well-posedness of (2.8) we define a bilinear form on the Hilbert space E_2 with domain

$$D(\mathfrak{a}) = V := \left(H^1(0,1)\right)^m \cap D(L). \tag{2.9}$$

as

$$\mathfrak{a}(u,v) := \sum_{j=1}^{m} \int_0^1 \mu_j c_j(x) u_j'(x) v_j'(x) dx + \sum_{j=1}^{m} \int_0^1 \mu_j p_j(x) u_j(x) v_j(x) dx - \langle Mq, r \rangle_{\mathbb{R}^n}, \tag{2.10}$$

where $Lu = q$ and $Lv = r$.

The next definition can be found e.g. in [40, Section 1.2.3].

Definition 2.2. From the form $\mathfrak{a}$ – using the Riesz representation theorem – we can obtain a unique operator $(B, D(B))$ in the following way:

$$\begin{aligned} D(B) &:= \{u \in V : \exists v \in E_2 \text{ s.t. } \mathfrak{a}(u,\varphi) = \langle v, \varphi \rangle_{E_2} \ \forall \varphi \in V\}, \\ Bu &:= -v. \end{aligned}$$

We say that the operator $(B, D(B))$ is *associated with the form* $\mathfrak{a}$.

In the following, we will claim that the operator associated with the form $\mathfrak{a}$ is $(A, D(A))$. Furthermore, we will state results regarding how the properties of $\mathfrak{a}$ and the matrix M carry on the properties of the operator A, obtaining the well-posedness of the abstract Cauchy-problem (2.8) on E_2 and even on L^p-spaces of the edges. The proofs of these statements combine techniques of [36] (where no p_j's on the right-hand-side of (2.1b) are considered) and techniques of [38] (where p_j's are considered for the heat equation but the matrix M is diagonal).

Proposition 2.3. *The operator associated to the form* $\mathfrak{a}$ (2.9)–(2.10) *is* $(A, D(A))$ *in* (2.7).

Proof. We can proceed similarly as in the proofs of [36, Lemma 3.4] and [38, Lemma 3.3]. □

Proposition 2.4. *The form* $\mathfrak{a}$ *is densely defined, continuous, closed and accretive, hence* $(A, D(A))$ *is densely defined, dissipative and sectorial. Furthermore,* $\mathfrak{a}$ *is symmetric, hence the operator* $(A, D(A))$ *is self-adjoint.*

Proof. The first three properties of $\mathfrak{a}$ (densely defined, continuous and closed) follow analogous to the proof of [38, Lemma 3.2]. Since M is dissipative (that is, negative semidefinite), and $p_j \geq 0$, $j = 1, \ldots, m$, the form $\mathfrak{a}$ is accretive, see the proofs of [36, Proposition 3.2] and [38, Lemma 3.2]. The symmetricity of $\mathfrak{a}$ follows from the fact that M is real and symmetric, see the proof of [36, Corollary 3.3]. The properties of A follow now by [40, Proposition 1.24, 1.51, Theorem 1.52]. □

As a corollary we obtain well-posedness of (2.8).

Proposition 2.5. *Assuming Assumption 2.1 on the matrix M, the operator* $(A, D(A))$ *defined in* (2.7) *generates a C_0 analytic, compact semigroup of contractions* $(T_2(t))_{t\geq 0}$ *on* E_2*. Hence, the abstract Cauchy problem* (2.8) *is well-posed on* E_2.

Proof. The claim follows from Proposition 2.4 and the fact that $(A, D(A))$ is resolvent compact. This is true since V is densely and compactly embedded in E_2 by the Rellich–Khondrakov Theorem, and we can use [24, Theorem 1.2.1]. □

In the following we will extend the semigroup $(T_2(t))_{t\geq 0}$ on L^p-spaces. To this end we define

$$E_p := \prod_{j=1}^{m} L^p(0,1;\mu_j dx), \qquad p \in [1,\infty]$$

and

$$\|u\|_{E_p}^p := \sum_{j=1}^{m} \|u_j\|_{L^p(0,1;\mu_j dx)}^p, \qquad u \in E_p, \quad p \in [1,\infty),$$

$$\|u\|_{E_\infty} := \max_{j=1,\dots,m} \|u_j\|_{L^\infty(0,1)}, \qquad u \in E_\infty.$$

We can characterize features of the semigroup $(T_2(t))_{t\geq 0}$ by those of $(\mathrm{e}^{tM})_{t\geq 0}$, the semigroup generated by the matrix M – hence, by properties of M. In particular, the following holds.

Proposition 2.6. *The semigroup $(T_2(t))_{t\geq 0}$ on E_2 associated with $\mathfrak{a}$ enjoys the following properties:*

- *$(T_2(t))_{t\geq 0}$ is positive if and only if the matrix M has positive off-diagonal – that is, if it generates a positive matrix semigroup $(\mathrm{e}^{tM})_{t\geq 0}$;*
- *Since M is negative semidefinite, the semigroup $(T_2(t))_{t\geq 0}$ is contractive on E_∞ if and only if*

$$b_{ii} + \sum_{k\neq i} |b_{ik}| \leq 0, \qquad i = 1,\dots,n,$$

that is $(\mathrm{e}^{tM})_{t\geq 0}$ is ℓ^∞-contractive.

Proof. It follows using analogous techniques as in the proof of [36, Theorem 3.5] and [38, Lemma 4.1, Proposition 5.3] □

To obtain the desired extension of the semigroup on L^p-spaces, we assume the following on the matrix M.

Assumption 2.7. *For the matrix $M = \left(b_{ij}\right)_{n\times n}$ we assume the following properties:*

1. *M satisfies Assumption 2.1;*
2. *For $i \neq k$, $b_{ik} \geq 0$, that is, M has positive off-diagonal;*
3. $$\sum_{k\neq i} b_{ik} \leq -b_{ii}, \quad i = 1,\dots,n,$$
   that is, the matrix is diagonally dominant.

Proposition 2.8. *If M satisfies Assumptions 2.7 then the semigroup $(T_2(t))_{t\geq 0}$ extends to a family of compact, contractive, positive one-parameter semigroups $(T_p(t))_{t\geq 0}$ on E_p, $1 \leq p \leq \infty$. Such semigroups are strongly continuous if $p \in [1,\infty)$, and analytic of angle $\frac{\pi}{2} - \arctan \frac{|p-2|}{2\sqrt{p-1}}$ for $p \in (1,\infty)$.*

Moreover, the spectrum of A_p is independent of p, where A_p denotes the generator of $(T_p(t))_{t\geq 0}$, $1 \leq p \leq \infty$.

Proof. It follows by [4, Section 7.2] as in [36, Theorem 4.1] and [38, Corollary 5.6]. □

We also can prove that the generators of the semigroups in the spaces E_p, $1 \leq p \leq \infty$ have in fact the same form as in E_2, with appropriate domain.

Lemma 2.9. *For all $p \in [1, \infty]$ the generator A_p of the semigroup $(T_p(t))_{t\geq 0}$ is given by the operator defined in* (2.5) *with domain*

$$D(A_p) = \left\{ u \in \prod_{j=1}^{m} W^{2,p}(0,1;\mu_j dx) \cap D(L) : MLu = Cu \right\}. \tag{2.11}$$

In particular, A_p has compact resolvent for $p \in [1, \infty]$.

Proof. See [36, Proposition 4.6] and [38, Lemma 5.7]. □

As a summary we obtain the following theorem.

Theorem 2.10. *The first order problem* (2.1) *is well-posed on E_p, $p \in [1, \infty)$, i.e., for all initial data* $\mathsf{u} \in E_p$ *the problem* (2.1) *admits a unique mild solution that continuously depends on the initial data.*

3 The stochastic Allen–Cahn equation on networks

3.1 An abstract stochastic Cauchy problem

Let $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space endowed with a right continuous filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$. Let $(W_H(t))_{t\in[0,T]}$ be a cylindrical Wiener process, defined on $(\Omega, \mathscr{F}, \mathbb{P})$, in some Hilbert space H with respect to the filtration $\mathbb{F}$; that is, $(W_H(t))_{t\in[0,T]}$ is $(\mathscr{F}_t)_{t\in[0,T]}$-adapted and for all $t > s$, $W_H(t) - W_H(s)$ is independent of $\mathscr{F}_s$. To be able to handle the stochastic Allen–Cahn equation on networks, first we cite a result of M. Kunze and J. van Neerven, regarding the following abstract equation

$$\begin{cases} dX(t) = [AX(t) + F(t, X(t))]dt + G(t, X(t))dW_H(t) \\ X(0) = \xi, \end{cases} \tag{SCP}$$

see [32, Section 3]. If we assume that $(A, D(A))$ generates a strongly continuous, analytic semigroup S on the Banach space E with $\|S(t)\| \leq Ke^{\omega t}$, $t \geq 0$ for some $K \geq 1$ and $\omega \in \mathbb{R}$, then for $\omega' > \omega$ the fractional powers $(\omega' - A)^\alpha$ are well-defined for all $\alpha \in (0, 1)$. In particular, the fractional domain spaces

$$E^\alpha := D((\omega' - A)^\alpha), \quad \|v\|_\alpha := \|(\omega' - A)^\alpha v\|, \quad v \in D((\omega' - A)^\alpha) \tag{3.1}$$

are Banach spaces. It is well-known (see e.g. [26, §II.4–5.]), that up to equivalent norms, these spaces are independent of the choice of ω'.

For $\alpha \in (0, 1)$ we define the extrapolation spaces $E^{-\alpha}$ as the completion of E under the norms $\|v\|_{-\alpha} := \|(\omega' - A)^{-\alpha} v\|$, $v \in E$. These spaces are independent of $\omega' > \omega$ up to an equivalent norm.

We fix $E^0 := E$.

Remark 3.1. If A is injective and $\omega = 0$ (hence, the semigroup S is bounded), then by [28, Chapter 6.2, Introduction] we can choose $\omega' = 0$. That is,

$$E^\alpha \cong D((-A)^\alpha), \qquad \alpha \in [0, 1).$$

To obtain the desired result for the solution of (SCP), one has to impose the following assumptions for the mappings in (SCP). These are – in the first and third cases slightly simplified versions of – Assumptions (A1), (A5), (A4), (F'), (F'') and (G'') in [32]. Let B be a Banach space, $\|\cdot\|$ will denote $\|\cdot\|_B$. For $u \in B$ we define the *subdifferential of the norm at u* as the set

$$\partial\|u\| := \{u^* \in B^* : \|u^*\| = 1 \text{ and } \langle u, u^*\rangle = 1\} \tag{3.2}$$

which is not empty by the Hahn–Banach theorem. Furthermore, let E be a UMD Banach space of type 2.

Assumptions 3.2.

1. *$(A, D(A))$ is densely defined, closed and sectorial on E.*
2. *For some $0 \le \theta < \frac{1}{2}$ we have continuous, dense embeddings*
$$E^\theta \hookrightarrow B \hookrightarrow E.$$
3. *Let S be the strongly continuous analytic semigroup generated by $(A, D(A))$. Then S restricts to a strongly continuous contraction semigroup S^B on B, in particular, $A|_B$ is dissipative.*
4. *The map $F\colon [0,T] \times \Omega \times B \to B$ is locally Lipschitz continuous in the sense that for all $r > 0$, there exists a constant $L_F^{(r)}$ such that*
$$\|F(t,\omega,u) - F(t,\omega,v)\| \le L_F^{(r)}\|u - v\|$$
for all $\|u\|, \|v\| \le r$ and $(t,\omega) \in [0,T] \times \Omega$ and there exists a constant $C_{F,0} \ge 0$ such that
$$\|F(t,\omega,0)\| \le C_{F,0}, \qquad t \in [0,T],\ \omega \in \Omega.$$
Moreover, for all $u \in B$ the map $(t,\omega) \mapsto F(t,\omega,u)$ is strongly measurable and adapted. Finally, for suitable constants $a, b \ge 0$ and $N \ge 1$ we have
$$\langle Au + F(t, u+v), u^*\rangle \le a(1 + \|v\|)^N + b\|u\|$$
for all $u \in D(A|_B)$, $v \in B$ and $u^* \in \partial\|u\|$, see (3.2).
5. *There exist constants a'', b'', $m' > 0$ such that the function $F\colon [0,T] \times \Omega \times B \to B$ satisfies*
$$\langle F(t,\omega,u+v) - F(t,\omega,v), u^*\rangle \le a''(1 + \|v\|)^{m'} - b''\|u\|^{m'}$$
for all $t \in [0,T]$, $\omega \in \Omega$, $u, v \in B$ and $u^* \in \partial\|u\|$, and
$$\|F(t,v)\| \le a''(1 + \|v\|)^{m'}$$
for all $v \in B$.
6. *Let $\gamma(H, E^{-\kappa_G})$ denote the space of γ-radonifying operators from H to $E^{-\kappa_G}$ for some $0 \le \kappa_G < \frac{1}{2}$, see e.g. [32, Section 3.1]. Then the map $G\colon [0,T] \times \Omega \times B \to \gamma(H, E^{-\kappa_G})$ is locally Lipschitz continuous in the sense that for all $r > 0$, there exists a constant $L_G^{(r)}$ such that*
$$\|G(t,\omega,u) - G(t,\omega,v)\|_{\gamma(H,E^{-\kappa_G})} \le L_G^{(r)}\|u - v\|$$

for all $\|u\|, \|v\| \leq r$ *and* $(t, \omega) \in [0, T] \times \Omega$. *Moreover, for all* $u \in B$ *and* $h \in H$ *the map* $(t, \omega) \mapsto G(t, \omega, u)h$ *is strongly measurable and adapted.*

Finally, G *is of linear growth, that is, for suitable constant* c',

$$\|G(t, \omega, u)\|_{\gamma(H, E^{-\kappa_G})} \leq c' \left(1 + \|u\|\right)$$

for all $(t, \omega, u) \in [0, T] \times \Omega \times B$.

Recall that a *mild solution* of (SCP) is a solution of the following implicit equation

$$\begin{aligned} X(t) &= S(t)\xi + \int_0^t S(t-s)F(s, X(s))\, ds + \int_0^t S(t-s)G(s, X(s))\, dW_H(s) \\ &=: S(t)\xi + S * F(\cdot, X(\cdot))(t) + S \diamond G(\cdot, X(\cdot))(t) \end{aligned} \tag{3.3}$$

where

$$S * f(t) = \int_0^t S(t-s) f(s)\, ds$$

denotes the "usual" convolution, and

$$S \diamond g(t) = \int_0^t S(t-s) g(s)\, dW_H(s)$$

denotes the stochastic convolution with respect to W_H.

The result of Kunze and van Neerven that will be useful for our setting is the following. We note that this was first proved in [32, Theorem 4.9] but with a typo in the statement which was later corrected in the recent arXiv preprint [33, Theorem 4.9].

Theorem 3.3 ([33, Theorem 4.9])**.** *Suppose that Assumptions 3.2 hold and let* $2 < q < \infty$, $0 \leq \theta < \frac{1}{2}$, $0 \leq \kappa_G < \frac{1}{2}$ *satisfy*

$$\theta + \kappa_G < \frac{1}{2} - \frac{1}{q}.$$

Then for all $\xi \in L^q(\Omega, \mathscr{F}_0, \mathbb{P}; B)$ *there exists a unique global mild solution*

$$X \in L^q\left(\Omega, C([0, T]; B)\right)$$

of (SCP)*. Moreover, for some constant* $C > 0$ *we have*

$$\mathbb{E}\|X\|^q_{C([0,T];B)} \leq C \cdot \left(1 + \mathbb{E}\|\xi\|^q\right).$$

3.2 Preparatory results

In order to apply the abstract result of Theorem 3.3 to the stochastic Allen–Cahn equation on a network we need to prove some preparatory results using the setting of Section 2.

On the edges of the graph G we will consider continuous functions that satisfy the continuity condition in the vertices, see Subsection 2.1. We will refer to such functions as *continuous functions on the graph* G and denote them by $C(G)$.

Definition 3.4. We define

$$C(G) := D(L),$$

see (2.4), which can be looked at as the Banach space of all continuous functions on the graph G, hence the norm on $C(G)$ can be defined as

$$\|u\|_{C(G)} = \max_{j=1,\dots,m} \sup_{[0,1]} |u_j|, \quad u \in C(G).$$

This space will play the role of the space B in our setting, hence we set

$$B := C(G) \text{ and } \|\cdot\|_{C(G)} := \|\cdot\|_B. \tag{3.4}$$

We will show that for θ big enough the continuous, dense embeddings

$$E_p^\theta \hookrightarrow B \hookrightarrow E_p$$

hold, where

$$E_p^\theta \text{ is defined for the operator } A_p \text{ on the Banach space } E_p \text{ as in (3.1).} \tag{3.5}$$

To do so, we first need a technical lemma, and define the maximal operator on E_p as

$$A_{p,\max} := \begin{pmatrix} \frac{d}{dx}\left(c_1 \frac{d}{dx}\right) - p_m & & 0 \\ & \ddots & \\ 0 & & \frac{d}{dx}\left(c_m \frac{d}{dx}\right) - p_m \end{pmatrix} \tag{3.6}$$

with domain

$$D(A_{p,max}) := \left(\prod_{j=1}^m W^{2,p}(0,1;\mu_j dx)\right) \cap D(L), \tag{3.7}$$

see (2.5) (2.6) in E_2. Hence, the domain of $A_{p,\max}$ only contains the continuity condition in the nodes.

Furthermore, define

$$W_0(G) := \prod_{j=1}^m W_0^{2,p}(0,1;\mu_j dx), \tag{3.8}$$

where

$$W_0^{2,p}(0,1;\mu_j dx) = W^{2,p}(0,1;\mu_j dx) \cap W_0^{1,p}(0,1;\mu_j dx), \qquad j = 1,\dots,m.$$

That is, $W_0(G)$ contains such vectors of functions that are twice weakly differentiable on each edge and continuous on the graph with Dirichlet boundary conditions.

Lemma 3.5.

$$D(A_{p,max}) \cong W_0(G) \times \mathbb{R}^n,$$

where the isomorphism is taken for $D(A_{p,\max})$ equipped with the operator graph norm.

Proof. We will use the setting of [27] for $A = A_{p,\max}$, $X = E_p$ and the boundary operator $L : D(L) \subset E_p \to \mathbb{R}^n =: Y$. Denote

$$A_0 := A_{p,\max}|_{\ker L},$$

which is the operator (3.6) with Dirichlet boundary conditions. Hence, it is a generator on E_p. Clearly

$$D(A_0) = W_0(G) \tag{3.9}$$

holds.

We now choose $\lambda \in \rho(A_0)$. Using [27, Lemma 1.2] we have that

$$D(A_{p,\max}) = D(A_0) \oplus \ker(\lambda - A_{p,\max}).$$

Furthermore, the map

$$L\colon \ker(\lambda - A_{p,\max}) \to \mathbb{R}^n \tag{3.10}$$

is an onto isomorphism, having the inverse

$$D_\lambda := (L|_{\ker(\lambda - A_{p,\max})})^{-1}\colon \mathbb{R}^n \to \ker(\lambda - A_{p,\max})$$

called *Dirichlet-operator*, see [27, (1.14)]. By [27, (1.15)],

$$D_\lambda L\colon D(A_{p,\max}) \to \ker(\lambda - A_{p,\max})$$

is the projection in $D(A_{p,\max})$ onto $\ker(\lambda - A_{p,\max})$ along $D(A_0)$. Since $D_\lambda L$ is continuous, by the properties of the direct sum, see e.g. [42, Theorem 2.5], we obtain that

$$D(A_{p,\max}) \cong D(A_0) \times \ker(\lambda - A_{p,\max})$$

holds. Now using (3.9) and that (3.10) is an isomorphism, the claim follows. □

Lemma 3.6. *For the space B defined in* (3.4)

$$B \cong (C_0[0,1])^m \times \mathbb{R}^n$$

holds.

Proof. Let $u \in B$ arbitrary and $r := Lu \in \mathbb{R}^n$. We can define the unique $v^u \in B$ such that v^u_j is a first order polynomial for each $j = 1, \dots, m$ taking values

$$v^u_j(\mathsf{v}_i) = r_i, \quad \text{for } \mathsf{e}_j \in \Gamma(\mathsf{v}_i)\ j = 1, \dots, m,\ i = 1, \dots, n.$$

Then $Lv^u = r$ and

$$u - v^u \in (C_0[0,1])^m.$$

Denote

$$B_1 := \{v^u : u \in B\} \subset B$$

a closed subspace. Clearly,

$$(C_0[0,1])^m \cap B_1 = \{0_B\}$$

and if $u \in B$ then $u = (u - v^u) + v^u$ with $u - v^u \in (C_0[0,1])^m$ and $v^u \in B_1$. Hence

$$B = (C_0[0,1])^m \oplus B_1.$$

By the construction of v^u follows that since $L : B \to \mathbb{R}^n$ is onto,

$$L|_{B_1}\colon B_1 \to \mathbb{R}^n$$

is a bijection. The operator $L|_{B_1}$ is also bounded for the norm of B induced on B_1. Hence, by the open mapping theorem, it is an isomorphism. Denoting its inverse by

$$L_1 := (L|_{B_1})^{-1} : \mathbb{R}^n \to B_1,$$

we obtain that

$$L_1 L\colon B \to B_1$$

is the continuous projection from B onto B_1 along $(C_0[0,1])^m$. Hence, we can use [42, Theorem 2.5] and obtain

$$B \cong (C_0[0,1])^m \times \mathbb{R}^n.$$

□

Corollary 3.7. *Let E_p^θ defined in (3.5). If $\theta > \frac{1}{2p}$ then the following continuous, dense embeddings are satisfied:*

$$E_p^\theta \hookrightarrow B \hookrightarrow E_p. \tag{3.11}$$

Proof. We know that $(A_p, D(A_p))$ is sectorial and maximal dissipative, hence it is injective and generates a contractive semigroup. By Remark 3.1 we have that

$$E_p^\theta \cong D((-A_p)^\theta)$$

for $\theta \in [0,1)$. It follows from [4, Theorem in §5.3.5] and [4, Theorem in §4.4.10] that for the complex interpolation spaces

$$D((-A_p)^\theta) \cong [D(-A_p), E_p]_\theta,$$

hence

$$E_p^\theta \cong [D(-A_p), E_p]_\theta$$

holds with equivalence of norms. Defining $(A_{p,\max}, D(A_{p,\max}))$ as in (3.6), (3.7) we have that

$$D(A_p) \hookrightarrow D(A_{p,\max})$$

holds. Hence

$$E_p^\theta \hookrightarrow \left[D(-A_{p,\max}), E_p\right]_\theta. \tag{3.12}$$

By Lemma 3.5,

$$D(-A_{p,\max}) \cong W_0(G) \times \mathbb{R}^n \tag{3.13}$$

holds, where $W_0(G)$ is defined in (3.8). Since $E_p \cong E_p \times \{0_{\mathbb{R}^n}\}$, using general interpolation theory, see e.g. [43, Section 4.3.3], we have that for $\theta > \frac{1}{2p}$

$$\left[W_0(G) \times \mathbb{R}^n, E_p \times \{0_{\mathbb{R}^n}\}\right]_\theta \hookrightarrow \left(\prod_{j=1}^m W_0^{2\theta,p}(0,1;\mu_j dx)\right) \times \mathbb{R}^n.$$

Thus, by (3.12) and (3.13)

$$E_p^\theta \hookrightarrow \left(\prod_{j=1}^m W_0^{2\theta,p}(0,1;\mu_j dx)\right) \times \mathbb{R}^n \tag{3.14}$$

holds. Hence,

$$E_p^\theta \hookrightarrow (C_0[0,1])^m \times \mathbb{R}^n \tag{3.15}$$

is true. Applying Lemma 3.6 we obtain that for $\theta > \frac{1}{2p}$

$$E_p^\theta \hookrightarrow B \tag{3.16}$$

is satisfied. Using Lemma 3.6 again, we have $B \hookrightarrow E_p$, and the claim follows. □

In the following we will prove that the part of the operator $(A_p, D(A_p))$ in B is the generator of a strongly continuous semigroup on B. First notice that by the form (2.11) of $D(A_p)$ and by (3.11)

$$D(A_p) \subset B \hookrightarrow E_p \tag{3.17}$$

holds.

Proposition 3.8. *The part of $(A_p, D(A_p))$ in B generates a positive strongly continuous semigroup of contractions on B.*

Proof. 1. We first prove that the semigroup $(T_p(t))_{t\geq 0}$ leaves B invariant. We take $u \in B$ and use that $(T_p(t))_{t\geq 0}$ is analytic on E_p (see Proposition 2.8). Hence, $T_p(t)u \in D(A_p)$. By (3.17) also

$$T_p(t)u \in B$$

holds.

2. In the next step we prove that $(T_p(t)|_B)_{t\geq 0}$ is a strongly continuous semigroup. By [26, Proposition I.5.3], it is enough to prove that there exist $K > 0$ and $\delta > 0$ and a dense subspace $D \subset B$ such that

(a) $\|T_p(t)\|_B \leq K$ for all $t \in [0, \delta]$, and

(b) $\lim_{t\downarrow 0} T_p(t)u = u$ for all $u \in D$.

To verify (a), we obtain by Proposition 2.8 that for $u \in B$

$$\|T_p(t)u\|_B = \|T_p(t)u\|_{E_\infty} = \|T_\infty(t)u\|_{E_\infty} \leq \|u\|_{E_\infty} = \|u\|_B,$$

hence

$$\|T_p(t)\|_B \leq 1 =: K, \quad t \geq 0.$$

To prove (b) take $\frac{1}{2p} < \theta < \frac{1}{2}$ arbitrary. By (3.11) we have that

$$D := E_p^\theta \hookrightarrow B$$

with dense, continuous embedding. Hence, there exists $C > 0$ such that for $u \in D$,

$$\begin{aligned}\|T_p(t)u - u\|_B &\leq C \cdot \|T_p(t)u - u\|_{E_p^\theta} \\ &= C \cdot \|T_p(t)(-A_p)^\theta u - (-A_p)^\theta u\|_{E_p} \to 0, \quad t \downarrow 0.\end{aligned}$$

Summarizing 1. and 2., and using (3.17), we can apply [26, Proposition in Section II.2.3] for $(A_p, D(A_p))$ and $Y = B$, and obtain that the part of $(A_p, D(A_p))$ in B generates a positive strongly continuous semigroup of contractions on B. □

Corollary 3.9. *The first order problem (2.1) is well-posed on B, i.e., for all initial data $\mathsf{u} \in B$ the problem (2.1) admits a unique mild solution that continuously depends on the initial data.*

3.3 Main results

In this subsection we first apply the above results to the following stochastic evolution equation, based on (2.1). This corresponds to a slightly more general version of (1.1), see (3.33) later.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space endowed with a right-continuous filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ for some $T > 0$ given. We consider the problem

$$\begin{cases} \dot{u}_j(t,x) = (c_j u_j')'(t,x) - p_j(x)u_j(t,x) & \\ \qquad + f_j(t,x,u_j(t,x)) & \\ \qquad + g_j(t,x,u_j(t,x))\dfrac{\partial w_j}{\partial t}(t,x), & t \in (0,T],\ x \in (0,1),\ j = 1,\dots,m, \quad (a) \\ u_j(t,\mathsf{v}_i) = u_\ell(t,\mathsf{v}_i) =: q_i(t), & t \in (0,T],\ \forall j,\ell \in \Gamma(\mathsf{v}_i),\ i = 1,\dots,n, \quad (b) \\ [Mq(t)]_i = -\sum_{j=1}^m \phi_{ij}\mu_j c_j(\mathsf{v}_i)u_j'(t,\mathsf{v}_i), & t \in (0,T],\ i = 1,\dots,n, \quad (c) \\ u_j(0,x) = \mathsf{u}_j(x), & x \in [0,1],\ j = 1,\dots,m, \quad (d) \end{cases} \tag{3.18}$$

where $\frac{\partial w_j}{\partial t}$, $j = 1, \ldots, m$, are independent space-time white noises on $[0, 1]$; written as formal derivatives of independent cylindrical Wiener-processes $(w_j(t))_{t\in[0,T]}$, defined on $(\Omega, \mathscr{F}, \mathbb{P})$, in the Hilbert space $L^2(0, 1; \mu_j dx)$ with respect to the filtration $\mathbb{F}$.

The functions $f_j \colon [0, T] \times \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ are polynomials of the form

$$f_j(t, \omega, x, \eta) = -a_{j,2k+1}(t, \omega, x)\eta^{2k+1} + \sum_{l=0}^{2k} a_{j,l}(t, \omega, x)\eta^l, \quad \eta \in \mathbb{R}, j = 1, \ldots, m \tag{3.19}$$

for some fixed integer k. For the coefficients we assume that there are constants $0 < c \leq C < \infty$ such that

$$c \leq a_{j,2k+1}(t, \omega, x) \leq C, \ \left|a_{j,l}(t, \omega, x)\right| \leq C, \text{ for all } j = 1, \ldots, m, \ l = 0, 2, \ldots, 2k,$$

for all $x \in [0, 1]$, $t \in [0, T]$ and almost all $\omega \in \Omega$, see [32, Example 4.2]. The coefficients $a_{j,l} \colon [0, T] \times \Omega \times [0, 1] \to \mathbb{R}$ are jointly measurable and adapted in the sense that for each j and l and for each $t \in [0, T]$, the function $a_{j,l}(t, \cdot)$ is $\mathscr{F}_t \otimes \mathcal{B}_{[0,1]}$-measurable, where $\mathcal{B}_{[0,1]}$ denotes the sigma-algebra of the Borel sets on $[0, 1]$.

We further assume a technical assumption regarding the graph structure that will play and important role in our setting.

Assumption 3.10. *For the coefficients in (3.19) we assume that*

$$\left(a_{1,l}(t, \omega, \cdot), \ldots, a_{m,l}(t, \omega, \cdot)\right)^\top \in B \text{ for all } l = 1, \ldots, 2k + 1,$$

$t \in [0, T]$ *and almost all* $\omega \in \Omega$.

Remark 3.11. If the coefficients in (3.19) do not depend on j – that is, they are the same on different edges –, and satisfy

$$a_l(t, \omega, \cdot) = a_{j,l}(t, \omega, \cdot) \in C[0, 1], \quad t \in [0, T], \omega \in \Omega, \quad j = 1, \ldots m, \ l = 1, \ldots, 2k + 1$$

and

$$a_l(t, \omega, 0) = a_l(t, \omega, 1), \text{ for all } l = 1, \ldots 2k + 1,$$

then Assumption 3.10 is fulfilled. This is the case e.g. if $a_l' s$ are constant (not depending on x).

For the functions g_j we assume

$$\begin{aligned} &g_j \colon [0, T] \times \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R}, \quad j = 1, \ldots, m \text{ are locally Lipschitz continuous} \\ &\text{and of linear growth in the fourth variable,} \\ &\text{uniformly with respect to the first three variables.} \end{aligned} \tag{3.20}$$

We further assume that the functions are jointly measurable and adapted in the sense that for each j and $t \in [0, T]$, $g_j(t, \cdot)$ is $\mathscr{F}_t \otimes \mathcal{B}_{[0,1]} \otimes \mathcal{B}_{\mathbb{R}}$-measurable, where $\mathcal{B}_{[0,1]}$ and $\mathcal{B}_{\mathbb{R}}$ denote the sigma-algebras of the Borel sets on $[0, 1]$ and $\mathbb{R}$, respectively.

The above assumptions on the coefficients on the edges, except for Assumption 3.10 which is specific for the graph setting, are analogous to those in [32, Section 5] and [33, Section 5].

To handle system (3.18), we rewrite it in the form of the abstract stochastic Cauchy-problem (SCP). To do so, we specify the functions appearing in (SCP) corresponding to (3.18).

The operator $(A, D(A)) = (A_p, D(A_p))$ will be the generator of the strongly continuous analytic semigroup $S := (T_p(t))_{t\geq 0}$ on the Banach space $E := E_p$ for some large $p \geq 2$, see Proposition 2.8 and Lemma 2.9. Hence, E is a UMD space of type 2.

For the function $F\colon [0,T] \times \Omega \times B \to B$ we have

$$F(t,\omega,u)(s) := \left(f_1(t,\omega,s,u_1(s)),\ldots,f_m(t,\omega,s,u_m(s))\right)^\top, \qquad s \in [0,1]. \tag{3.21}$$

Since B is an algebra, Assumption 3.10 assures that F maps $[0,T] \times \Omega \times B$ into B.

To define the operator G we argue in analogy with [33, Section 5]. First define

$$H := E_2$$

the product L^2-space, see (2.3), which is a Hilbert space. We further define the multiplication operator $\Gamma\colon [0,T] \times B \to \mathcal{L}(H)$ as

$$[\Gamma(t,u)h]\,(s) := \begin{pmatrix} g_1(t,s,u_1(s)) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & g_m(t,s,u_m(s)) \end{pmatrix} \cdot \begin{pmatrix} h_1(s) \\ \vdots \\ h_m(s) \end{pmatrix}, \qquad s \in (0,1), \tag{3.22}$$

for $u \in B$, $h \in H$. Because of the assumptions (3.20) on the functions g_j, Γ clearly maps into $\mathcal{L}(H)$.

Let $(A_2, D(A_2))$ be the generator on $H = E_2$, see Proposition 2.5, and pick $\kappa_G \in (\frac{1}{4}, \frac{1}{2})$. By (3.14) in the proof of Corollary 3.7 we have that there exists a continuous embedding

$$\imath\colon E_2^{\kappa_G} \to \left(\prod_{j=1}^m H_0^{2\kappa_G}(0,1;\mu_j dx)\right) \times \mathbb{R}^n =: \mathcal{H},$$

where $\mathcal{H}$ is a Hilbert space. Applying the steps (3.15) and (3.16) of Corollary 3.7 we obtain that $\mathcal{H} \hookrightarrow B$ holds, and by (3.11), there exists a continuous embedding

$$\jmath\colon \mathcal{H} \to E_p$$

for $p \geq 2$ arbitrary.

Define now G by

$$(-A_p)^{-\kappa_G} G(t,u)h := \jmath\,\imath\,(-A_2)^{-\kappa_G}\Gamma(t,u)h, \qquad u \in B,\ h \in H. \tag{3.23}$$

Proposition 3.12. *Let $p \geq 2$ and $\kappa_G \in (\frac{1}{4}, \frac{1}{2})$ be arbitrary. Then the operator G defined in (3.23) maps $[0,T] \times B$ into $\gamma(H, E_p^{-\kappa_G})$.*

Proof. We can argue as in [39, Section 10.2]. Using [39, Lemma 2.1(4)], we obtain in a similar way as in [39, Corollary 2.2]) that $\jmath \in \gamma(\mathcal{H}, E_p)$, since $2\kappa_G > \frac{1}{2}$ holds. Hence, by the definition of G and the ideal property of γ-radonifying operators, the mapping G takes values in $\gamma(H, E_p^{-\kappa_G})$. $\square$

The driving noise process W_H is defined by

$$W_H(t) = \begin{pmatrix} w_1(t) \\ \vdots \\ w_m(t) \end{pmatrix}, \qquad t \in [0,T], \tag{3.24}$$

and thus $(W_H(t))_{t\in[0,T]}$ is a cylindrical Wiener process, defined on $(\Omega, \mathscr{F}, \mathbb{P})$, in the Hilbert space H with respect to the filtration $\mathbb{F}$.

We will state now the result regarding system (SCP) corresponding to (3.18).

Theorem 3.13. *Let F, G and W defined in (3.21), (3.23) and (3.24), respectively. Let $q > 4$ be arbitrary. Then for every $\xi \in L^q(\Omega, \mathscr{F}_0, \mathbb{P}; B)$ a unique mild solution X of equation (SCP) exists globally and belongs to $L^q(\Omega; C([0,T]; B))$.*

Proof. The condition $q > 4$ allows us to choose $2 \leq p < \infty$, $\theta \in [0, \frac{1}{2})$ and $\kappa_G \in (\frac{1}{4}, \frac{1}{2})$ such that

$$\theta > \frac{1}{2p} \tag{3.25}$$

and

$$0 \leq \theta + \kappa_G < \frac{1}{2} - \frac{1}{q}.$$

We will apply Theorem 3.3 with θ and κ_G having the properties above. To this end we have to check Assumptions 3.2 for the mappings in (SCP), taking $A = A_p$ and $E = E_p$ for the p chosen above. Assumption (1) is satisfied because of the generator property of A_p, see Proposition 2.8. Assumption (2) is satisfied since (3.25) holds and we can use Corollary 3.7. Assumption (3) is satisfied by the statement of Proposition 3.8. Using that the functions f_j are polynomials of the 4th variable of the same degree $2k+1$ (see (3.19)), a similar computation as in [32, Example 4.2] and [32, Example 4.5], using techniques from [23, Section 4.3], shows that Assumptions (4) and (5) are satisfied for F with $N = m' = 2k+1$. By Proposition 3.12, G takes values in $\gamma(H, E_p^{-\kappa_G})$ with $H = E_2$ and κ_G chosen above. Using the assumptions (3.20) on the functions g_j and the proof of [39, Theorem 10.2], we obtain that G is locally Lipschitz continuous and of linear growth as a map $[0,T] \times B \to \gamma(H, E_p^{-\kappa_G})$, hence Assumption (6) holds. ☐

In the following theorem we will state a result regarding Hölder regularity of the mild solution of (SCP) corresponding to (3.18), see (3.3).

Theorem 3.14. *Let $q > 4$ be arbitrary, $\lambda, \eta > 0$ and $p \geq 2$ such that $\lambda + \eta > \frac{1}{2p}$. We assume that $\xi \in L^{(2k+1)q}(\Omega; E_p^{\lambda+\eta})$, where k is the constant appearing in (3.19). If the inequality*

$$\lambda + \eta < \frac{1}{4} - \frac{1}{q} \tag{3.26}$$

is fulfilled, then the mild solution X of (SCP) from Theorem 3.13 satisfies

$$X \in L^q(\Omega; C^\lambda([0,T], E_p^\eta)).$$

Proof. Using the continuous embedding (3.11), we have that

$$\xi \in L^{(2k+1)q}(\Omega; B)$$

holds. Since $(2k+1)q > 4$, by Theorem 3.13 there exists a global mild solution

$$X \in L^{(2k+1)q}(\Omega; C([0,T], B)).$$

This solution satisfies the following implicit equation (see (3.3)):

$$X(t) = S(t)\xi + S * F(\cdot, X(\cdot))(t) + S \diamond G(\cdot, X(\cdot))(t), \tag{3.27}$$

where S denotes the semigroup generated by A_p on E_p, $*$ denotes the usual convolution, $\diamond$ denotes the stochastic convolution with respect to $\mathcal{W}$. In the following we have to estimate the $L^q(\Omega; C^\lambda([0,T], E_p^\eta))$-norm of X, and we will do this using the triangle-inequality in (3.27).

For the qth power of first term we have

$$\begin{aligned}\mathbb{E}\|S(\cdot)\xi\|^q_{C^\lambda([0,T],E_p^\eta)} &= \mathbb{E}\left(\sup_{t,s\in[0,T]}\frac{\|S(t)\xi - S(s)\xi\|_{E_p^\eta}}{|t-s|^\lambda}\right)^q \\ &\le \mathbb{E}\left(\sup_{h\in[0,T]}\frac{\|S(h)\xi-\xi\|_{E_p^\eta}}{|h|^\lambda}\right)^q \\ &= \mathbb{E}\left(\sup_{h\in[0,T]}\frac{\|S(h)(-A_p)^\eta\xi-(-A_p)^\eta\xi\|_{E_p}}{|h|^\lambda}\right)^q. \end{aligned}\tag{3.28}$$

By assumption, $(-A_p)^\eta\xi \in D((-A_p)^\lambda)$ holds. Applying [26, Proposition II.5.33] we obtain that $(-A_p)^\eta\xi$ lies in the Hölder space of order λ on E_p, denoted by C_p^λ. Hence,

$$\sup_{h\in[0,T]}\frac{\|S(h)(-A_p)^\eta\xi-(-A_p)^\eta\xi\|_{E_p}}{|h|^\lambda} = \|(-A_p)^\eta\xi\|_{\mathsf{F}_{p,\lambda}} < \infty,$$

where $\|\cdot\|_{\mathsf{F}_{p,\lambda}}$ denotes the Favard norm of order λ on E_p, see [26, Definition II.5.10]. Furthermore, because of the continuous inclusion $D((-A_p)^\lambda) \hookrightarrow C_p^\lambda$, we have that there exists $c = c(\lambda)$ such that

$$\|(-A_p)^\eta\xi\|_{\mathsf{F}_{p,\lambda}} \le c\cdot\|(-A_p)^\eta\xi\|_{E_p^\lambda} = c\cdot\|(-A_p)^{\lambda+\eta}\xi\|_{E_p}.$$

Hence,

$$\mathbb{E}\|S(\cdot)\xi\|^q_{C^\lambda([0,T],E_p^\eta)} \le c\cdot\mathbb{E}\|(-A_p)^{\lambda+\eta}\xi\|^q_{E_p} < \infty$$

by assumption.

To estimate the qth power of the second term

$$\mathbb{E}\|S * F(\cdot,X(\cdot))\|^q_{C^\lambda([0,T],E_p^\eta)}$$

we choose $\theta > \frac{1}{2p}$ such that

$$\lambda+\eta+\theta < 1-\frac{1}{q}.$$

We will use [39, Lemma 3.6] with this θ, $\alpha = 1$, and q instead of p, and obtain that there exist constants $C \ge 0$ and $\varepsilon > 0$ such that

$$\|S * F(\cdot,X(\cdot))\|_{C^\lambda([0,T],E_p^\eta)} \le CT^\varepsilon\|F(\cdot,X(\cdot))\|_{L^q(0,T;E_p^{-\theta})}.\tag{3.29}$$

We have to estimate the expectation of the qth power on the right-hand-side of (3.29). By Corollary 3.7 we obtain

$$B \hookrightarrow E_p \hookrightarrow E_p^{-\theta},$$

since $\theta > \frac{1}{2p}$ holds and $(\omega'-A_p)^{-\theta}$ is an isomorphism between $E_p^{-\theta}$ and E_p. Using this and Assumptions 3.2(5) with $m' = 2k+1$ (which holds by the proof of Theorem 3.13), we have

$$\begin{aligned}\mathbb{E}\|F(\cdot,X(\cdot))\|^q_{L^q(0,T;E_p^{-\theta})} &= \mathbb{E}\int_0^T \|F(s,X(s))\|^q_{E_p^{-\theta}}\,ds \\ &\lesssim \mathbb{E}\int_0^T \|F(s,X(s))\|^q_B\,ds \\ &\lesssim \mathbb{E}\int_0^T \left(1+\|X(s)\|_B^{(2k+1)q}\right)ds \\ &\lesssim 1+\mathbb{E}\sup_{t\in[0,T]}\|X(t)\|_B^{(2k+1)q},\end{aligned}$$

where $\lesssim$ denotes that the expression on the left-hand-side is less or equal to a constant times the expression on the right-hand-side. This implies that for each $T > 0$ there exists $C_T > 0$ such that

$$\left(\mathbb{E}\|S * F(\cdot, X(\cdot))\|^q_{C^\lambda([0,T],E_p^\eta)}\right)^{\frac{1}{q}} \leq C_T \cdot \left(1 + \|X(t)\|^{2k+1}_{L^{(2k+1)q}(\Omega;C([0,T],B))}\right), \tag{3.30}$$

and the right-hand-side is finite.

To estimate the stochastic convolution term in (3.27) we first fix $0 < \alpha < \frac{1}{2}$ such that

$$\lambda + \eta + \frac{1}{4} < \alpha - \frac{1}{q}$$

holds. We now choose $\kappa_G \in (\frac{1}{4}, \frac{1}{2})$ such that

$$\lambda + \eta + \kappa_G < \alpha - \frac{1}{q}$$

is satisfied. Applying [39, Proposition 4.2] with $\theta = \kappa_G$ and q instead of p, we have that there exist $\varepsilon > 0$ and $C \geq 0$ such that

$$\mathbb{E}\left\|S \diamond G(\cdot, X(\cdot))\right\|^q_{C^\lambda([0,T],E_p^\eta)} \leq C^q T^{\varepsilon q} \int_0^T \mathbb{E}\left\|s \mapsto (t-s)^{-\alpha} G(s, X(s))\right\|^q_{\gamma(L^2(0,t;H),E_p^{-\kappa_G})} dt.$$

In the following we proceed similarly as done in the proof of [32, Theorem 4.3], with $N = 1$ and q instead of p. Since $E_p^{-\kappa_G}$ is a Banach space of type 2 (because E_p is of that type), the continuous embedding

$$L^2(0, t; \gamma(H, E_p^{-\kappa_G})) \hookrightarrow \gamma(L^2(0, t; H), E_p^{-\kappa_G})$$

holds. Using this, Young's inequality and the properties of G, respectively, we obtain the following estimates

$$\begin{aligned}
\mathbb{E}\left\|S \diamond G(\cdot, X(\cdot))\right\|^q_{C^\lambda([0,T],E_p^\eta)} &\lesssim T^{\varepsilon q} \int_0^T \mathbb{E}\left\|s \mapsto (t-s)^{-\alpha} G(s, X(s))\right\|^q_{L^2(0,t;\gamma(H,E_p^{-\kappa_G}))} dt \\
&= T^{\varepsilon q} \mathbb{E} \int_0^T \left(\int_0^t (t-s)^{-2\alpha} \left\|G(s, X(s))\right\|^2_{\gamma(H,E_p^{-\kappa_G})} ds\right)^{\frac{q}{2}} dt \\
&\leq T^{\varepsilon q} \left(\int_0^T t^{-2\alpha}\, dt\right)^{\frac{q}{2}} \mathbb{E} \int_0^T \left\|G(t, X(t))\right\|^q_{\gamma(H,E_p^{-\kappa_G})} dt \\
&\leq T^{(\frac{1}{2}-\alpha+\varepsilon)q} (c')^q \cdot \mathbb{E} \int_0^T (1 + \|X(t)\|_B)^q\, dt \\
&\lesssim T^{(\frac{1}{2}-\alpha+\varepsilon)q+1} (c')^q \cdot \left(1 + \mathbb{E}\|X(t)\|^q_{C([0,T],B)}\right).
\end{aligned}$$

Hence, for each $T > 0$ there exists constant $C'_T > 0$ such that

$$\left(\mathbb{E}\left\|S \diamond G(\cdot, X(\cdot))\right\|^q_{C^\lambda([0,T],E_p^\eta)}\right)^{\frac{1}{q}} \leq C'_T \cdot \left(1 + \|X(t)\|_{L^{(2k+1)q}(\Omega;C([0,T],B))}\right)^{2k+1}. \tag{3.31}$$

In summary, by (3.28), (3.30) and (3.31), we obtain that $X \in L^q(\Omega; C^\lambda([0,T], E_p^\eta))$ holds, hence the proof is completed. □

We are now in the position to finally consider (1.1). Let

$$\beta := \max_{1\le j\le m} \beta_j.$$

We also introduce

$$f_j(\eta) := f(\eta) = -\eta^3 + \beta^2\eta. \tag{3.32}$$

and

$$\varrho_j := \beta^2 - \beta_j^2 \ge 0.$$

With these notations, we can rewrite (1.1) in an equivalent form as

$$\begin{cases} \dot{u}_j(t,x) = (c_j u_j')'(t,x) - \tilde{p}_j(x)u_j(t,x) & \\ \qquad\qquad + f_j(u_j(t,x)) & \\ \qquad\qquad + g_j(t,x,u_j(t,x))\dfrac{\partial w_j}{\partial t}(t,x), & t\in(0,T],\ x\in(0,1),\ j=1,\dots,m,\\ u_j(t,\mathsf{v}_i) = u_\ell(t,\mathsf{v}_i) =: q_i(t), & t\in(0,T],\ \forall j,\ell\in\Gamma(\mathsf{v}_i),\ i=1,\dots,n,\\ [Mq(t)]_i = -\sum_{j=1}^m \phi_{ij}\mu_j c_j(\mathsf{v}_i)u_j'(t,\mathsf{v}_i), & t\in(0,T],\ i=1,\dots,n,\\ u_j(0,x) = \mathsf{u}_j(x), & x\in[0,1],\ j=1,\dots,m, \end{cases} \tag{3.33}$$

with $\tilde{p}_j(x) := p_j(x) + \varrho_j$, $j = 1,\dots m$.

We define the operator A_p on E_p as in (2.5) with $\tilde{p}_j$'s instead of p_j's and with domain (2.11).

Theorem 3.15. *Let F, G and W defined in (3.21), (3.23) and (3.24), respectively, for the system (3.33). Let $q > 4$ be arbitrary. Then for every $\xi \in L^q(\Omega, \mathscr{F}_0, \mathbb{P}; B)$ a unique mild solution X of equation (SCP) corresponding to (3.33), which is equivalent to (1.1), exists globally and belongs to $L^q(\Omega; C([0,T]; B))$. Let $\lambda, \eta > 0$, $p \ge 2$ be arbitrary constants such that $\lambda + \eta > \frac{1}{2p}$. If $\xi \in L^{3q}(\Omega; E_p^{\lambda+\eta})$ and the inequality*

$$\lambda + \eta < \frac{1}{4} - \frac{1}{q}$$

is fulfilled, then $X \in L^q(\Omega; C^\lambda([0,T], E_p^\eta))$.

Proof. First note that the coefficients $\tilde{p}_j$ stay nonnegative as the constants ϱ_j are nonnegative. Furthermore, the nonlinear terms $f_j = f$ in (3.32) are of the form (3.19) with $k = 1$ and constant coefficients. Hence, Assumption 3.10 is fulfilled by Remark 3.11. The statement then follows from Theorems 3.13 and 3.14. □

3.4 Concluding remarks

In equation (3.18a) we could have prescribed coloured noise instead of white noise on the edges of the graph. That is, we could set

$$\begin{aligned} \dot{u}_j(t,x) &= (c_j u_j')'(t,x) - p_j(x)u_j(t,x) \\ &\quad + f_j(t,x,u_j(t,x)) \\ &\quad + g_j(t,x,u_j(t,x))R_j\frac{\partial w_j}{\partial t}(t,x), \qquad t\in(0,T],\ x\in(0,1),\ j=1,\dots,m, \end{aligned} \tag{3.34}$$

with $R_j \in \gamma(L^2(0,1;\mu_j dx), L^p(0,1;\mu_j dx))$. Then we define

$$R := \begin{pmatrix} R_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & R_m \end{pmatrix} \in \gamma(H, E_p)$$

with $H = E_2$ and $p \geq 2$ arbitrary. Using this, we can define the operator $G : [0,T] \times B \to \gamma(H, E_p)$ as

$$G(t,u)h := \Gamma(t,u)Rh, \quad h \in H,$$

where the operator $\Gamma : [0,T] \times B \to \mathcal{L}(H)$ is defined in (3.22). It is easy to see that G satisfies Assumptions 3.2(6) with $\kappa_G = 0$. For example, if $u, v \in B$ with $\|u\|, \|v\| \leq r$, then

$$\begin{aligned} \|G(t,u) - G(t,v)\|_{\gamma(H,E_p)} &\leq \|\Gamma(t,u) - \Gamma(t,v)\|_{\mathcal{L}(E_p)} \cdot \|R\|_{\gamma(H,E_p)} \\ &\leq L^{(r)} \cdot \|u - v\|_B \cdot \|R\|_{\gamma(H,E_p)} \end{aligned}$$

where $L^{(r)}$ is the maximum of the Lipschitz-constants of the functions g_j on the ball of radius r.

If setting (3.34) instead of (3.18a), Theorem 3.13 remains true as stated; that is, for $q > 4$, but one may use a simpler Hilbert space machinery; that is, one may set $p = 2$ in the proof. However, in the coloured noise case, Theorem 3.13 is true also for $q > 2$. But this can only be shown by choosing $p > 2$ large enough in the proof and hence, in this case, the Banach space arguments are crucial.

In Theorem 3.14, if one takes $p = 2$ (Hilbert space) and $q > 4$, then the statement is true for $\lambda + \eta > \frac{1}{4}$ with

$$\lambda + \eta < \frac{1}{2} - \frac{1}{q} \tag{3.35}$$

instead of (3.26). In this case R will be a Hilbert–Schmidt operator whence the covariance operator of the driving process is trace-class. However, the statement of the theorem remains true for $q > 2$ as well assuming (3.35) instead of (3.26), but only for the Banach space E_p for p large enough so that $\lambda + \eta > \frac{1}{2p}$.

The statements of Theorem 3.15 could also be changed accordingly.

Acknowledgements

The authors would like to thank the anonymous referee for her/his useful comments that helped them to improve the presentation of the paper.

M. Kovács acknowledges the support of the Marsden Fund of the Royal Society of New Zealand through grant no. 18-UOO-143, the Swedish Research Council (VR) through grant no. 2017-04274 and the NKFIH through grant no. 131545.

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Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **8**, 1–11; https://doi.org/10.14232/ejqtde.2020.1.8 www.math.u-szeged.hu/ejqtde/

Blow-up analysis for a porous media equation with nonlinear sink and nonlinear boundary condition

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Received 20 November 2020, appeared 7 February 2021
Communicated by László Simon

Abstract. In this paper, we study porous media equation $u_t = \Delta u^m - u^p$ with nonlinear boundary condition $\frac{\partial u}{\partial \nu} = ku^q$. We determine some sufficient conditions for the occurrence of finite time blow-up or global existence. Moreover, lower and upper bounds for blow-up time are also derived by using various inequality techniques.

Keywords: porous media equation, nonlinear boundary condition, bounds for blow-up time.

2020 Mathematics Subject Classification: 35B44, 35K40.

1 Introduction

In this paper, we are interested in investigating the blow-up phenomena of the following porous media equation with nonlinear sink and nonlinear boundary condition:

$$\begin{cases} u_t = \Delta u^m - u^p, & (x,t) \in \Omega \times (0,t^*), \\ \frac{\partial u}{\partial \nu} = ku^q, & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $m > 1$ and $p > 1, q \geq 1$, k is a positive constant, $\Omega \subset \mathbb{R}^3$ is a star-shaped domain with smooth boundary. ν is the unit outward normal vector on $\partial\Omega$, $u_0(x) > 0$ is the initial value. t^* is the blow-up time if the solutions blow up. It is well known that the data m, p, q may greatly affect the behavior of $u(x,t)$ as time evolves.

The mathematical investigation of the phenomenon of blow-up of solutions to parabolic equations and systems has received much attention in the recent literature. We refer to the readers the books of Straughan [14] and Quittner and Souplet [13], as well as papers of Weissler [15, 16], and so on. The determination of sufficient conditions for blow-up and the existence or nonexistence of global solution to problem, as well as bounds for the blow-up time have been the focus of some of these studies [1, 5–7, 18].

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For the initial-boundary value problem of porous media equation

$$u_t = \Delta u^m + f(u) \tag{1.2}$$

where $f(u) \geq 0$, Wu and Gao [17] established the blow-up criterion of equation (1.2) by using the method of energy. Besides, there are many references for the blow-up behavior of its solutions [4,8]. The methods used in the study of blow-up often lead to upper bound for the blow-up time when blow-up occur. However, in applied problems, because of the explosive nature of the solution, a lower bound on blow-up time is more important. Then, there are many papers giving the estimate of the lower bound of blow-up time [2,3,9,10,12]. In [9], the authors gave the estimations of the lower bound for blow-up time for problem (1.2) under Robin boundary conditions, by using various inequalities. When $m = 1$, Payne and Philippin etc. [10] studied blow-up phenomena of the classical solution of the following initial-boundary problem

$$u_t = \Delta u - f(u) \tag{1.3}$$

under the help of energy method and Sobolev type inequality, they gave the lower bound of blow-up time when condition for blow-up holds.

However, to our best knowledge, there is no paper where the blow-up phenomenon is studied with $m > 1$ and nonlinear sink as a reaction term. So, it is natural to consider problem (1.1). Methods used in this paper are motivated by the aforementioned papers. Because of the difference between the diffusion term and the reaction term, we will study the blow-up phenomena of (1.1) by modifying their techniques.

In Sections 2 and 3, by using energy method and various inequality techniques, we determine a criterion which implies blow-up, and drive upper and lower bounds for t^*; in Section 4, a criterion for boundedness of the solution in all time $t > 0$ is determined; In the last section, a relevant example will be listed to illustrate applications of our results.

2 Blow-up and upper bound estimation of t^*

In this section we establish a blow-up criterion for problem (1.1) and derive an upper bound for blow-up time, by using the auxiliary function method.

Theorem 2.1. *Let $u(x,t)$ be a nonnegative classical solution of problem (1.1) and assume $m + q - 1 \geq p$. Then $u(x,t)$ will blow-up in finite time t^*, and*

$$t^* \leq \frac{2m\varphi_0^{-a}}{(m+1)^2 a(1+a)M},$$

where a, M and φ_0^{-a} are some constants which will be given in the later proof.

Proof. Denote

$$\varphi(t) = \int_\Omega u^{m+1} dx. \tag{2.1}$$

Taking the derivative of (2.1), we have

$$\begin{aligned}\varphi'(t) &= (m+1)\int_\Omega u^m u_t dx \\ &= (m+1)\int_\Omega u^m(\Delta u^m - u^p)dx \\ &= (m+1)\int_\Omega u^m \Delta u^m dx - (m+1)\int_\Omega u^{m+p}dx \\ &= (m+1)mk\int_{\partial\Omega} u^{2m+q-1}ds - (m+1)\int_\Omega |\nabla u^m|^2 dx - (m+1)\int_\Omega u^{m+p}dx.\end{aligned} \tag{2.2}$$

Moreover, by using the notation

$$\psi(t) = \frac{2m^2k}{2m+q-1}\int_{\partial\Omega} u^{2m+q-1}ds - \int_\Omega |\nabla u^m|^2 dx - \frac{2m}{m+p}\int_\Omega u^{m+p}dx, \tag{2.3}$$

since $m+q-1 \geq p$, we have

$$\varphi'(t) \geq \frac{(m+1)(2m+q-1)}{2m}\psi(t). \tag{2.4}$$

From (2.3) one obtains

$$\psi'(t) = 2m^2k\int_{\partial\Omega} u^{2m+q-2}u_t ds - \int_\Omega |\nabla u^m|_t^2 dx - 2m\int_\Omega u^{m+p-1}u_t dx, \tag{2.5}$$

because

$$\nabla(u_t^m \nabla u^m) = u_t^m \Delta u^m + \frac{1}{2}|\nabla u^m|_t^2. \tag{2.6}$$

Integrate both sides of (2.6), then we obtain

$$\begin{aligned}\int_\Omega |\nabla u^m|_t^2 dx &= 2\int_{\partial\Omega} u_t^m \frac{\partial u^m}{\partial \nu}ds - 2\int_\Omega u_t^m \Delta u^m dx \\ &= 2m^2k\int_{\partial\Omega} u^{2m+q-2}u_t ds - 2m\int_\Omega u^{m-1}\Delta u^m u_t dx.\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.5), we have

$$\begin{aligned}\psi'(t) &= 2m\int_\Omega u^{m-1}\Delta u^m u_t dx - 2m\int_\Omega u^{m+p-1}u_t dx \\ &= 2m\int_\Omega u^{m-1}u_t^2 dx > 0.\end{aligned} \tag{2.8}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}(\varphi'(t))^2 &= \left[(m+1)\int_\Omega u^m u_t dx\right]^2 \\ &\leq (m+1)^2\int_\Omega u^{m+1}dx\int_\Omega u^{m-1}u_t^2 dx \\ &= \frac{(m+1)^2}{2m}\varphi(t)\psi'(t).\end{aligned} \tag{2.9}$$

Thus (2.4) implies

$$(\varphi'(t))^2 \geq \frac{(m+1)(2m+q-1)}{2m}\varphi'(t)\psi(t). \tag{2.10}$$

We get from (2.9) and (2.10)

$$\varphi(t)\psi'(t) \geq \frac{(2m+q-1)}{m+1}\varphi'(t)\psi(t),$$

by using the notation $\frac{2m+q-1}{m+1} = 1+a$, we find

$$\varphi(t)\psi'(t) \geq (1+a)\varphi'(t)\psi(t).$$

From the above inequality, we obtain

$$(\varphi^{-(1+a)}\psi)' \geq 0,$$

hence

$$\varphi^{-(1+a)}\psi \geq \varphi_0^{-(1+a)}\psi_0 = M,$$

where $\varphi_0 = \varphi(0)$ and $\psi_0 = \psi(0)$. Combining the above formula with (2.10), we find

$$\varphi'(t) \geq \frac{(m+1)(2m+q-1)}{2m}\psi(t) \geq \frac{(m+1)^2}{2m}(1+a)M\varphi^{1+a}, \tag{2.11}$$

then we have

$$\varphi^{-a}(t) \leq \varphi_0^{-a} - \frac{(m+1)^2}{2m}a(1+a)Mt. \tag{2.12}$$

Therefore

$$t^* \leq \frac{2m\varphi_0^{-a}}{(m+1)^2 a(1+a)M}. \tag{2.13}$$

□

3 Lower bound for the blow-up time

In this section, we estimate the lower bound of the blow-up time by constructing some auxiliary functions and using different inequality techniques, such as Sobolev type inequality and Hölder inequality etc. Our theorem is given as follows.

Theorem 3.1. *Assume that $u(x,t)$ is a nonnegative classical solution of problem (1.1), further it blows up at finite time t^*. Then*

$$t^* \geq \int_{\phi(0)}^{\infty} \frac{d\eta}{k_1\eta^{\frac{3}{2}} + k_2\eta^{\frac{3(n-m+1)}{n}} + k_3\eta - k_4\eta^{\frac{n+p-1}{n}}},$$

where $n = 2(m+2q-3)$ and $\phi(0), k_1, k_2, k_3, k_4$ are constants, defined in the proof later.

Proof. We define

$$\phi(t) = \int_\Omega u^{2(m+2q-3)}dx = \int_\Omega u^n dx. \tag{3.1}$$

The derivative of (3.1) w.r.t. t can be written as follows

$$\begin{aligned}
\phi'(t) &= n\int_\Omega u^{n-1}u_t dx \\
&= n\int_\Omega u^{n-1}(\Delta u^m - u^p)dx \\
&= n\int_\Omega u^{n-1}\Delta u^m dx - n\int_\Omega u^{n+p-1}dx \\
&= nmk\int_{\partial\Omega} u^{n+m+q-2}ds - n(n-1)m\int_\Omega u^{n+m-3}|\nabla u|^2 dx - n\int_\Omega u^{n+p-1}dx.
\end{aligned} \tag{3.2}$$

To estimate $\int_{\partial\Omega} u^{n+m+q-2} ds$, we can refer to the Lemma A.1 in [11], and obtain

$$\int_{\partial\Omega} u^{n+m+q-2} ds \leq \frac{N}{\rho_0} \int_\Omega u^{n+m+q-2} dx + \frac{(n+m+q-2)d}{\rho_0} \int_\Omega u^{n+m+q-3} |\nabla u| dx, \tag{3.3}$$

where

$$\rho_0 = \min_{\partial\Omega} (x \cdot \nu), \qquad d = \max_{\partial\Omega} |x|.$$

Note that ρ_0 is positive since Ω is star-shaped by assumption.

Applying the Hölder inequality, we have

$$\begin{aligned} \int_\Omega u^{n+m+q-2} dx &\leq \left(\int_\Omega u^n dx\right)^{\frac{q-1}{m+2q-3}} \left(\int_\Omega u^{n+m+2q-3} dx\right)^{\frac{m+q-2}{m+2q-3}} \\ &= \left(\int_\Omega u^n dx\right)^{\frac{2(q-1)}{n}} \left(\int_\Omega u^{\frac{3n}{2}} dx\right)^{\frac{2(m+q-2)}{n}} \\ &\leq \frac{2(q-1)}{n} \int_\Omega u^n dx + \frac{2(m+q-2)}{n} \int_\Omega u^{\frac{3n}{2}} dx. \end{aligned} \tag{3.4}$$

Using Cauchy's inequality with ϵ and inverse Hölder inequality, we get

$$\int_\Omega u^{n+m+q-3} |\nabla u| dx \leq \frac{1}{4\epsilon} \int_\Omega u^{n+m+2q-3} dx + \epsilon \int_\Omega u^{n+m-3} |\nabla u|^2 dx, \tag{3.5}$$

and

$$\int_\Omega u^{n+p-1} dx \geq |\Omega|^{\frac{1-p}{n}} \left(\int_\Omega u^n dx\right)^{\frac{n+p-1}{n}}. \tag{3.6}$$

First taking (3.4) and (3.5) into (3.3), then taking (3.3) and (3.6) into (3.2), (3.2) becomes

$$\begin{aligned} \phi'(t) \leq & \left[\frac{kmd\epsilon(n+m+q-2)}{\rho_0} - m(n-1)\right] n \int_\Omega u^{n+m-3} |\nabla u|^2 dx \\ & + \left[\frac{kmnd(n+m+q-2)}{4\epsilon\rho_0} + \frac{2kmN(m+q-2)}{\rho_0}\right] \int_\Omega u^{\frac{3n}{2}} dx \\ & + \frac{2mkN(q-1)}{\rho_0} \int_\Omega u^n dx - n|\Omega|^{\frac{1-p}{n}} \left(\int_\Omega u^n dx\right)^{\frac{n+p-1}{n}}. \end{aligned} \tag{3.7}$$

Now we estimate $\int_\Omega u^{\frac{3n}{2}} dx$, using Sobolev type inequality (see (A.5) in [11]) which holds if $N = 3$ and obtain for arbitrary $\mu > 0$

$$\begin{aligned} \int_\Omega u^{\frac{3n}{2}} dx \leq & \frac{1}{3^{\frac{3}{4}}} \left[\frac{3}{2\rho_0} \int_\Omega u^n dx + \frac{n}{2}\left(1 + \frac{d}{\rho_0}\right) \int_\Omega u^{n-1} |\nabla u| dx\right]^{\frac{3}{2}} \\ \leq & \frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}} \Bigg\{ \left(\frac{3}{2\rho_0}\right)^{\frac{3}{2}} \left(\int_\Omega u^n dx\right)^{\frac{3}{2}} + \left[\frac{n}{2}(1 + \frac{d}{\rho_0})\right]^{\frac{3}{2}} \frac{1}{4\mu^3} |\Omega|^{\frac{3(m-1)}{n}} \left(\int_\Omega u^n dx\right)^{\frac{3(n-m+1)}{n}} \\ & + \left[\frac{n}{2}\left(1 + \frac{d}{\rho_0}\right)\right]^{\frac{3}{2}} \frac{3\mu}{4} \int_\Omega u^{n+m-3} |\nabla u|^2 dx \Bigg\}. \end{aligned} \tag{3.8}$$

By substituting (3.8) into (3.7), we obtain

$$\begin{aligned}\phi'(t) \leq &\left\{\left[\frac{kmnd\epsilon(n+m+q-2)}{\rho_0} - mn(n-1)\right]\right.\\ &+\left[\frac{kmnd(n+m+q-2)}{4\epsilon\rho_0} + \frac{2kmN(m+q-2)}{\rho_0}\right]\\ &\left.\times\left[\frac{n}{2}(1+\frac{d}{\rho_0})\right]^{\frac{3}{2}}\frac{3\mu}{4}\frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}}\right\}\int_\Omega u^{n+m-3}|\nabla u|^2 dx\\ &+k_1\phi^{\frac{3}{2}} + k_2\phi^{\frac{3(n-m+1)}{n}} + k_3\phi - k_4\phi^{\frac{n+p-1}{n}}.\end{aligned} \tag{3.9}$$

For $\epsilon > 0$ small enough, choosing an appropriate $\mu > 0$ such that $k_0 \leq 0$, this leads to

$$\phi'(t) \leq k_1\phi^{\frac{3}{2}} + k_2\phi^{\frac{3(n-m+1)}{n}} + k_3\phi - k_4\phi^{\frac{n+p-1}{n}}, \tag{3.10}$$

where

$$\begin{aligned}k_0 = &\left\{\left[\frac{kmnd\epsilon(n+m+q-2)}{\rho_0} - mn(n-1)\right] + \left[\frac{kmnd(n+m+q-2)}{4\epsilon\rho_0} + \frac{2kmN(m+q-2)}{\rho_0}\right]\right.\\ &\left.\times\left[\frac{n}{2}\left(1+\frac{d}{\rho_0}\right)\right]^{\frac{3}{2}}\frac{3\mu}{4}\frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}}\right\},\\ k_1 = &\left[\frac{kmnd(n+m+q-2)}{4\epsilon\rho_0} + \frac{2kmN(m+q-2)}{\rho_0}\right]\frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}}\left(\frac{3}{2\rho_0}\right)^{\frac{3}{2}},\\ k_2 = &\left[\frac{kmnd(n+m+q-2)}{4\epsilon\rho_0} + \frac{2kmN(m+q-2)}{\rho_0}\right]\frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}}\left[\frac{n}{2}\left(1+\frac{d}{\rho_0}\right)\right]^{\frac{3}{2}}\frac{1}{4\mu^3}|\Omega|^{\frac{3(m-1)}{n}},\\ k_3 = &\frac{2kmN(q-1)}{\rho_0},\\ k_4 = &\,n|\Omega|^{\frac{1-p}{n}}.\end{aligned}$$

Integrating (3.10) from 0 to t^*, we obtain

$$t^* \geq \int_{\phi(0)}^{\infty}\frac{d\eta}{k_1\eta^{\frac{3}{2}} + k_2\eta^{\frac{3(n-m+1)}{n}} + k_3\eta - k_4\eta^{\frac{n+p-1}{n}}}. \tag{3.11}$$

□

4 Non-existence of blow-up

In this section we show that if the classical solution exists then it may not blow-up when the exponents satisfy $p > m + 2(q-1)$. We define

$$\varphi(t) = \int_\Omega u^{m+1}dx. \tag{4.1}$$

We establish the following theorem.

Theorem 4.1. *Let $p > m + 2(q-1)$, if $u(x,t)$ is a classical solution of (1.1) for $t < t^* \leq \infty$ then $\varphi(t)$ is bounded for all $t < t^*$.*

Proof. Assume that $u(x,t)$ is a classical solution of (1.1) for $t < t^* \leq \infty$. Taking the derivative of (4.1), by (2.2) we have

$$\varphi'(t) = (m+1)mk \int_{\partial\Omega} u^{2m+q-1} ds - (m+1) \int_{\Omega} |\nabla u^m|^2 dx - (m+1) \int_{\Omega} u^{m+p} dx. \tag{4.2}$$

To estimate $\int_{\partial\Omega} u^{2m+q-1} ds$, we obtain

$$\begin{aligned} \int_{\partial\Omega} u^{2m+q-1} ds &\leq \frac{N}{\rho_0} \int_{\Omega} u^{2m+q-1} dx + \frac{(2m+q-1)d}{\rho_0} \int_{\Omega} u^{2m+q-2} |\nabla u| dx \\ &= \frac{N}{\rho_0} \int_{\Omega} u^{2m+q-1} dx + \frac{(2m+q-1)d}{\rho_0 m} \int_{\Omega} u^{m+q-1} |\nabla u^m| dx. \end{aligned} \tag{4.3}$$

Applying Cauchy's inequality with β, we have

$$\int_{\Omega} u^{m+q-1} |\nabla u^m| dx \leq \beta \int_{\Omega} u^{2(m+q-1)} dx + \frac{1}{4\beta} \int_{\Omega} |\nabla u^m|^2 dx. \tag{4.4}$$

Choosing $\beta = \frac{(2m+q-1)kd}{4\rho_0}$, and inserting (4.4) into (4.3), then inserting (4.3) into (4.2), we get

$$\varphi'(t) \leq (m+1) \left[\frac{kmN}{\rho_0} \int_{\Omega} u^{2m+q-1} dx + \frac{kd\beta(2m+q-1)}{\rho_0} \int_{\Omega} u^{2(m+q-1)} dx - \int_{\Omega} u^{m+p} dx \right]. \tag{4.5}$$

Using Hölder's inequality and Young's inequality with ε, we obtain

$$\begin{aligned} \int_{\Omega} u^{2(m+q-1)} dx &\leq \left(\int_{\Omega} u^{m+p} dx \right)^{\alpha} \left(\int_{\Omega} u^{2m+q-1} dx \right)^{1-\alpha} \\ &\leq \alpha \varepsilon^{\frac{1}{\alpha}} \int_{\Omega} u^{m+p} dx + (1-\alpha) \varepsilon^{\frac{1}{\alpha-1}} \int_{\Omega} u^{2m+q-1} dx, \end{aligned} \tag{4.6}$$

where $\alpha = \frac{q-1}{p-(m+q-1)}$ and $0 < \alpha < 1$ by the assumption of the theorem.

Combining (4.6) with (4.5), we find

$$\varphi'(t) \leq (m+1) \left[H \int_{\Omega} u^{2m+q-1} dx - W \int_{\Omega} u^{m+p} dx \right], \tag{4.7}$$

where

$$H = \left[(1-\alpha) \varepsilon^{\frac{1}{\alpha-1}} \frac{kd\beta(2m+q-1)}{\rho_0} + \frac{kmN}{\rho_0} \right],$$

$$W = \left[1 - \alpha \varepsilon^{\frac{1}{\alpha}} \frac{kd\beta(2m+q-1)}{\rho_0} \right],$$

we may choose ε so small that $W > 0$ holds.

Using Hölder's inequality again

$$\int_{\Omega} u^{2m+q-1} dx \leq |\Omega|^{\frac{p-(m+q-1)}{m+p}} \left(\int_{\Omega} u^{m+p} dx \right)^{\frac{2m+q-1}{m+p}}, \tag{4.8}$$

thus

$$\int_\Omega u^{m+p}dx \geq |\Omega|^{\frac{-p+(m+q-1)}{2m+q-1}} \left(\int_\Omega u^{2m+q-1}dx\right)^{\frac{m+p}{2m+q-1}}, \tag{4.9}$$

where $|\Omega|$ denotes the measure of Ω. Inserting (4.9) into (4.7), we have

$$\varphi'(t) \leq (m+1)\int_\Omega u^{2m+q-1}dx \left[H - W|\Omega|^{\frac{-p+(m+q-1)}{2m+q-1}} \left(\int_\Omega u^{2m+q-1}dx\right)^{\frac{p-(m+q-1)}{2m+q-1}}\right]. \tag{4.10}$$

Application of Hölder's inequality leads to

$$\varphi(t) = \int_\Omega u^{m+1}dx \leq \left(\int_\Omega u^{2m+q-1}dx\right)^{\frac{m+1}{2m+q-1}} |\Omega|^{\frac{m+q-2}{2m+q-1}}. \tag{4.11}$$

From the above equation, we obtain

$$\left(|\Omega|^{\frac{-(m+q-2)}{2m+q-1}} \int_\Omega u^{m+1}dx\right)^{\frac{2m+q-1}{m+1}} \leq \int_\Omega u^{2m+q-1}dx.$$

Thus from (4.10) we derive

$$\varphi'(t) \leq (m+1)\int_\Omega u^{2m+q-1}dx \left[H - W|\Omega|^{\frac{m+q-1-p}{m+1}} \varphi(t)^{\frac{p-(m+q-1)}{m+1}}\right]. \tag{4.12}$$

Since $p > m+2(q-1) \geq m+q-1$, from (4.12) one can conclude that $\varphi(t)$ is bounded for $t < t^* \leq +\infty$. In fact, if for some $t_0 < t^*$, $\varphi(t_0)$ is so large that $\left[H - W|\Omega|^{\frac{m+q-1-p}{m+1}} \varphi(t_0)^{\frac{p-(m+q-1)}{m+1}}\right]$ is negative, then $\varphi'(t) < 0$ for all $t_0 < t < t^*$ with the property $\varphi(t) > \varphi(t_0)$ since the exponent of $\varphi(t)$ is positive. Consequently, the continuously differentiable function $\varphi(t)$ is (strictly) monotone decreasing in $[t_0, t^*)$, thus $\varphi(t) \leq \varphi(t_0)$ if $t_0 < t < t^*$. □

Remark 4.2. For $q = 1$, we can see, $p = m$ is the blow-up exponent. But for $q > 1$ and $m+q-1 < p < m+2(q-1)$, we do not assert whether the solutions blow-up in finite time with nonlinear boundary condition. Due to technical reasons up to now, we can not give a positive or negative answer.

5 Example and applications

In this part, we give an example to illustrate applications of Theorem 2.1 and Theorem 3.1.

Example 5.1. Let $u(x,t)$ is a solution of the following problem

$$\begin{cases} u_t = \Delta u^3 - u^3, & (x,t) \in \Omega \times (0,t^*), \\ \frac{\partial u}{\partial \nu} = u^2, & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) = 0.5 - |x|^2 > 0, & x \in \Omega, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^3 \mid |x|^2 = \sum_{i=1}^3 x_i^2 < 0.0001\}$ is a ball in $\mathbb{R}^3$. Now $m = 3, q = 2$, $p = 3$, $k = 1$, $u_0 = 0.5 - |x|^2, N = 3, \rho_0 = 0.01, d = 0.01, |\Omega| = 4.1888 \times 10^{-6}$.

First, we get the upper bound of blow-up time through the following calculations

$$\begin{aligned}
\varphi(0) &= \int_\Omega u_0^{m+1}dx \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{0.01} (0.5-|r|^2)^4 r^2 dr \\
&= 4\pi \int_0^{0.01} (0.5-|r|^2)^4 r^2 dr = 2.6167\times 10^{-7},
\end{aligned}$$

$$\begin{aligned}
\psi(0) &= \frac{2m^2k}{2m+q-1}\int_{\partial\Omega} u_0^{2m+q-1}ds - \int_\Omega |\nabla u_0^m|^2 dx - \frac{2m}{m+p}\int_\Omega u_0^{m+p}dx \\
&= \frac{18}{7}\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{0.01} (0.5-|r|^2)^7 r^2 dr \\
&\quad - 9\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{0.01} (0.5-|r|^2)^4 |\nabla(0.5-|r|^2)|^2 r^2 dr \\
&\quad - \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{0.01} (0.5-|r|^2)^6 r^2 dr \\
&= \frac{72}{7}\pi \int_0^{0.01} (0.5-|r|^2)^7 r^2 dr - 144\pi \int_0^{0.01} (0.5-|r|^2)^4 r^4 dr \\
&\quad - 4\pi \int_0^{0.01} (0.5-|r|^2)^6 r^2 dr = 1.8111\times 10^{-8}.
\end{aligned}$$

Taking M and a into (2.13), then

$$t^* \le \frac{2m\varphi_0}{(m+1)^2 a(1+a)\psi(0)} = 4.1280. \tag{5.1}$$

Next, we obtain the lower bound of blow-up time by the following calculations

$$\begin{aligned}
\phi(0) &= \int_\Omega u_0^{2(m+2q-3)}dx \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} sin\varphi d\varphi \int_0^{0.01} (0.5-|r|^2)^8 r^2 dr \\
&= 4\pi \int_0^{0.01} (0.5-|r|^2)^8 r^2 dr = 1.6347\times 10^{-8}.
\end{aligned}$$

We choose $\epsilon = 0.1$, $\mu = 0.0022$, and calculate that

$$k_1 = 6.9069\times 10^6, \quad k_2 = 1.8015\times 10^8, \quad k_3 = 1800, \quad k_4 = 176.8348.$$

Then

$$t^* \ge \int_{\phi(0)}^{\infty} \frac{d\eta}{k_1\eta^{\frac{3}{2}} + k_2\eta^{\frac{3(n-m+1)}{n}} + k_3\eta - k_4\eta^{\frac{n+p-1}{n}}} = 0.0012. \tag{5.2}$$

Therefore, combining (5.1) with (5.2), we get

$$0.0012 \le t^* \le 4.1280.$$

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 9, 1–19; https://doi.org/10.14232/ejqtde.2021.1.9 www.math.u-szeged.hu/ejqtde/

Permanence and exponential stability for generalised nonautonomous Nicholson systems

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Received 10 November 2020, appeared 7 February 2021
Communicated by Leonid Berezansky

Abstract. The paper is concerned with nonautonomous generalised Nicholson systems under conditions which imply their permanence: by refining the assumptions for permanence, explicit lower and upper uniform bounds for all positive solutions are provided, as well as criteria for the global exponential stability of these systems. In particular, for periodic systems, conditions for the existence of a globally exponentially attractive positive periodic solution are derived.

Keywords: delay differential equations, Nicholson systems, exponential stability, permanence.

2020 Mathematics Subject Classification: 34K12, 34K25, 34K20, 92D25.

1 Introduction

In a recent paper [9], the permanence for a family of multidimensional nonautonomous and noncooperative delay differential equations (DDEs), which includes a large spectrum of structured models used in population dynamics and other fields, was investigated. Once the permanence is established, several question about the global behaviour of solutions arise. To further analyse the stability and other features of such models, it is, however, clear that the conditions to be imposed depend heavily on the shape and properties of the nonlinear terms.

Nicholson-type systems constitute a specific case included in such family. Here, we consider a nonautonomous generalised Nicholson system with bounded distributed delays given by

$$\begin{aligned} x_i'(t) = &- d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)x_j(t) \\ &+ \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^{t} \lambda_{ik}(s)x_i(s)e^{-c_{ik}(s)x_i(s)}\,ds, \quad t \geq t_0,\ i = 1, \dots, n, \end{aligned} \tag{1.1}$$

where all the coefficients and delays are continuous, nonnegative and satisfy some additional conditions described in the next section.

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Since the introduction of the classic Nicholson's blowflies equation

$$x'(t) = -dx(t) + px(t-\tau)e^{-ax(t-\tau)} \quad (a, d, p, \tau > 0), \tag{1.2}$$

by Gurney et al. [12], as a model based on the experimental data of Nicholson [18] and constructed to study the Australian sheep blowfly pest, the original equation (1.2) as well as a large number of modified and generalised scalar models have been extensively used in population dynamics and other mathematical biology contexts – yet, many open problems concerning the asymptotic behaviour of solutions to scalar Nicholson equations remain unsolved [1]. In recent years, Nicholson-type systems have received much attention in view of their applications as models for populations structured in several patches or classes (see e.g. [2] for some concrete applications). Significant progress has been made, addressing topics such as the extinction, permanence, existence of positive equilibria or periodic solutions, stability of solutions, global attractivity of equilibria or periodic solutions. Systems with autonomous coefficients (and either autonomous or time-dependent delays) were investigated in [2,3,6,7,11,14,25], whereas the works [4,8–10,15,16,21,22,24] were concerned with nonautonomous versions of such systems.

The purpose of this paper is to complement the studies in [8,9], with more results on the large time behaviour of solutions to (1.1), by providing criteria for their global exponential stability, as well as explicit uniform lower and upper bounds for all positive solutions. The results on stability are obtained by refining the assumptions for permanence established previously in [9]. In [8], the existence of a positive periodic solution for *periodic* Nicholson's blowflies systems was analysed, and, in the case of systems with all discrete delays multiples of the period, criteria for the global attractivity of such a positive periodic solution established. Here, we provide sufficient conditions for the *exponential stability* of any positive solution of (1.1), without any constraint on the type of delays.

We emphasize that, in spite of the recent interest in *nonautonomous* Nicholson systems, only a few authors have exhibited criteria for their stability, usually for periodic or almost periodic Nicholson equations or systems with *discrete* time-delays; see [5,8,13,15–17,21,23,24] and references therein. Typically, conditions have been imposed in such a way that convenient lower and upper bounds for all solutions hold. Here, as we shall see, the permanence is still a key ingredient to prove the stability, however, only an explicit *upper bound* for solutions of such systems will be required. The criteria enhance and extend some recent achievements in the literature in several ways: not only are the imposed assumptions less restrictive than the ones found in recent papers, but (1.1) is much more general: namely, it incorporates *distributed* delays, not all coefficients are required to be bounded and the global exponential stability is studied for a model that is not necessarily periodic or almost periodic.

This paper is organized as follows: Section 2 is devoted to the study of uniform lower and upper bounds for the positive solutions of (1.1). Section 3 addresses the global stability of (1.1). Examples and a comparison with recent results in the literature [13,16,21,23] are also given, in particular for periodic systems. A brief section of conclusions ends the paper.

2 Permanence: uniform bounds for the solutions

For simplicity of exposition, and without loss of generality, take $t_0 = 0$ in (1.1) and let $\tau = \sup\{\tau_{ik}(t) : t \geq 0, i = 1, \dots, n, k = 1, \dots, m_i\} > 0$. Take $C := C([-\tau, 0]; \mathbb{R}^n)$ with the supremum norm $\|\phi\| = \max_{\theta \in [-\tau, 0]} |\phi(\theta)|$ as the phase space. In abstract form, system (1.1) is written as the DDE

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)x_j(t) + f_i(t, x_{i,t}), \quad t \geq 0,\ i = 1, \dots, n, \tag{2.1}$$

where the nonlinearities take the form

$$f_i(t, x_{i,t}) = \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^{t} \lambda_{ik}(s)x_i(s)e^{-c_{ik}(s)x_i(s)}\, ds, \quad i = 1, \dots, n. \tag{2.2}$$

For (1.1), define the $n \times n$ matrices

$$\begin{aligned} D(t) &= \operatorname{diag}\left(d_1(t), \dots, d_n(t)\right), \quad A(t) = \left[a_{ij}(t)\right] \\ B(t) &= \operatorname{diag}\left(\beta_1(t), \dots, \beta_n(t)\right), \quad t \geq 0, \end{aligned} \tag{2.3}$$

where we may suppose that $a_{ii}(t) \equiv 0$ (since $a_{ii}(t)$ may be incorporated in $d_i(t)$) and $\beta_i(t)$ denotes

$$\beta_i(t) := \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^{t} \lambda_{ik}(s)\, ds, \quad t \geq 0, \quad i = 1, \dots, n;$$

The following assumptions will be considered:

(h1) $d_i(t), a_{ij}(t), b_{ik}(t), \tau_{ik}(t),\ \lambda_{ik}(t), c_{ik}(t)$ are continuous and nonnegative with $d_i(t) > 0$, $c_{ik}(t) \geq \underline{c_i} > 0, \beta_i(t) > 0$, $\tau_{ik}(t) \in [0, \tau]$, $c_{ik}(t)$ are *bounded*, for $i, j = 1, \dots, n, k = 1, \dots, m_i$ and $t \geq 0$;

(h2) there is a positive vector u such that $\liminf_{t\to\infty} \left[D(t) - A(t)\right]u > 0$;

(h3) there are a positive vector v and $T > 0, \alpha > 1$ such that $B(t)v \geq \alpha[D(t) - A(t)]v$ for $t \geq T$.

The particular case of (1.1) with $c_{ik}(t) \equiv 1$ for $1 \leq i \leq n,\ 1 \leq k \leq m_i$, is expressed by

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{k=1}^{m_i} b_{ik}(t) \int_{t-\tau_{ik}(t)}^{t} \lambda_{ik}(s)h(x_i(s))\, ds, \quad 1 \leq i \leq n, \tag{2.4}$$

for $h(x) = xe^{-x}$, $x \geq 0$. Note that the nonlinearity h is unimodal, $e^{-1} = h(1) = \max_{x\geq 0} h(x)$, $h(\infty) = 0$ and $x = 2$ is its unique inflexion point.

We now set the usual orders in $\mathbb{R}^n$ and C. $\mathbb{R}^n$ may be seen as the subset of constant functions in C. We suppose that $\mathbb{R}^n$ is equipped with the maximum norm $|\cdot|$. Let $\mathbb{R}^+ = [0, \infty)$. A vector $v \in \mathbb{R}^n$ is nonnegative, with notation $v \geq 0$ (respectively, positive, denoted by $v > 0$), if $v \in (\mathbb{R}^+)^n$ (respectively $v \in (0, \infty)^n$). We denote $\vec{1} = (1, \dots, 1)$. Consider the cone $C^+ = C([-\tau, 0]; (\mathbb{R}^+)^n)$ of nonnegative functions in C and the partial order in C yielded by C^+: $\phi \leq \psi$ if and only if $\psi - \phi \in C^+$. Thus, $\phi \geq 0$ if and only if $\phi \in C^+$. We write $\phi > 0$ if $\phi(\theta) > 0$ for $-\tau \leq \theta \leq 0$. The relations $\leq$ and $<$ are defined in the obvious way. For $u, v \in \mathbb{R}^n$ with $u \leq v$, $[u, v] \subset C$ denotes the ordered interval $[u, v] = \{\phi \in C : u \leq \phi \leq v\}$.

Due to the real-world interpretation of our models, we take

$$C_0^+ = \{\phi \in C^+ : \phi(0) > 0\}$$

as the set of admissible initial conditions, and only consider solutions $x(t) = x(t, t_0, \phi)$ of (1.1) with initial conditions $x_{t_0} = \phi,\ \phi \in C_0^+$. It is clear that such solutions are defined and positive on $\mathbb{R}^+$.

The definitions of permanence and global stability are recalled below.

Definition 2.1. Consider a DDE $x'(t) = f(t, x_t)$ in C for which all solutions $x(t) = x(t, 0, \phi)$ with $\phi \in C_0^+$ are defined on $\mathbb{R}^+$. The DDE is said to be **permanent** if there exist positive constants m, M such that all solutions $x(t) = x(t, 0, \phi)$ with $\phi \in C_0^+$ satisfy

$$m \le \liminf_{t\to\infty} x_i(t), \quad \limsup_{t\to\infty} x_i(t) \le M \quad \text{for } i = 1, \dots, n.$$

For short, we say that $x'(t) = f(t, x_t)$ is **globally attractive** (in C_0^+) if all positive solutions are globally attractive: for any $\phi, \psi \in C_0^+$,

$$x(t, 0, \phi) - x(t, 0, \psi) \to 0 \quad \text{as } t \to \infty;$$

and the DDE $x'(t) = f(t, x_t)$ is said to be (eventually) **globally exponentially stable** if there exist $\delta > 0, M > 0$ such that, for any $\phi \in C_0^+$, there is $T \ge 0$ such that

$$|x(t, t_0, \phi) - x(t, t_0, \psi)| \le M e^{-\delta(t-t_0)} \|\phi - \psi\|, \quad \text{for } t \ge t_0 \ge T,\ \psi \in C_0^+.$$

Note that δ, M do not depend on t_0, ϕ, though a priori T depends on ϕ.

Although the nonlinear terms in (1.1) are nonmonotone, results for cooperative systems from [19] will be used.

Definition 2.2. A DDE $x'(t) = f(t, x_t)$ is **cooperative** if $f = (f_1, \dots, f_n)$ satisfies the *quasi-monotone condition* (Q) in [19], as follows:

if $\phi, \psi \in C^+$ and $\phi \ge \psi$, then $f_i(t, \phi) \ge f_i(t, \psi)$ for $t \ge 0$, whenever $\phi_i(0) = \psi_i(0)$ for some i.

In [9], the permanence of generalised Nicholson systems was established.

Theorem 2.3 ([9, Corollary 3])**.** *Assume (h1)–(h3) and that $\beta_i(t)$ are bounded on $\mathbb{R}^+$. Then (1.1) is permanent.*

Remark 2.4. When $\liminf_{t\to\infty} d_i(t) > 0$, for all i, Theorem 2.3 is still valid if one replaces (h2) by the assumptions $D(t)u \ge \alpha A(t)u, t \gg 1$, for some vector $u > 0$ and constant $\alpha > 1$. Similarly, (h3) can be replaced by the condition $\liminf_{t\to\infty} \big[B(t) - D(t) + A(t)\big]v > 0$, for some vector $v > 0$, when $\beta_i(t)$ are all bounded. In fact, if $\beta_i(t)$ are bounded below and above by positive constants, for all i, conditions $\liminf_{t\to\infty} \big[B(t) - D(t) + A(t)\big]v > 0$ and (h3) are equivalent. See [9] for details.

Remark 2.5. In fact, instead of (2.1), more general Nicholson systems with possible delays in the linear terms were considered in [9]:

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} L_{ij}(t)x_{j,t} + f_i(t, x_{i,t}), \quad t \ge 0,\ i = 1, \dots, n, \tag{2.5}$$

where f_i are as in (2.2) and $L_{ij}(t)$ are linear bounded functionals, *nonnegative* (i.e. $L_{ij}(t)(\psi) \ge 0$ for $\psi \in C([-\tau, 0]; \mathbb{R}^+)$) and continuous in t. With $\|L_{ij}(t)\| = a_{ij}(t)$, the permanence of such systems was also established in [9], if in addition to (h1)–(h3) $a_{ij}(t)$ are bounded and $\beta_i(t)$ bounded below and above by positive constants.

When (h2) and (h3) are satisfied simultaneously by a same vector $v = (v_1, \dots, v_n) > 0$, there are δ, α such that

$$\liminf_{t\to\infty} \Big(d_i(t)v_i - \sum_j a_{ij}(t)v_j \Big) \geq \delta > 0, \quad \liminf_{t\to\infty} \frac{\beta_i(t)v_i}{d_i(t)v_i - \sum_j a_{ij}(t)v_j} \geq \alpha > 1, \quad i = 1, \dots, n.$$

This motivates the following definition: for $t \geq 0$ and $v = (v_1, \dots, v_n) > 0$ such that $[D(t) - A(t)]v \neq 0$, set

$$\gamma_i(t, v) = \frac{\beta_i(t)v_i}{d_i(t)v_i - \sum_j a_{ij}(t)v_j}, \quad i = 1, \dots, n. \tag{2.6}$$

For the particular case $v = \vec{1} := (1, \dots, 1)$, we obtain

$$\gamma_i(t) := \gamma_i(t, \vec{1}) = \frac{\beta_i(t)}{d_i(t) - \sum_j a_{ij}(t)}, \quad i = 1, \dots, n. \tag{2.7}$$

Next result gives sufficient conditions, expressed in terms of $\gamma_i(t, v)$, for the positive invariance of some specific intervals under (1.1), and also provides explicit uniform lower and upper bounds for all solutions.

Theorem 2.6. *For (1.1), assume (h1), and that $c_{ik}(t)$ are bounded below and above on $\mathbb{R}^+$ by positive constants, and denote $\underline{c_i}, \overline{c_i}$ such that*

$$0 < \underline{c_i} \leq c_{ik}(t) \leq \overline{c_i} \quad \text{for } t \in \mathbb{R}^+,\ 1 \leq i \leq n,\ 1 \leq k \leq m_i.$$

Suppose that there are constants a, b with $0 < a \leq b$, $t_0 \geq 0$ and a vector $v = (v_1, \dots, v_n) > 0$ such that

$$e^a \leq \gamma_i(t, v) \leq e^b, \quad 1 \leq i \leq n,\ t \geq t_0, \tag{2.8}$$

and define

$$\underline{C} = \underline{C}(v) := \min_{1\leq i\leq n} (\underline{c_i} v_i), \quad \overline{C} = \overline{C}(v) := \max_{1\leq i\leq n} (\overline{c_i} v_i). \tag{2.9}$$

Then:

(a) The ordered interval $[m\overline{C}^{-1}v, \underline{C}^{-1}e^{b-1}v] = \{\phi = (\phi_1, \dots, \phi_n) \in C : m\overline{C}^{-1}v_i \leq \phi_i \leq \underline{C}^{-1}e^{b-1}v_i,$ $i = 1, \dots, n\} \subset C$, where $m\underline{C}^{-1}\overline{C} \in (0,1)$ is such that

$$m \leq a \quad \text{and} \quad h(m\overline{c_i}v_i\underline{C}^{-1}) \leq h(\overline{c_i}v_i\underline{C}^{-1}e^{b-1}), \quad i = 1, \dots, n, \tag{2.10}$$

is positively invariant for (1.1) and $t \geq t_0$.

(b) If $\beta_i(t)$ are also bounded below and above by positive constants, any positive solution $x(t) = (x_1(t), \dots, x_n(t))$ of (1.1) satisfies

$$m\overline{C}^{-1}v_i \leq \liminf_{t\to\infty} x_i(t) \leq \limsup_{t\to\infty} x_i(t) \leq e^{b-1}\underline{C}^{-1}v_i, \quad i = 1, \dots, n. \tag{2.11}$$

Proof. (a) Write (1.1) as $x'(t) = F(t, x_t)$, with the components F_i of F given by

$$F_i(t, \phi) = -d_i(t)\phi_i(0) + \sum_j a_{ij}(t)\phi_j(0) + \sum_{k=1}^{m_i} b_{ik}(t) \int_{-\tau_{ik}(t)}^0 \lambda_{ik}(t+s)h_{ik}(t+s, \phi_i(s))\, ds, \quad i = 1, \dots, n,$$

where $h_{ik}(s,x) := xe^{-c_{ik}(s)x}$. Let $\underline{c_i}, \overline{c_i}$ be such $0 < \underline{c_i} \le c_{ik}(s) \le \overline{c_i}$ for $s \in \mathbb{R}^+$ and all i,k, and take the functions $h_i^-(x) = xe^{-\overline{c_i}x}, h_i^+(x) = xe^{-\underline{c_i}x}$. For $h(x) = xe^{-x}$ as before, we have $h_i^-(x) = (\overline{c_i})^{-1}h(\overline{c_i}x), h_i^+(x) = (\underline{c_i})^{-1}h(\underline{c_i}x)$. Clearly, $h_i^-(x) \le h_{ik}(s,x) \le h_i^+(x) \le (\underline{c_i}e)^{-1}$, for $s,x \ge 0$.

We know already that the set $(0,\infty)^n$ is forward invariant. We now compare the solutions of (1.1) from above with the solutions of the *cooperative* system $x'(t) = F^u(t,x_t)$, where the components of F^u are given by $F_i^u(t,\phi) = -d_i(t)\phi_i(0) + \sum_j a_{ij}(t)\phi_j(0) + \beta_i(t)(\underline{c_i}e)^{-1}$. Clearly $F_i(t,\phi) \le F_i^u(t,\phi)$ for all $\phi \in C^+$. From [19], this implies that $x(t,t_0,\phi,F) \le x(t,t_0,\phi,F^u)$, where $x(t,t_0,\phi,F)$ and $x(t,t_0,\phi,F^u)$ are the solutions of $x'(t) = F(t,x_t)$ and $x'(t) = F^u(t,x_t)$ with initial condition $x_{t_0} = \phi \in C_0^+$, respectively. If $\phi \in [0, \underline{C}^{-1}e^{b-1}v]$ and $\phi_i(0) = \underline{C}^{-1}e^{b-1}v_i$ for some i, the use of (2.8) implies

$$\begin{aligned}
F_i^u(t,\phi) &\le \underline{C}^{-1}e^{b-1}\Big[-d_i(t)v_i + \sum_j a_{ij}(t)v_j\Big] + \beta_i(t)(\underline{c_i}e)^{-1}\\
&\le \Big[d_i(t)v_i - \sum_j a_{ij}(t)v_j\Big]\Big[-\underline{C}^{-1}e^{b-1} + \gamma_i(t,v)(\underline{c_i}v_ie)^{-1}\Big]\\
&\le e^{b-1}\Big[d_i(t)v_i - \sum_j a_{ij}(t)v_j\Big](-\underline{C}^{-1} + (\underline{c_i}v_i)^{-1}) \le 0.
\end{aligned}$$

From [19, p. 82], the set $(0, \underline{C}^{-1}e^{b-1}v] \subset C$ is positively invariant for (1.1).

Next, we start by observing that, for any $a,b > 0$ with $a \le b$, we have $a < e^{a-1} \le e^{b-1}$ for all $a \ne 1$. By considering the cases $a < e^{b-1} \le 1$, $a < 1 \le e^{b-1}$ or $1 \le a < e^{b-1}$, it is possible to choose $m \in (0, \underline{C}\overline{C}^{-1})$ such that conditions (2.10) are fulfilled. We get

$$h_i^-(\underline{C}^{-1}e^{b-1}v_i) = (\overline{c_i})^{-1}h(\underline{C}^{-1}e^{b-1}\overline{c_i}v_i) \ge (\overline{c_i})^{-1}h(m\underline{C}^{-1}\overline{c_i}v_i) = h_i^-(m\underline{C}^{-1}v_i).$$

As $1 > m\overline{c_i}v_i\underline{C}^{-1}$ and h is increasing on $(0,1)$, for ϕ_i such that $m\overline{C}^{-1}v_i \le \phi_i(s) \le \underline{C}^{-1}e^{b-1}v_i$ we therefore obtain

$$h_i^-(\phi_i(s)) \ge h_i^-(m\underline{C}^{-1}v_i)$$

and $F_i(t,\phi) \ge F_i^l(t,\phi) := -d_i(t)\phi_i(0) + \sum_j a_{ij}(t)\phi_j(0) + \beta_i(t)h_i^-(m\underline{C}^{-1}v_i)$ for $i = 1,\dots,n$.

Consider the interval $\hat{I} = [m\overline{C}^{-1}v, \underline{C}^{-1}e^{b-1}v] \subset C$. For $\phi \in \hat{I}$ with $\phi_i(0) = m\overline{C}^{-1}v_i$ for some i, the lower bound in (2.8) leads to

$$\begin{aligned}
F_i^l(t,\phi) &\ge \Big[d_i(t)v_i - \sum_j a_{ij}(t)v_j\Big]\Big[-m\overline{C}^{-1} + \gamma_i(t,v)v_i^{-1}h_i^-(m\overline{C}^{-1}v_i)\Big]\\
&\ge \big[d_i(t)v_i - \sum_j a_{ij}(t)v_j\big]\Big[-m\overline{C}^{-1} + \gamma_i(t,v)(v_i\overline{c_i})^{-1}h(m\overline{C}^{-1}\overline{c_i}v_i)\Big]\\
&= m\overline{C}^{-1}\big[d_i(t)v_i - \sum_j a_{ij}(t)v_j\big]\Big[-1 + \gamma_i(t,v)e^{-m\overline{C}^{-1}\overline{c_i}v_i}\Big]\\
&\ge m\overline{C}^{-1}\big[d_i(t)v_i - \sum_j a_{ij}(t)v_j\big]\big[-1 + e^ae^{-m}\big] \ge 0.
\end{aligned}$$

Hence, from [19] it follows that $\hat{I}$ is positively invariant for (1.1).

(b) Next, assume also that $0 < \underline{\beta} \le \beta_i(t) \le \overline{\beta}$ for $t \ge 0$. From (2.8),

$$d_i(t)v_i - \sum_j a_{ij}(t)v_j \ge e^{-b}\underline{\beta}v_i,\ \ \beta_i(t)v_i \ge e^a[d_i(t)v_i - \sum_j a_{ij}(t)v_j],$$

hence (h2)–(h3) are satisfied. From Theorem 2.3, (2.4) is permanent.

Fix any positive solution $x(t)$ of (2.4), define

$$\bar{x}_i = \limsup_{t\to\infty} x_i(t), \quad i = 1,\dots,n,$$

and let $\max_j(v_j^{-1}\bar{x}_j) = v_i^{-1}\bar{x}_i$ for some i. By the fluctuation lemma, there exists a sequence $t_k \to \infty$ such that $x_i(t_k) \to \bar{x}_i$ and $x_i'(t_k) \to 0$. Without loss of generality, we can also suppose that $v_i^{-1}x_i(t_k) = \max_{1\le j\le n} v_j^{-1}x_j(t_k)$ for k large – otherwise, we choose $t_k \to \infty$ such that, for some subsequence, $v_i^{-1}x_i(t_k) = \max_{1\le j\le n}\max_{t\in[k\tau,(k+1)\tau]} v_j^{-1}x_j(t)$. Thus, reasoning as in (a),

$$\begin{aligned} x_i'(t_k) &\le -d_i(t_k)x_i(t_k) + \sum_{j=1}^n a_{ij}(t_k)v_i^{-1}v_jx_i(t_k) + \beta_i(t_k)(\underline{c_i}e)^{-1} \\ &\le v_i^{-1}\Big(d_i(t_k)v_i - \sum_{j=1}^n a_{ij}(t_k)v_j\Big)\Big[-x_i(t_k) + \gamma_i(t,v)(\underline{c_i}e)^{-1}\Big] \\ &\le v_i^{-1}\Big(d_i(t_k)v_i - \sum_{j=1}^n a_{ij}(t_k)v_j\Big)\Big[-x_i(t_k) + e^{b-1}(\underline{c_i}v_i)^{-1}v_i\Big] \\ &\le v_i^{-1}\Big(d_i(t_k)v_i - \sum_{j=1}^n a_{ij}(t_k)v_j\Big)\Big[-x_i(t_k) + e^{b-1}\underline{C}^{-1}v_i\Big]. \end{aligned} \tag{2.12}$$

Consider a subsequence of (t_k), still denoted by (t_k), for which $d_i(t_k) - \sum_{j=1}^n a_{ij}(t_k) \to \ell > 0$. By letting $k \to \infty$, we obtain $0 \le -\bar{x}_i + e^{b-1}\underline{C}^{-1}v_i$, thus $\bar{x}_i \le e^{b-1}\underline{C}^{-1}v_i$. For $j \ne i$, it follows that $\bar{x}_j \le v_jv_i^{-1}\bar{x}_i \le e^{b-1}\underline{C}^{-1}v_j$.

Proceeding as in (a), in a similar way one can now show that $\liminf_{t\to\infty} x(t) \ge m\overline{C}^{-1}v$ for all positive solutions. This proves (b). □

Remark 2.7. For the simpler case (2.4), where the nonlinearities are all given in terms of $h(x) = xe^{-x}$, under (h1) and

$$e^a \le \gamma_i(t) \le e^b, \quad 1 \le i \le n,\ t \ge t_0 \tag{2.13}$$

(i.e., $v = \vec{1}$ in $\gamma_i(t,v)$), we have $\overline{C} = \underline{C} = 1$; thus, the interval $[m, e^{b-1}]^n$ is forward invariant, where $m > 0$ is chosen so that $m < 1, m \le a$ and $h(m) \le h(e^{b-1})$.

We also derive the following auxiliary result.

Lemma 2.8. *For (1.1), assume (h1) and that $0 < \underline{c_i} \le c_{ik}(t) \le \overline{c_i}$ for $t \in \mathbb{R}^+, 1 \le i \le n, 1 \le k \le m_i$. Suppose also that there are a vector $v = (v_1,\dots,v_n) > 0, t \ge t_0$ and a constant γ such that*

$$0 < \gamma_i(t,v) \le \gamma, \quad 1 \le i \le n,\ t \ge t_0. \tag{2.14}$$

For $\underline{C}, \overline{C}$ as in (2.9), the interval $(0, \gamma(\underline{C}e)^{-1}v] \subset C$ is positively invariant for (1.1) ($t \ge t_0$). In particular, if (2.14) holds with

$$\gamma < 2e\underline{C}\,\overline{C}^{-1},$$

there exist solutions of (1.1) such that $0 < x_i(t) < 2(\overline{c_i})^{-1}$, $t \ge t_0$, $i = 1,\dots,n$.

Proof. The invariance of the interval $I := (0, \gamma(\underline{C}e)^{-1}v]$ for (1.1) was shown in the above proof. If in addition $\gamma < 2e\underline{C}\overline{C}^{-1}$, then $I \subset (0, 2\overline{C}^{-1}v)$, and in particular the solutions with initial conditions $\phi \in I$ satisfy $0 < \overline{c_i}x_i(t) < 2$ for $t \ge 0, 1 \le i \le n$. □

Remark 2.9. Consider e.g. the Nicholson system (2.4). If $0 < \gamma_i(t,v) \le e^b v_i$ for all i, for some $b > 0$ and a vector $v = (v_1, \dots, v_n) > 0$, from the proof of Theorem 2.6 the interval $(0, e^{b-1}v]$ is positively invariant. With $v = \vec{1}$ and $0 < \gamma_i(t) \le \gamma < e$ and the boundedness conditions in Theorem 2.6 (b), $\limsup_{t\to\infty} x_i(t) \le \gamma e^{-1} < 1$ for all positive solution; this means that (1.1) has a cooperative behaviour, because the nonlinearity $h(x)$ is monotone on $[0,1]$. Note, however, that (2.8) with e.g. $v = \vec{1}$ and $e < \gamma = e^b$ does not imply that the interval $[1,b]^n$ is positively invariant. In fact, for simplicity take $n = 1$ and consider the Nicholson equation $x'(t) = -d(t)x(t) + e^b d(t)x(t-\tau)e^{-x(t-\tau)}$, for some $b > 1$. For an initial condition $1 \le \phi \le b$ such that $\phi(0) = b$ and $\phi(-\tau) = 1$, then $x'(0) = d(0)[-b + e^{b-1}] = d(0)e^b[-h(b) + h(1)] > 0$, thus $x(t) > b$ for $t > 0$ sufficiently small. Nevertheless, we conjecture that if (2.13) is satisfied with $\gamma < e^2$ and all coefficients are bounded, then all positive solutions of (2.4) satisfy $\limsup_{t\to\infty} x_i(t) < 2$ for all i. See also Remark 3.9.

3 Stability

In this section, sufficient conditions for the global exponential stability of Nicholson systems (1.1) are established.

In the sequel, the following auxiliary lemma will play an important role.

Lemma 3.1 ([8]). *Fix $m \in (0,1)$ and define $G_m : (0,2) \times [0,\infty) \to \mathbb{R}$ by*

$$G_m(x,y) = \begin{cases} \frac{h(y)-h(x)}{y-x}, & y \ne x \\ (1-x)e^{-x}, & y = x \end{cases}$$

where $h(x) = xe^{-x}, x \ge 0$. Then, $G_m(x,y)$ is continuous and, for any $x \in (0,2)$, there is $\mathfrak{M}_m(x) := \max_{y\ge m} |G_m(x,y)| < e^{-x}$.

As a consequence, for a function $h_c(x) := xe^{-cx} = c^{-1}h(cx)$ for some $c > 0$, it follows that for any fixed $x \in (0, 2c^{-1})$ and $m \in (0, c^{-1})$, we have

$$|h_c(y) - h_c(x)| \le \mathfrak{M}_m(cx)|y - x| \quad \text{for all } y \ge m, \tag{3.1}$$

where $\mathfrak{M}_m(x)$ is the function defined in the lemma above. Moreover, $\mathfrak{M}_m : (0,2) \to (0, e^{-2})$ is continuous.

We first establish a criterion for the global attractivity of (1.1).

Theorem 3.2. *Consider (1.1) under (h1)–(h3) and suppose that the coefficients $\beta_i(t), c_{ik}(t)$ are all bounded below and above by positive constants on $\mathbb{R}^+$, for all i,k. Assume in addition that there exists a positive solution $x^*(t)$ such that*

$$\limsup_{t\to\infty} c_{ik}(t)x_i^*(t) < 2, \quad i = 1,\dots,n,\ k = 1,\dots,m_i. \tag{3.2}$$

Then, any two positive solutions $x(t), y(t)$ of (1.1) satisfy

$$\lim_{t\to\infty}(x(t) - y(t)) = 0.$$

Proof. From Theorem 2.3, system (1.1) is permanent. Let $h_{ik}(t,x) = xe^{-c_{ik}(t)x}$ for $t, x \ge 0$ and all i,k.

Write $0 < \underline{\beta} \le \beta_i(t) \le \overline{\beta}, 0 < \underline{c} \le \underline{c_i} \le c_{ik}(t) \le \overline{c_i} \le \overline{c}$ for $t \in \mathbb{R}^+$ and all i,k. From the permanence of (1.1), there are m, M with $0 < m < 1 \le M$, such that any solution $x(t) = x(t,0,\phi)$ with $\phi \in C_0^+$ satisfies $m \le x_i(t) \le M$ for $i = 1,\dots,n$ and $t \ge T$, for some $T = T(\phi) > 0$. Fix a positive solution $x^*(t)$ as in (3.2), let $m_0 := \underline{c}m$ and $\varepsilon > 0$ small, so that $m_0 \le c_{ik}(t)x_i^*(t) \le 2-\varepsilon$ for all $i = 1,\dots,n, k = 1,\dots,m_i$ and $t \gg 1$. In Lemma 3.1, take the function $\mathfrak{M} := \mathfrak{M}_{m_0}$.

Effecting the changes of variables $z_i(t) = \frac{x_i(t)}{x_i^*(t)} - 1$ $(1 \le i \le n)$, system (1.1) becomes

$$\begin{aligned} z_i'(t) &= \frac{1}{x_i^*(t)}\Big\{x_i'(t) - (1+z_i(t))(x_i^*)'(t)\Big\} \\ &= \frac{1}{x_i^*(t)}\Big\{ - d_i^*(t)z_i(t) + \sum_j a_{ij}(t)x_j^*(t)z_j(t) \\ &\quad + \sum_{k=1}^{m_i} b_{ik}(t)\int_{t-\tau_{ik}(t)}^t \lambda_{ik}(s)\Big[h_{ik}\big(s, x_i^*(s)(1+z_i(s))\big) - h_{ik}\big(s, x_i^*(s)\big)\Big]ds\Big\}, \end{aligned} \tag{3.3}$$

for $i = 1,\dots,n, t \ge 0$, where

$$d_i^*(t) = \sum_j a_{ij}(t)x_j^*(t) + \sum_{k=1}^{m_i} b_{ik}(t)\int_{t-\tau_{ik}(t)}^t \lambda_{ik}(s)h_{ik}\big(s, x_i^*(s)\big)\,ds.$$

Let $z(t) = (z_1(t),\dots,z_n(t))$ be any solution of (3.3) with initial condition $z_0 \ge -1$, $z(0) > -1$. Define $-v_i = \liminf_{t\to\infty} z(t), u_i = \limsup_{t\to\infty} z(t)$. From the permanence of (1.1), in particular $-1 < -v_i \le u_i < \infty$ and, as observed, $x_i^*(t) \ge m$ and $x_i^*(t)(1+z_i(t)) \ge m$ for $t > 0$ large. Consider $u = \max_i u_i, v = \max_i v_i$. A priori, $-v, u$ can be both nonnegative, both nonpositive, or have different signs, nevertheless it is sufficient to show that $\max(u,v) = 0$.

Let $\max(u,v) = u$. In this case, $u \ge 0$. Assume for the sake of contradiction that $u > 0$. Choose i such that $u = u_i$ and take a sequence $t_k \to \infty$ with $z_i(t_k) \to u, z_i'(t_k) \to 0$.

From (3.1), we have

$$\Big|h_{ip}\big(s, x_i^*(s)(1+z_i(s))\big) - h_{ip}\big(s, x_i^*(s)\big)\Big| \le \mathfrak{M}(c_{ip}(s)x_i^*(s))x_i^*(s)|z_i(s)|,$$

for $1 \le i \le n, 1 \le p \le m_i$ and $s \ge 0$ sufficiently large. As previously, for k large we may suppose that $z_j(t_k) \le z_i(t_k)$ for all j, and from (3.3) we get

$$\begin{aligned} z_i'(t_k) &\le \frac{1}{x_i^*(t_k)}\Big[- d_i^*(t_k) + \sum_j a_{ij}(t_k)x_j^*(t_k)\Big]z_i(t_k) \\ &\quad + \sum_{p=1}^{m_i} b_{ip}(t_k)\int_{t_k-\tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s)\mathfrak{M}\big(c_{ip}(s)x_i^*(s)\big)x_i^*(s)|z_i(s)|\,ds \\ &= \frac{1}{x_i^*(t_k)}\Big\{ - z_i(t_k)\sum_{p=1}^{m_i} b_{ip}(t_k)\int_{t_k-\tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s)h_{ip}\big(s, x_i^*(s)\big)\,ds \\ &\quad + \sum_{p=1}^{m_i} b_{ip}(t_k)\int_{t_k-\tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s)\mathfrak{M}\big(c_{ip}(s)x_i^*(s)\big)x_i^*(s)|z_i(s)|\,ds\Big\} \\ &= \frac{1}{x_i^*(t_k)}\sum_{p=1}^{m_i} b_{ip}(t_k)\int_{t_k-\tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s)x_i^*(s)\Big[- z_i(t_k)e^{-c_{ip}(s)x_i^*(s)} + \mathfrak{M}\big(c_{ip}(s)x_i^*(s)\big)|z_i(s)|\Big]ds. \end{aligned} \tag{3.4}$$

By the mean value theorem for integrals, we obtain

$$z_i'(t_k) \le \frac{1}{x_i^*(t_k)} \sum_{p=1}^{m_i} x_i^*(s_{k,p}) B_{kp} b_{ip}(t_k) \int_{t_k-\tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s)\, ds, \tag{3.5}$$

where

$$B_{kp} = -z_i(t_k) e^{-c_{ip}(s_{k,p}) x_i^*(s_{k,p})} + \mathfrak{M}\big(c_{ip}(s_{k,p}) x_i^*(s_{k,p})\big) |z_i(s_{k,p})|,$$

for some $s_{k,p} \in [t_k - \tau_{ip}(t_k), t_k]$.

For some subsequence of $(s_{k,p})_{k\in\mathbb{N}}$ $(1 \le p \le m_i)$, still denoted by $(s_{k,p})$, there exist the limits $\lim_k c_{ip}(s_{k,p}) x_i^*(s_{k,p}) = \xi_p \in [m_0, 2-\varepsilon]$ and $\lim_k z_i(s_{k,p}) = w_p \in [-v, u]$. Since $\mathfrak{M}(x)$ is continuous, this leads to

$$\lim_k B_{kp} = -u e^{-\xi_p} + \mathfrak{M}(\xi_p)|w_p| \le \big(-e^{-\xi_p} + \mathfrak{M}(\xi_p)\big) u < 0,$$

since Lemma 3.1 asserts that $\mathfrak{M}(\xi) < e^{-\xi}$ for any $\xi \in (0,2)$. In particular, $B_{kp} < 0$ for k large, $p = 1, \dots, m_i$, and from (3.5) we derive that

$$z_i'(t_k) \le \frac{m}{M} \beta_i(t_k) \max_{1\le p\le m_i} B_{kp} \le \frac{m}{M} \underline{\beta_i} \max_{1\le p\le m_i} B_{kp}.$$

By letting $k \to \infty$, this estimate yields

$$0 \le \max_{1\le p\le m_i} \big(-e^{-\xi_p} + \mathfrak{M}(\xi_p)\big) u < 0,$$

which is not possible. Thus, $u = 0$.

Similarly, consider the situation when $\max(u, v) = v$ (which implies $v \ge 0$), and suppose that $v > 0$. By choosing i such that $v = v_i$ and a sequence $t_k \to \infty$ with $z_i(t_k) \to -v$, $z_i'(t_k) \to 0$, for any $\varepsilon > 0$ small and k sufficiently large, reasoning as above we obtain

$$\begin{aligned} z_i'(t_k) &\ge -\frac{1}{x_i^*(t_k)} \sum_{p=1}^{m_i} b_{ip}(t_k) \int_{t_k-\tau_{ip}(t_k)}^{t_k} \lambda_{ip}(s) x_i^*(s) \Big[z_i(t_k) e^{-c_{ip}(s) x_i^*(s)} + \mathfrak{M}\big(c_{ip}(s) x_i^*(s)\big) |z_i(s)| \Big] ds \\ &\ge -\frac{m}{M} \beta_i(t_k) \max_{1\le p\le m_i} C_{kp}, \end{aligned}$$

where now

$$C_{kp} = z_i(t_k) e^{-c_{ip}(s_{k,p}) x_i^*(s_{k,p})} + \mathfrak{M}\big(c_{ip}(s_{k,p}) x_i^*(s_{k,p})\big) |z_i(s_{k,p})|$$

for some subsequences $s_{k,p} \in [t_k - \tau_{ip}(t_k), t_k]$. In an analogous way, by taking convergent subsequences of the sequences $c_{ip}(s_{k,p}) x_i^*(s_{k,p})$ and $z_i(s_{k,p})$, we obtain a contradiction from Lemma 3.1. Consequently, $v = 0$. This completes the proof. □

Note that hypotheses (h2), (h3) in the statement of Theorem 3.2 were imposed only to derive the permanence of (1.1). In fact, the above proof applies if, instead of the permanence, all solutions are bounded and persistent; in other words, if for any $\phi \in C_0^+$ there are constants $m(\phi), M(\phi)$, such that $0 < m(\phi) \le \liminf_{t\to\infty} x(t, 0, \phi) \le \limsup_{t\to\infty} x(t, 0, \phi) \le M(\phi)$.

We are ready to state our main result, on the global exponential stability of (1.1).

Theorem 3.3. *Suppose that the hypotheses of Theorem 3.2 are satisfied. Then, (1.1) is (eventually) globally exponentially stable: there exist $\delta > 0, L > 0$ such that, for any $\phi^* \in C_0^+$, there is $T = T(\phi^*)$ such that*

$$|x(t, t_0, \phi) - x(t, t_0, \phi^*)| \le L e^{-\delta(t-t_0)} \|x_{t_0}(0, \phi) - x_{t_0}(0, \phi^*)\|, \quad t \ge t_0 \ge T,\ \phi \in C_0^+. \tag{3.6}$$

Proof. As in the proof of Theorem 3.2, take m, M so that any positive solution $x(t)$ of (1.1) satisfies

$$m < \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) < M, \quad i = 1, \dots, n,$$

and consider the previous notation for $\underline{\beta}, m_0 := \underline{c}m$ and $\mathfrak{M} := \mathfrak{M}_{m_0}$. Since $\mathfrak{M}(\xi) < e^{-\xi}$ on $(0, 2)$, from the continuity of $\mathfrak{M}$ it follows that, for any $\varepsilon > 0$ small, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\delta + \frac{m}{M}\underline{\beta}\Big[\mathfrak{M}(\xi)e^{\delta\tau} - e^{-\xi}\Big] < 0 \quad \text{for all } \xi \in [m_0, 2-\varepsilon]. \tag{3.7}$$

From Theorem 3.2, if $x^*(t)$ is a solution as in (3.2), any positive solution of (1.1) also satisfies (3.2).

Fix any positive solution $x^*(t) = x(t, 0, \phi^*)$ of (1.1) with $\phi^* \in C_0^+$, and take $T = T(\phi^*) \ge \tau$ and $\varepsilon > 0$ in such a way that $m \le x_i^*(t) \le M, m_0 \le c_{ik}(t)x_i^*(t) \le 2 - \varepsilon$ for all $t \ge T - \tau, i = 1, \dots, n,\ k = 1, \dots, m_i$. Consider any other positive solution $x(t) = x(t, 0, \phi)$ with $\phi \in C_0^+$, and any $t_0 \ge T$; in particular note that $x^*(t) > 0$ for $t \ge t_0$.

Next, effect the changes of variables $z_i(t) = e^{\delta t}\big(\frac{x_i(t)}{x_i^*(t)} - 1\big)$ $(1 \le i \le n)$, where $\delta > 0$ satisfies (3.7). Keeping the notations in Theorem 3.2, the transformed system is

$$\begin{aligned} z_i'(t) = \delta z_i(t) + \frac{1}{x_i^*(t)}\bigg\{ &- d_i^*(t)z_i(t) + \sum_j a_{ij}(t)x_j^*(t)z_j(t) \\ &+ e^{\delta t}\sum_{k=1}^{m_i} b_{ik}(t)\int_{t-\tau_{ik}(t)}^{t} \lambda_{ik}(s)\Big[h_{ik}\big(s, x_i^*(s)(1 + e^{-\delta s}z_i(s))\big) - h_{ik}\big(s, x_i^*(s)\big)\Big]ds\bigg\}, \end{aligned} \tag{3.8}$$

We now claim that the solution of (3.8) with initial condition $z_{t_0} = \psi$ satisfies

$$|z(t, t_0, \psi)| \le \|\psi\|, \quad t \ge t_0. \tag{3.9}$$

Otherwise, suppose that there exist $t_1 > t_0$ and $i \in \{1, \dots, n\}$ such that

$$|z(t_1)| = |z_i(t_1)| > \|\psi\|, \quad |z_j(t)| < |z_i(t_1)|, \quad \text{for } t \in [t_0 - \tau, t),\ 1 \le j \le n.$$

Consider the case $z_i(t_1) > 0$ (the case $z_i(t_1) < 0$ is analogous). From the definition of t_1, we have $z_i'(t_1) \ge 0$. On the other hand, from (3.7), (3.8) and reasoning as in (3.4), we obtain

$$\begin{aligned} z_i'(t_1) &\le \frac{1}{x_i^*(t_1)}\bigg\{\Big[\delta x_i^*(t_1) - \Big(d_i^*(t_1) - \sum_j a_{ij}(t_1)x_j^*(t_1)\Big)\Big]z_i(t_1) \\ &\quad + e^{\delta t_1}\sum_{k=1}^{m_i} b_{ik}(t_1)\int_{t_1-\tau_{ik}(t_1)}^{t_1} \lambda_{ik}(s)\mathfrak{M}\big(c_{ik}(s)x_i^*(s)\big)e^{-\delta s}x_i^*(s)|z_i(s)|ds\bigg\}, \\ &\le \frac{1}{x_i^*(t_1)}\bigg\{\Big[\delta x_i^*(t_1) - \sum_{k=1}^{m_i} b_{ik}(t_1)\int_{t_1-\tau_{ik}(t_1)}^{t_1} \lambda_{ik}(s)h_{ik}\big(s, x_i^*(s)\big)\,ds\Big]z_i(t_1) \\ &\quad + e^{\delta\tau}\sum_{k=1}^{m_i} b_{ik}(t_1)\int_{t_1-\tau_{ik}(t_1)}^{t_1} \lambda_{ik}(s)\mathfrak{M}\big(c_{ik}(s)x_i^*(s)\big)x_i^*(s)|z_i(s)|\,ds\bigg\} \\ &\le \frac{z_i(t_1)}{x_i^*(t_1)}\bigg\{\delta x_i^*(t_1) + \sum_{k=1}^{m_i} b_{ik}(t_1)\int_{t_1-\tau_{ik}(t_1)}^{t_1} \lambda_{ik}(s)x_i^*(s)\Big[-e^{-c_{ik}(s)x_i^*(s)} + e^{\delta\tau}\mathfrak{M}\big(c_{ik}(s)x_i^*(s)\big)\Big]\,ds\bigg\} \\ &< \frac{z_i(t_1)}{x_i^*(t_1)}\bigg\{\delta x_i^*(t_1) - \frac{\delta M}{m\underline{\beta}}\sum_{k=1}^{m_i} b_{ik}(t_1)\int_{t_1-\tau_{ik}(t_1)}^{t_1} \lambda_{ik}(s)x_i^*(s)\,ds\bigg\} \\ &\le \delta M\frac{z_i(t_1)}{x_i^*(t_1)}\bigg(1 - \frac{\beta_i(t_1)}{\underline{\beta}}\bigg) \le 0, \end{aligned}$$

which is a contradiction. Thus (3.9) holds.

Going back to the solution $x(t)$, for $t \geq t_0$ and $i = 1, \dots, n$ we have

$$e^{\delta t}|x_i(t) - x_i^*(t)| = x_i^*(t)|z_i(t)| \leq \|z_{t_0}\| = \sup_{\theta \in [-\tau, 0]} \left\{ e^{\delta(t_0+\theta)} \left(\frac{x_{t_0}(0,\phi)}{x_{t_0}(0,\phi^*)} - 1 \right) \right\}$$
$$\leq e^{\delta t_0} \frac{M}{m} \|x_{t_0}(0,\phi) - x_{t_0}(0,\phi^*)\|.$$

The proof of (3.6) is complete. □

From the above results, a pratical criterion to deduce the global exponential stability of (1.1) is given below.

Theorem 3.4. *For (1.1), assume (h1), suppose that $\beta_i(t), c_{ik}(t)$ are all bounded below and above by positive constants. Assume also that there are a vector $v = (v_1, \dots, v_n) > 0$ and constants α, γ such that*

$$1 < \alpha \leq \gamma_i(t,v) \leq \gamma < 2e\,\underline{C}\,\overline{C}^{-1}, \quad 1 \leq i \leq n,\ t \gg 1, \tag{3.10}$$

where $0 < \underline{c_i} \leq c_{ik}(t) \leq \overline{c_i}$ for $t \in \mathbb{R}^+$ and all i,k and $\underline{C} = \underline{C}(v)\ \overline{C} = \overline{C}(v)$ are as in (2.9). Then, (1.1) is (eventually) globally exponentially stable.

Proof. Clearly, $\beta_i(t) \geq \underline{\beta_i} > 0$ on $\mathbb{R}^+$ and $1 < \alpha \leq \gamma_i(t,v) \leq \gamma$ $(1 \leq i \leq n)$ imply that (h2), (h3) hold. The result follows immediately from Theorem 3.3 and Lemma 2.8. □

Remark 3.5. If all the coefficients are bounded, one can easily check that Theorems 2.6, 3.3 and 3.4 are still valid for systems of the form (2.5).

For Nicholson systems (2.4), the above results are written in a simpler form.

Corollary 3.6. *For (2.4), assume (h1) and suppose that there are a vector $v = (v_1, \dots, v_n) > 0$ and a constant $\gamma < 2e|v|^{-1} \min_{1 \leq i \leq n} v_i$ such that*

$$0 < \gamma_i(t,v) \leq \gamma, \quad 1 \leq i \leq n,\ t \gg 1, \tag{3.11}$$

where $|v| = \max_{1 \leq i \leq n} v_i$. Then, there are positive solutions of (1.1) satisfying $x_i(t) < 2$ for all $t \geq 0, i = 1, \dots, n$. If in addition, $\beta_i(t)$ are bounded below and above by positive constants and

$$\gamma_i(t,v) \geq \alpha > 1, \quad 1 \leq i \leq n,\ t \gg 1,$$

for some α, then (2.4) is is (eventually) globally exponentially stable. In particular, this is the case if

$$1 < \alpha \leq \gamma_i(t) \leq \gamma < 2e, \quad 1 \leq i \leq n,\ t \gg 1. \tag{3.12}$$

Example 3.7. Consider the planar system

$$\begin{aligned} x_1'(t) &= -t^\eta x_1(t) + (t^\eta - 1)x_2(t) + \frac{\beta}{\sigma_1(t)} \int_{t-\sigma_1(t)}^t x_1(s)e^{-x_1(s)}\,ds, \\ x_2'(t) &= -t^\eta x_2(t) + (t^\eta - 1)x_1(t) + \frac{\beta}{\sigma_2(t)} \int_{t-\sigma_2(t)}^t x_2(s)e^{-x_2(s)}\,ds, \end{aligned} \qquad t \geq 1, \tag{3.13}$$

where $\eta > 0, \beta > 1$, the delays $\sigma_i(t)$ are positive, continuous and bounded, $i = 1,2$. With the previous notations, $d_i(t) = t^\eta, a_{ii}(t) = 0, \beta_i(t) \equiv \beta > 1$, $i = 1,2$ and $a_{12}(t) = a_{21}(t) = t^\eta - 1$, thus $\gamma_1(t) = \gamma_2(t) = \beta$. For this concrete example, if $\beta \in (1, e^2)$, there exists a positive equilibrium $x^* = (\log\beta, \log\beta) < (2,2)$. From Theorem 3.3, we deduce that all positive solutions $x(t)$ converge exponentially to x^* as $t \to \infty$.

In the case of *periodic* Nicholson systems, we also obtain the following result.

Corollary 3.8. *Consider a periodic Nicholson system* (1.1)*, with $d_i(t)$, $a_{ij}(t)$, $b_{ik}(t)$, $\tau_{ik}(t)$, $\lambda_{ik}(t)$, $c_{ik}(t)$ continuous, nonnegative and ω-periodic functions (for some $\omega > 0$), with $d_i(t)$, $\beta_i(t)$, $c_{ik}(t)$ positive, for all i, j, k. If there exist a vector $v > 0$ such that*

$$\begin{cases} \min_{t\in[0,\omega]} \gamma_i(t,v) > 1, \\ \max_{t\in[0,\omega]} \gamma_i(t,v) < 2e\,\underline{C}\,\overline{C}^{-1}, \quad 1 \le i \le n, \end{cases} \tag{3.14}$$

then there exists a positive ω-periodic solution of (1.1)*, which is globally exponentially stable.*

Proof. By [8], it turns out that the sufficient conditions for permanence also imply the existence of a positive periodic solution. The result is an immediate consequence of Theorem 3.4. □

Remark 3.9. For the periodic Nicholson system with discrete delays multiple of period given by

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \beta_i(t)x_i(t - m_i\omega)e^{-c_i(t)x_i(t-m_i\omega)}, \quad 1 \le i \le n. \tag{3.15}$$

with $m_i \in \mathbb{N}, \omega > 0$ and $d_i(t) > 0$, $a_{ij}(t) \ge 0$, $\beta_i(t), c_i(t) > 0$ continuous ω-periodic functions, the existence and global attractivity of a positive periodic solution was proven in [8] under the condition

$$\begin{cases} \min_{t\in[0,\omega]} \gamma_i(t,v) > 1, \\ \max_{t\in[0,\omega]} \gamma_i(t,v) < \exp(2\,\underline{C}\,\overline{C}^{-1}), \quad 1 \le i \le n, \end{cases} \tag{3.16}$$

for some vector $v > 0$ and $\underline{C}, \overline{C}$ defined as in (2.9). Clearly, $ex \le e^x$ for $x \ge 0$. We conclude that Corollay 3.8 extends the result in [8] to more general systems (1.1) – with global *exponential* stability, rather than global attractivity –, however, under the more restrictive assumption of $\gamma_i := \max_{t\in[0,\omega]} \gamma_i(t,v) < 2e\,\underline{C}\,\overline{C}^{-1}$, instead of $\gamma_i < e^{2\underline{C}\,\overline{C}^{-1}}$. The key point to establish the result in [8] under the latter assumption was the following: as the delays are multiple of the period, an ω-periodic solution $x^*(t)$ for (3.15) is also an ω-periodic solution for the corresponding ODE

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \beta_i(t)x_i(t)e^{-c_i(t)x_i(t)}, \quad 1 \le i \le n. \tag{3.17}$$

From this fact, one easily deduces that $\max_{t\ge 0}\big(c_i(t)x_i^*(t)\big) < 2$ for all i, provided that (3.16) holds. Whether Theorem 3.4 is still valid for a general system (1.1) with (3.10) replaced by

$$1 < \alpha \le \gamma_i(t,v) \le \gamma < e^{2\underline{C}\,\overline{C}^{-1}}, \quad 1 \le i \le n, \ t \gg 1,$$

(conf. Remark 2.9) is an interesting open problem. We conjecture that the answer is affirmative, at least if some further constraints on α are prescribed.

We now apply our results to Nicholson equations and systems with *discrete delays*, and compare the above criteria with some more results in the literature. The corollary below addresses the scalar case, a similar one can be written for systems with $n > 1$.

Corollary 3.10. *Consider the scalar Nicholson equation*

$$x'(t) = -d(t)x(t) + \sum_{k=1}^{m} \beta_k(t)x(t-\tau_k(t))e^{-c_k(t)x(t-\tau_k(t))}, \tag{3.18}$$

where $d, \beta_k, \tau_k, c_k : \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous functions and* $d(t) > 0$ *on* $\mathbb{R}^+$*,* $\tau_k(t) \in [0,\tau]$ *(for some* $\tau > 0$*) and* $c_k(t), \beta(t) := \sum_{k=1}^{m} \beta_k(t)$ *are bounded above and below by positive constants. If*

$$1 < \alpha \le \frac{\sum_{k=1}^{m} \beta_k(t)}{d(t)} \le \gamma < 2e\frac{\min_{1\le k\le m} \underline{c_k}}{\max_{1\le k\le m} \overline{c_k}}, \quad t \ge 0, \tag{3.19}$$

where $\underline{c_k} = \inf_{t\ge 0} c_k(t), \overline{c_k} = \sup_{t\ge 0} c_k(t)$*, then* (3.18) *is globally exponentially stable.*

If all the coefficients and delays d, β_k, τ_k, c_k are ω-periodic, Corollary 3.8 implies the existence of a globally exponentially stable ω-periodic positive solution to (3.18). We stress that the *periodic* equation (3.18) was studied in [16] and its stability established. Denote $\kappa \in (0,1), \tilde{\kappa} \in (1,\infty)$ the constants which satisfy

$$h'(\kappa) = -h'(2), \quad h(\kappa) = h(\tilde{\kappa}). \tag{3.20}$$

The approximate values of $\kappa, \tilde{\kappa}$ were evaluated in [23]: $\kappa \approx 0.7215, \tilde{\kappa} \approx 1.3423$. Assuming that

$$\frac{\sum_{k=1}^{m} \beta_k(t)}{d(t)} < e^2, \quad t \in [0,\omega], \tag{3.21}$$

and that there is $M > \kappa$ such that

$$1 \le \min_{1\le k\le m} \underline{c_k} \le \max_{1\le k\le m} \overline{c_k} \le \frac{\tilde{\kappa}}{M} \tag{3.22}$$

and

$$\frac{1}{eM}\sum_{k=1}^{m} \frac{\beta_k(t)}{c_k(t)} < d(t) < e^{-\kappa}\sum_{k=1}^{m} \frac{\beta_k(t)}{c_k(t)}, \quad t \in [0,\omega], \tag{3.23}$$

Liu [16] used a Lyapunov functional to show that there exists an ω-periodic positive solution of (3.18) which is globally exponentially stable. A similar approach was used by Liu in [17], for an almost periodic version of (3.18) with a nonlinear density-dependent mortality term $-d_1(t) + d_2(t)e^{-x(t)}$, instead of $-d(t)x(t)$.

In fact, in order to prove the above exponential stability under the conditions (3.21)–(3.23), Liu [16] started by establishing that the ordered interval $[\kappa, M]$ in $C = ([-\tau,0];\mathbb{R})$ is positively invariant. For the periodic case, by itself, the constraint (3.21) is weaker than the second inequality in (3.19). However, not only is the requirement (3.22) a strong restriction to the application of Liu's criterion, but, if (3.22) holds, our assumption (3.19) simply reads as

$$\frac{1}{eM}\frac{\tilde{\kappa}}{2}\sum_{k=1}^{m} \beta_k(t) < d(t) < \sum_{k=1}^{m} \beta_k(t), \quad t \in [0,\omega].$$

In this situation, we always have $\sum_{k=1}^{m} \beta_k(t) > e^{-\kappa}\sum_{k=1}^{m} \frac{\beta_k(t)}{c_k(t)}$; if one can choose M in (3.22) such that $\tilde{\kappa}^2 < 2M$, i.e., if $\max_{1\le k\le m} \overline{c_k} < 2/\tilde{\kappa} \approx 1.490$, then $\frac{1}{eM}\frac{\tilde{\kappa}}{2}\sum_{k=1}^{m} \beta_k(t) < \frac{1}{eM}\sum_{k=1}^{m} \frac{\beta_k(t)}{c_k(t)}$, and our result strongly improves the criterion in [16]. For instance, with $c_k(t) \equiv 1$, our hypothesis (3.19) reads as

$$1 < \frac{\sum_{k=1}^{m} \beta_k(t)}{d(t)} < 2e \quad \text{for } t \in [0,\omega];$$

on the other hand, we may take $M = \tilde{\kappa}$ in (3.22), and conditions (3.21), (3.23) are equivalent to

$$e^{\kappa} \le \frac{\sum_{k=1}^{m} \beta_k(t)}{d(t)} \le e\tilde{\kappa} \quad \text{for } t \in [0, \omega],$$

which is much more restrictive than (3.19).

More recently, Wang et al. [23] generalised the scalar version (3.18) by considering the following multi-dimentional model with patch structure:

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij}(t)x_j(t) + \sum_{p=1}^{m} \beta_{ip}(t)x_i(t - \tau_{ip}(t))e^{-c_{ip}(t)x_i(t-\tau_{ip}(t))}, \quad i = 1, \dots, n, \tag{3.24}$$

where $d_i, a_{ij}, \beta_{ip}, \tau_{ip}, c_{ip} : \mathbb{R} \to \mathbb{R}^+$ are continuous, pseudo almost periodic functions, $d_i(t) > 0$ and satisfies some further properties, and $\inf_{t\ge t_0} \beta_i(t) > 0$ where $\beta_i(t) := \sum_{p=1}^{m} \beta_{ip}(t)$, for all i, j, p. With $\kappa, \tilde{\kappa}$ defined as in (3.20), in [23] the authors assumed the following set of assumptions, for $1 \le i \le n, 1 \le p \le m$:

$$\begin{aligned}
&1 \le \inf_{t\in\mathbb{R}} c_{ip}(t) \le \sup_{t\in\mathbb{R}} c_{ip}(t) \le M^{-1}\tilde{\kappa}, \quad \text{for some } M > \kappa,\\
&\sup_{t\in\mathbb{R}} \left\{ -d_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij}(t) + \frac{1}{eM}\sum_{p=1}^{m} \frac{\beta_{ip}(t)}{c_{ip}(t)} \right\} < 0,\\
&\inf_{t\in\mathbb{R}} \left\{ -d_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij}(t) + e^{-\kappa}\sum_{p=1}^{m} \frac{\beta_{ip}(t)}{c_{ip}(t)} \right\} > 0,
\end{aligned} \tag{3.25}$$

and showed that:

(i) all solutions $x(t) = x(t, t_0, \phi)$ of (3.24) with initial conditions $\phi \in C_0^+$ satisfy

$$\kappa \le \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) \le M, \quad i = 1, \dots, n; \tag{3.26}$$

(ii) there exists a positive pseudo almost periodic solution $x^*(t)$ of (3.24), which satisfies $\kappa \le x_i^*(t) \le M$ for all $t \in \mathbb{R}$ and $i = 1, \dots, n$;

(iii) $x^*(t)$ is globally exponentially stable.

See also [5, 24] for similar criteria. Recently, some of the constraints in [23] were slightly loosened in [13].

With our methodology, under the condition $\inf_{t\in\mathbb{R}} c_{ip}(t) \ge 1$ and taking e.g. $v = \vec{1}$ in (3.10), from Theorem 3.4 we obtain that system (3.24) is globally exponentially stable provided that

$$\left.\begin{aligned}
&\inf_{t\in\mathbb{R}} \frac{\sum_{p=1}^{m} \beta_{ip}(t)}{d_i(t) - \sum_{j=1, j\neq i}^{n} a_{ij}(t)} > 1,\\
&\sup_{t\in\mathbb{R}} \frac{\sum_{p=1}^{m} \beta_{ip}(t)}{d_i(t) - \sum_{j=1, j\neq i}^{n} a_{ij}(t)} < \frac{2e}{\max\limits_{i,p} \sup\limits_{t\in\mathbb{R}} c_{ip}(t)},
\end{aligned}\right. \quad 1 \le i \le n,\ t \gg 1. \tag{3.27}$$

As in the previous scalar case, one easily verifies that for most situations conditions (3.27) are less restrictive than (3.25).

We finish this section with a couple of simple examples.

Example 3.11. Consider the following ω-periodic Nicholson-type system with discrete delays:

$$\begin{aligned}
x_1'(t) &= -d_1(t)x_1(t) + b_1(t)x_2(t - \sigma_1(t)) + c_1(t)x_1(t - \tau_1(t))e^{-x_1(t-\tau_1(t))}\\
x_2'(t) &= -d_2(t)x_2(t) + b_2(t)x_2(t - \sigma_2(t)) + c_2(t)x_2(t - \tau_2(t))e^{-x_2(t-\tau_2(t))},
\end{aligned} \tag{3.28}$$

where $d_i(t), b_i(t), c_i(t), \sigma_i(t), \tau_i(t)$ $(i = 1, 2)$ are positive, continuous and ω-periodic functions. Applying Corollary 3.8 with $v = (1, v_2)$ (conf. also Remark 3.5)), we derive that (3.28) has a globally exponentially stable positive ω-periodic solution if there exists a positive constant v_2 such that

$$1 < \frac{c_1(t)}{d_1(t) - b_1(t)v_2} < 2e\frac{\min\{1, v_2\}}{\max\{1, v_2\}}, \quad 1 < \frac{c_2(t)v_2}{d_2(t)v_2 - b_2(t)} < 2e\frac{\min\{1, v_2\}}{\max\{1, v_2\}}, \quad t \in [0, \omega].$$

In particular, this assertion is valid if

$$1 < \frac{c_i(t)}{d_i(t) - b_i(t)} < 2e, \quad t \in [0, \omega], \ i = 1, 2. \tag{3.29}$$

In the case of (3.28) with $\sigma_i(t) \equiv 0$ and a unique constant delay in the nonlinear part, i.e., $\tau_i(t) \equiv \tau > 0$, by using the continuation theorem of coincidence degree and a Lyapunov functional, Troib [21] established sufficient conditions for the existence and global attractivity of a positive ω-periodic solution. As analysed in [8] with more detail, we can assert that the results in [21] not only do not apply to the framework of nonconstant delays $\tau_i(t)$, nor to other simple situations, but also the assumed constraints are more restrictive than (3.29).

Example 3.12. As a particular case of (3.28), consider the π-periodic system

$$\begin{aligned} x_1'(t) &= -(1 + \cos^2 t)x_1(t) + c_1(1 + \sin^2 t)x_2(t) + \beta_1(1 + \cos^2 t)x_1(t - \tau_1(t))e^{-x_1(t-\tau_1(t))} \\ x_2'(t) &= -(1 + \sin^2 t)x_2(t) + c_2(1 + \cos^2 t)x_1(t) + \beta_2(1 + \sin^2 t)x_2(t - \tau_2(t))e^{-x_2(t-\tau_2(t))} \end{aligned} \tag{3.30}$$

where $c_i, \beta_i > 0$ and the delays $\tau_i(t)$ are π-periodic, continuous and nonnegative, $i = 1, 2$. With the previous notation, for $v = (1, v_2) > 0$ we have

$$\begin{aligned} \gamma_1(t, v) &:= \frac{\beta_1(1 + \cos^2 t)}{1 + \cos^2 t - v_2 c_1(1 + \sin^2 t)} \\ \gamma_2(t, v) &:= \frac{\beta_2 v_2(1 + \sin^2 t)}{v_2(1 + \sin^2 t) - c_2(1 + \cos^2 t)}. \end{aligned} \tag{3.31}$$

If $4c_1c_2 < 1$, choosing v_2 such that $2c_2 < v_2 < (2c_1)^{-1}$, we obtain

$$0 < \alpha_i \le \gamma_i(t, v) \le \gamma_i, \quad \text{for } t \in [0, \pi], \ i = 1, 2,$$

where

$$\alpha_1 = \frac{\beta_1}{1 - \frac{1}{2}v_2c_1}, \quad \gamma_1 = \frac{\beta_1}{1 - 2v_2c_1}, \quad \alpha_2 = \frac{\beta_2}{1 - \frac{1}{2}v_2^{-1}c_2}, \quad \gamma_2 = \frac{\beta_2}{1 - 2v_2^{-1}c_2}.$$

In particular, with $c_i < \frac{1}{2}$, $i = 1, 2$, one can take $v_2 = 1$; if in addition $c_i < 2(2e - 1)(8e - 1)^{-1} \approx 0.4277$ and β_i is chosen so that $1 - \frac{1}{2}c_i < \beta_i < 2e(1 - 2c_i)$ for $i = 1, 2$, we obtain $1 < \alpha_i < \gamma_i < 2e$, $i = 1, 2$, therefore there exists a positive π-periodic solution $x^*(t)$ which is globally exponentially attractive.

4 Conclusions

This paper concerns the global asymptotic behaviour of positive solutions for a very broad family of Nicholson systems (1.1). Uniform lower and upper bounds for all solutions, as well as their global exponential stability are established, which generalise most of the results in recent literature. We observe that systems (1.1) incorporate distributed delays, whereas most authors only consider systems (3.24) with discrete delays. Moreover, as mentioned in Remarks 2.5 and 3.5, if $a_{ij}(t)$ are bounded, all the results apply to systems (2.5) with delays in the linear terms. The assumptions and proofs presented here rely heavily on the special properties of the Ricker nonlinearity $h(x) = xe^{-x}$, $x \geq 0$.

Some authors [7, 15, 24], have considered autonomous or nonautonomous Nicholson systems with discrete delays under restrictions on the coefficients implying that the systems have a monotone behaviour. In recent papers [5, 13, 16, 21, 23], conditions have been imposed for systems (3.24) in such a way that the estimates (3.26) should hold, where $1 \leq \inf_{t\in\mathbb{R}} c_{ip}(t) \leq \sup_{t\in\mathbb{R}} c_{ip}(t) \leq M^{-1}\tilde{\kappa}$ for some $M > \kappa$, for $\kappa, \tilde{\kappa}$ defined in (3.20), – and thus all positive solutions must satisfy

$$\kappa \leq \liminf_{t\to\infty} c_{ip}(t)x_i(t) \leq \limsup_{t\to\infty} c_{ip}(t)x_i(t) \leq \tilde{\kappa},$$

for all i, p. These estimates have been used in order to derive that, since $h(x) \geq h(\kappa)$ and $|h'(x)| \leq e^{-2}$ for $x \in [\kappa, \tilde{\kappa}]$, any two solutions $x(t), y(t)$ must satisfy

$$|h_{ip}(t, x_i(s)) - h_{ip}(t, y_i(s))| \leq e^{-2}|x_i(s) - y_i(s)|, \quad i = 1, \dots, n,$$

for all t and $s \in [t-\tau, t]$, where $h_{ip}(t, x(s)) = x(s)e^{-c_{ip}(t)x(s)}$. Our approach is essentially new: assuming the permanence, the exponential stability of (1.1) is proven using solely an explicit *upper bound* for solutions of such systems. Basically, we only need to assert the existence of (at least) one positive solution satisfying $\limsup_{t\to\infty} c_{ip}(t)x_i(t) < 2$ for all i, p. In Theorem 3.4 we have imposed condition (3.10), which guarantees that such a solution exists. As mentioned in Remark 3.9, an interesting open problem is whether such a condition can be replaced by the less restrictive assumption $1 < \alpha \leq \gamma_i(t, v) \leq \gamma < e^{2\underline{C}\,\overline{C}^{-1}}$.

Clearly, the method developed here can be further exploited, to study the global attractivity and exponential stability of other systems with patch structure – such as Mackey–Glass type systems –, or modified Nicholson systems with either nonlinear density-dependent mortality terms or harvesting terms, as in [5, 20, 22, 25]. In other words, under the conditions for permanence established in [9] and with suitable changes, the approach herein carries over to more general settings, and can be used to treat n-dimensional systems

$$x_i'(t) = -d_i(t, x_i(t)) + \sum_{j=1}^{n} L_{ij}(t)x_{j,t} + f_i(t, x_{i,t}), \quad t \geq 0,\ i = 1, \dots, n,$$

where $d_i(t, x) \geq 0, d_i(t, x) = O(x)$ at zero, the linear functionals $L_{ij}(t)$ are nonnegative and the nonlinearities f_i incorporate one or several monotone, or unimodal terms.

Acknowledgements

This work was supported by National Funding from FCT – Fundação para a Ciência e a Tecnologia (Portugal) under project UIDB/04561/2020.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **10**, 1–17; https://doi.org/10.14232/ejqtde.2021.1.10 www.math.u-szeged.hu/ejqtde/

Coupled nonautonomous inclusion systems with spatially variable exponents

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Received 21 November 2020, appeared 12 February 2021
Communicated by Christian Pötzsche

Abstract. A family of nonautonomous coupled inclusions governed by $p(x)$-Laplacian operators with large diffusion is investigated. The existence of solutions and pullback attractors as well as the generation of a generalized process are established. It is shown that the asymptotic dynamics is determined by a two dimensional ordinary nonautonomous coupled inclusion when the exponents converge to constants provided the absorption coefficients are independent of the spatial variable. The pullback attractor and forward attracting set of this limiting system is investigated.

Keywords: nonautonomous parabolic problems, variable exponents, pullback attractors, omega limit sets, upper semicontinuity.

2020 Mathematics Subject Classification: 35K55, 35K92, 35A16, 35B40, 35B41, 37B55.

1 Introduction

It is a well-known fact that many models of chemical, biological and ecological problems involve reaction-diffusion systems. For example, Fisher's equation:

$$w_t - D\frac{\partial^2 w}{\partial x^2} = aw(1-w).$$

A general reaction-diffusion system has the form

$$u_t - \mathcal{D}\Delta u = \mathbf{f}(u) \tag{RD}$$

where u is a vector representing chemical concentrations and $\mathcal{D}$ is a matrix of diffusion coefficients, assumed constant, and the second term represents chemical reactions. The form of $\mathbf{f}$ depends on the system being studied (it is typically nonlinear). Large diffusion phenomena many times appears in these systems. A shadow system, as a limiting system of reaction-diffusion model for algal bloom in which the diffusion rate tends to infinity, has been proposed in [27] to study whether or not stable nonconstant equilibrium solutions of the

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system exist. Large diffusion phenomena also appear in applications of chemical fluid flows [30].

When the diffusion does not follow a linear or a uniform structure the problem (RD) becomes

$$u_t - \mathcal{D}\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \mathbf{f}(u).$$

Partial differential problems with variable exponents have application in electrorheological fluids (see [19,31,32]) and image processing (see [13,22]). Another important application is modelling of flow in porous media [1,2]. Some other applications of equations with variable exponent growth conditions are magnetostatics [12] and capillarity phenomena [5].

Sometimes it is necessary to consider a multivalued right-hand side when uncertainties or discontinuities appear in the reaction term, while coupled systems occur when different phenomena interact. In these cases we have to work with differential inclusions instead of differential equations (see, for example, [3,9,14,15,20,23,28,29,42] and the references therein). Such inclusions have been used for modelling processes of combustion in porous media [20] and the surface temperature on Earth [9,15]. Moreover, differential inclusions appear in numerous applications such as the control of forest fires [7], conduction of electrical impulses in nerve axons [40,41]. In climatology, the energy balance models may lead to evolution differential inclusions which involve the p-Laplacian [16,17]. A degenerate parabolic-hyperbolic problem with a differential inclusion appears in a glaciology model [18].

We will consider the following nonautonomous coupled inclusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + C_1(t)|u|^{p(x)-2}u \in F(u,v), & t > \tau, \\ \dfrac{\partial v}{\partial t} - D\,\mathrm{div}(|\nabla v|^{q(x)-2}\nabla v) + C_2(t)|v|^{q(x)-2}v \in G(u,v), & t > \tau, \\ (u(\tau), v(\tau)) \text{ in } L^2(\Omega) \times L^2(\Omega), \end{cases} \tag{S}$$

on a bounded smooth domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with homogeneous Neumann boundary conditions. Here $D \in [1,\infty)$, F and G are bounded upper semicontinuous and positively sublinear multivalued maps, and the exponents $p(\cdot), q(\cdot) \in C(\overline{\Omega})$ satisfy

$$p^+ := \max_{x\in\overline{\Omega}} p(x) \ > \ p^- := \min_{x\in\overline{\Omega}} p(x) > 2, \ q^+ > q^- > 2.$$

In addition, the absorption coefficients $C_1, C_2 : [\tau, T] \times \Omega \to \mathbb{R}$ are functions in $L^\infty([\tau,T]\times\Omega)$ satisfying

(C1) there is a positive constant, γ such that $0 < \gamma \leq C_i(t,x)$ for almost all $(t,x) \in [\tau,T] \times \Omega$, $i = 1,2$.

(C2) $C_i(t,x) \geq C_i(s,x)$ for a.a. $x \in \Omega$ and $t \leq s$ in $[\tau,T]$, $i = 1,2$.

The authors of [21] considered this problem for only one equation with the external function globally Lipschitz, while those of [35] considered the autonomous version of this problem with $C_i(t,x) \equiv 1$. Nonautonomous equations of p-Laplacian type were previously considered in [24,38].

We will prove existence of strong global solutions for problem (S) and that these multivalued problems define exact generalized processes. The main tool used is a compactness result established in [36], which is a generalization of Baras' Theorem for the case that the main operator is time-dependent. In addition, we prove the existence of a pullback attractor and, when considering large diffusion and letting the exponents go to constants, we explore the robustness of the family of pullback attractors with respect to its limit problem which governs the whole asymptotic dynamics of the system.

The paper is organized as follows. In Section 2 we present some preliminaries. Section 3 is devoted to prove existence of global solutions for the system and in Section 4 we prove that problem (S) defines an exact generalized process which possess a pullback attractor. Finally, in Section 5 we consider the case when $D \to +\infty$ and the exponents converge to constants and investigate the dynamics of the limiting two dimensional ordinary nonautonomous coupled inclusion.

2 Preliminaries

Definition 2.1 ([43]). A subset K in $L^1(a,b;X)$ is uniformly integrable if, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\int_E \|f(t)\|_X dt \leq \varepsilon$ uniformly for $f \in K$ for each measurable subset E in $[a,b]$ with Lebesgue measure less than $\delta(\varepsilon)$.

Remark 2.2 ([8]). Since $[a,b]$ is compact, each uniformly integrable subset in $L^1(a,b;X)$ is bounded with respect to the norm of $L^1(a,b;X)$.

Consider the following IVP:

$$\begin{cases} \dfrac{du^n}{dt}(t) + A(t)u^n(t) \ni f_n(t), \quad t > \tau, \\ u^n(\tau) = u_{0_n}, \end{cases} \tag{$P_{t,n}$}$$

where for each $t > \tau$, $A(t)$ is maximal monotone in a Hilbert space H, $f_n \in K \subset L^1(\tau,T;H)$ and $u_{0_n} \in H$. In addition, suppose $\mathcal{D}(A(t)) = \mathcal{D}(A(\tau))$, $\forall\, t,\tau \in \mathbb{R}$, and $\overline{\mathcal{D}(A(t))} = H$, for all $t \in \mathbb{R}$.

Definition 2.3. A function $u^n : [\tau,T] \to H$ is called a strong solution of ($P_{t,n}$) on $[\tau,T]$ if

(i) $u^n \in \mathcal{C}([\tau,T];H)$;

(ii) u^n is absolutely continuous on any compact subset of (τ,T);

(iii) $u^n(t)$ is in $\mathcal{D}(A(t))$ for a.e. $t \in [\tau,T]$, $u^n(\tau) = u_{0_n}$, and satisfies the inclusion in ($P_{t,n}$) for a.e. $t \in [\tau,T]$.

We now present abstract conditions on the family of the operators $\{A(t)\}_{t>0}$ and f_n such that problem ($P_{t,n}$) has, for each $n \in \mathbb{N}$, a unique strong solution u^n on $[\tau,T]$. We are interested in the case where $A(t) = \partial\phi^t$, i.e., the evolution problem of the form

$$\frac{du}{dt}(t) + \partial\phi^t(u(t)) \ni f(t), \qquad \tau \leq t \leq T, \tag{E}$$

in a real Hilbert space H, where, for almost every $t \in [0,T]$, $A(t) := \partial\phi^t$ is the subdifferential of a lower semicontinuous, proper and convex function ϕ^t from H into $(-\infty,\infty]$. In this case, $A(t)$ is a maximal monotone operator.

Condition A: Let $T > \tau$ be fixed.

(I) There is a set $Z \subset\,]\tau,T]$ of zero measure such that ϕ^t is a lower semicontinuous proper convex function from H into $(-\infty,\infty]$ with a non-empty effective domain for each $t \in [\tau,T] - Z$.

(*II*) For any positive integer r there exist a constant $K_r > 0$, an absolutely continuous function $g_r : [\tau, T] \to \mathbb{R}$ with $g_r' \in L^\beta(\tau, T)$ and a function of bounded variation $h_r : [\tau, T] \to \mathbb{R}$ such that if $t \in [\tau, T] - Z$, $w \in D(\phi^t)$ with $|w| \le r$ and $s \in [t, T] - Z$, then there exists an element $\tilde{w} \in D(\phi^s)$ satisfying

$$|\tilde{w} - w| \le |g_r(s) - g_r(t)|(\phi^t(w) + K_r)^\alpha,$$
$$\phi^s(\tilde{w}) \le \phi^t(w) + |h_r(s) - h_r(t)|(\phi^t(w) + K_r),$$

where α is some fixed constant with $0 \le \alpha \le 1$ and

$$\beta := \begin{cases} 2 & \text{if } 0 \le \alpha \le \frac{1}{2}, \\ \dfrac{1}{1-\alpha} & \text{if } \frac{1}{2} \le \alpha \le 1. \end{cases}$$

Proposition 2.4 ([44]). *Suppose that **Condition A** is satisfied. Then for each $f \in L^2(\tau, T; H)$ and $u_0 \in \overline{D(\phi^\tau)}$, the equation (E) has a unique strong solution u on $[\tau, T]$ with $u(\tau) = u_0$.*

Moreover, u has the following properties:

(i) For all $t \in (\tau, T] - Z$ $u(t)$ is in $D(\phi^t)$ and satisfies $t\phi^t(u(t)) \in L^\infty(\tau, T)$ and $\phi^t(u(t)) \in L^1(\tau, T)$. Furthermore, for any $\tau < \delta < T$, $\phi^t(u(t))$ is of bounded variation on $[\delta, T] - Z$.

(ii) For any $\tau < \delta < T$, u is strongly absolutely continuous on $[\delta, T]$, and $t^{1/2}\frac{du}{dt} \in L^2(\tau, T; H)$.

In particular, if $u_0 \in D(\phi^\tau)$, then u satisfies

(i)′ For all $t \in [\tau, T] - Z$, $u(t)$ is in $D(\phi^t)$ and $\phi^t(u(t))$ is of bounded variation on $[\tau, T] - Z$.

(ii)′ u is strongly absolutely continuous on $[\tau, T]$ and satisfies $\frac{du}{dt} \in L^2(\tau, T; H)$.

For our specific problem, we consider $H := L^2(\Omega)$ with a bounded smooth domain $\Omega \subset \mathbb{R}^n$, $n \ge 1$, $p(\cdot) \in C(\bar{\Omega}, \mathbb{R})$, $p^+ := \max_{x\in\bar{\Omega}} p(x) \ge p^- := \min_{x\in\bar{\Omega}} p(x) > 2$, where $C : [\tau, T] \times \Omega \to \mathbb{R}$ is a function in $L^\infty([\tau, T] \times \Omega)$ satisfying conditions (C1) and (C2).

Consider the Lebesgue space with variable exponents

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable, } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

Define $\rho(u) := \int_\Omega |u(x)|^{p(x)} dx$ and

$$\|u\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \rho\left(\frac{u}{\lambda}\right) \le 1 \right\}$$

for $u \in L^{p(\cdot)}(\Omega)$. The generalized Sobolev space is defined as

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

It is well-known that $Y_p := W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{Y_p} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

Consider the operator $A(t)$ defined in Y_p such that for each $u \in Y_p$ associate the following element of its dual space ${Y_p}^*$, $A(t)u : Y_p \to \mathbb{R}$ given by

$$\langle A(t)u, v \rangle_{{Y_p}^*, Y_p} := D \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla v(x)\, dx + \int_\Omega C(t,x)|u(x)|^{p(x)-2} u(x) v(x)\, dx.$$

It was shown in [21] that the operator $A(t) : Y_p \to {Y_p}^*$ is monotone, hemicontinuous and coercive. Moreover, we have the following estimates on the operator.

Lemma 2.5 ([21]). *Let $u \in Y_p := W^{1,p(\cdot)}(\Omega)$. For each $t \geq 0$ we have*

$$\langle A(t)u, u\rangle_{Y_p{}^*,Y_p} \geq \frac{\min\{1,\gamma\}}{2^{p^+}} \begin{cases} \|u\|_{Y_p}^{p^+}, & \text{if } \|u\|_{Y_p} < 1, \\ \|u\|_{Y_p}^{p^-}, & \text{if } \|u\|_{Y_p} \geq 1. \end{cases} \tag{2.1}$$

It is easy to see that the operator $A(t) : H \to H$ defined by

$$A(t)u := -D\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + C(t)|u|^{p(x)-2}u,$$

satisfies **Condition A** and, consequently, by applying Proposition 2.4 we have that problem (E) has a unique strong solution.

We will also consider the following IVP:

$$\begin{cases} \dfrac{du}{dt} + A(t)u \ni 0, \quad t > \tau, \\ u(\tau) = u_0, \end{cases} \tag{P_t}$$

where for each $t > \tau$, $A(t)$ is maximal monotone in a Hilbert space H.

Definition 2.6. Define $\{V(t,\tau);\ V(t,\tau) : H \longrightarrow H,\ t \geq \tau\}$ by $V(t,\tau)(u_0) = u(t,u(\tau)) = u(t,u_0)$, where $u(t,u_0)$ is the unique strong solution of problem (P_t), and call $\{V(t,\tau);\ V(t,\tau) : H \longrightarrow H, t \geq \tau\}$ the evolution process generated by $A := \{A(t)\}_{t>\tau}$ in H. We say that the evolution process is compact if $V(t,\tau)$ is a compact operator for each $t > \tau$.

Let us review the concept of an evolution process in the next

Definition 2.7. An evolution process in a metric space X is a family $\{U(t,\tau) : X \to X,\ t \geq \tau \in \mathbb{R}\}$ satisfying:

i) $U(\tau,\tau) = I$;

ii) $U(t,\tau) = U(t,s)U(s,\tau)$, $\tau \leq s \leq t$.

Varying f_n and u_{0_n} in ($P_{t,n}$) we obtain a family of problems and consequently a family of solutions. Consider the following solution sets

$$M(K) := \{u^n; u^n \text{ is the unique strong solution of } (P_{t,n}), \text{ with } f_n \in K \text{ and } u_{0_n} \in H\}.$$

Theorems in [36] establish conditions which ensure that the set $M(K)$ possesses some property of compactness.

We now review some concepts and results from the literature which will be useful in the sequel to understand the conditions on the multivalued functions F and G. We refer the reader to [3, 4, 43] for more details about multivalued analysis theory. Let X be a real Banach space and M a Lebesgue measurable subset in $\mathbb{R}^q$, $q \geq 1$.

Definition 2.8. The map $G : M \to \mathcal{P}(X)$ is called measurable if for each closed subset C in X the set $G^{-1}(C) = \{y \in M; G(y) \cap C \neq \emptyset\}$ is Lebesgue measurable.

If G is a univalued map, the above definition is equivalent to the usual definition of a measurable function.

Definition 2.9. By a *selection* of $E : M \to \mathcal{P}(X)$ we mean a function $f : M \to X$ such that $f(y) \in E(y)$ a.e. $y \in M$, and we denote by Sel E the set Sel $E := \{f,\ f : M \to X$ is a measurable selection of $E\}$.

In what follows U denotes a topological space.

Definition 2.10. A mapping $G : U \to \mathcal{P}(X)$ is called upper semicontinuous [weakly upper semicontinuous] at $u \in U$, if

(i) $G(u)$ is nonempty, bounded, closed and convex.

(ii) For each open subset [open set in the weak topology] D in X satisfying $G(u) \subset D$, there exists a neighborhood V of u such that $G(v) \subset D$, for each $v \in V$.

If G is upper semicontinuous [weakly upper semicontinuous] at each $u \in U$, then it is called upper semicontinuous [weakly upper semicontinuous] on U.

Definition 2.11. $F, G : H \times H \to \mathcal{P}(H)$ are said to be bounded if, whenever B_1, B_2 are bounded, then $F(B_1, B_2) = \bigcup_{(u,v)\in B_1\times B_2} F(u,v)$ and $G(B_1, B_2) = \bigcup_{(u,v)\in B_1\times B_2} G(u,v)$ are bounded in H.

In order to obtain global solutions we impose suitable conditions on the external forces F and G.

Definition 2.12. The pair (F, G) of mappings $F, G : H \times H \to \mathcal{P}(H)$, which maps bounded subsets of $H \times H$ into bounded subsets of H, is called positively sublinear if there exist $a > 0$, $b > 0$, $c > 0$ and $m_0 > 0$ such that for each $(u, v) \in H \times H$ with $\|u\| > m_0$ or $\|v\| > m_0$ for which either there exists $f_0 \in F(u, v)$ satisfying $\langle u, f_0\rangle > 0$ or there exists $g_0 \in G(u, v)$ with $\langle v, g_0\rangle > 0$, we have both

$$\|f\| \le a\|u\| + b\|v\| + c \quad \text{and} \quad \|g\| \le a\|u\| + b\|v\| + c$$

for each $f \in F(u, v)$ and each $g \in G(u, v)$.

3 Existence of solution

Now we will establish the existence of a global solution for the system (S). The idea is to show that an appropriately defined multivalued map has at least one fix point whose existence is equivalent to the existence of at least one solution of (S).

We can rewrite the system in an abstract form as

$$\begin{cases} u_t + A(t)u \in F(u,v), & t > \tau, \\ v_t + B(t)v \in G(u,v), & t > \tau, \\ (u(\tau), v(\tau)) = (u_\tau, v_\tau) \in H \times H, \end{cases} \tag{$\tilde{S}$}$$

where, for each $t > \tau$, $A(t)$ and $B(t)$ are univalued maximal monotone operators in a real separable Hilbert space H of subdifferential type, i.e., $A(t) = \partial\varphi^t$, $B(t) = \partial\psi^t$ with φ^t, ψ^t non-negative maps satisfying **Condition A** with $\partial\varphi^t(0) = \partial\psi^t(0) = 0$, $\forall\ t \in \mathbb{R}$ and F and G are bounded, upper semicontinuous and positively sublinear multivalued maps.

Definition 3.1. A strong solution of ($\tilde{S}$) is a pair (u,v) satisfying: $u,v \in C([\tau,T];H)$ for which there exist $f,g \in L^1(\tau,T;H)$, $f(t) \in F(u(t),v(t))$, $g(t) \in G(u(t),v(t))$ *a.e.* in (τ,T), and such that (u,v) is a strong solution (see Definition 2.3) over (τ,T) to the system (P_1) below:

$$\begin{cases} u_t + A(t)u = f, \\ v_t + B(t)v = g, \\ u(\tau) = u_0, v(\tau) = v_0. \end{cases} \tag{P_1}$$

We obtain the global existence for our system ($\tilde{S}$) by applying the following

Theorem 3.2 ([36]). *Let $A = \{A(t)\}_{t>\tau}$ and $B = \{B(t)\}_{t>\tau}$ be families of univalued operators $A(t) = \partial\varphi^t$, $B(t) = \partial\psi^t$ with φ^t, ψ^t non negative maps satisfying* ***Condition A*** *with $\partial\varphi^t(0) = \partial\psi^t(0) = 0$. Also suppose each one A and B generates a compact evolution process, and let $F,G : H \times H \to \mathcal{P}(H)$ be upper semicontinuous and bounded multivalued maps. Then given a bounded subset $B_0 \subset H \times H$, there exists $T_0 > \tau$ such that for each $(u_0,v_0) \in B_0$ there exists at least one strong solution (u,v) of ($\tilde{S}$) defined on $[\tau,T_0]$. If, in addition, the pair (F,G) is positively sublinear, given $T > \tau$, the same conclusion is true with $T_0 = T$.*

4 Exact generalized process and pullback attractor

We will prove that the system ($\tilde{S}$) generates an exact generalized process. Let us review this concept in the following

Definition 4.1 ([37]). Let (X,ρ) be a complete metric space. A *generalized process* $\mathcal{G} = \{\mathcal{G}(\tau)\}_{\tau\in\mathbb{R}}$ on X is a family of function sets $\mathcal{G}(\tau)$ consisting of maps $\varphi : [\tau,\infty) \to X$, satisfying the properties:

[P1] For each $\tau \in \mathbb{R}$ and $z \in X$ there exists at least one $\varphi \in \mathcal{G}(\tau)$ with $\varphi(\tau) = z$;

[P2] If $\varphi \in \mathcal{G}(\tau)$ and $s \geq 0$, then $\varphi^{+s} \in \mathcal{G}(\tau+s)$, where $\varphi^{+s} := \varphi_{|[\tau+s,\infty)}$;

[P3] If $\{\varphi_j\} \subset \mathcal{G}(\tau)$ and $\varphi_j(\tau) \to z$, then there exists a subsequence $\{\varphi_\mu\}$ of $\{\varphi_j\}$ and $\varphi \in \mathcal{G}(\tau)$ with $\varphi(\tau) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geq \tau$.

Definition 4.2 ([37]). A generalized process $\mathcal{G} = \{\mathcal{G}(\tau)\}_{\tau\in\mathbb{R}}$ which satisfies the concatenation property:

[P4] If $\varphi,\psi \in \mathcal{G}$ with $\varphi \in \mathcal{G}(\tau)$, $\psi \in \mathcal{G}(r)$ and $\varphi(s) = \psi(s)$ for some $s \geq r \geq \tau$, then $\theta \in \mathcal{G}(\tau)$, where

$$\theta(t) := \begin{cases} \varphi(t), & t \in [\tau,s], \\ \psi(t), & t \in (s,\infty), \end{cases} \tag{4.1}$$

is called an *exact (or strict) generalized process.*

Property [P1] follows from the existence of a solution for the system ($\tilde{S}$), which was guaranteed in the previous section.

Let $D(u(\tau),v(\tau))$ be the set of solutions of ($\tilde{S}$) with initial data (u_τ,v_τ). Moreover, let us consider $G(\tau) := \bigcup_{(u_\tau,v_\tau)\in H\times H} D(u(\tau),v(\tau))$ and $G := \{G(\tau)\}_{\tau\in\mathbb{R}}$.

Theorem 4.3 ([36]). *Under the conditions of Theorem 3.2, G is an exact generalized process.*

The authors of [36] provided a result that gives sufficient conditions on $A = \{A(t)\}_{t>\tau}$ to ensure that the evolution process $\{V(t,\tau)\}_{t\geq\tau}$ generated by A (see Definition 2.6) is compact. Suppose that the following conditions are true for A:

(i) $\mathcal{D}(A(t)) = V$ for all $t \in [\tau, T]$ with V compactly embedded into H and $\overline{V} = H$, where V is a reflexive Banach space and H a Hilbert space;

(ii) for each $t \in [\tau, T]$, $A(t) = \partial\varphi^t$, with $\varphi^t(\cdot) := \varphi(t, \cdot) : H \to \mathbb{R} \cup \{\infty\}$ a convex, proper and lower semicontinuous map;

(iii) there exist constants $\alpha, \alpha_1, \alpha_2 > 0$ such that for each $t \in [\tau, T]$, $\alpha\|w\|_V^{\alpha_1} \leq \varphi^t(w)$ if $\|w\|_V < 1$ and $\alpha\|w\|_V^{\alpha_2} \leq \varphi^t(w)$ if $\|w\|_V \geq 1$;

(iv) for each $t \in [\tau, T]$, $\varphi^t(x) \geq 0$ for all $x \in H$ and $\varphi^t(0) = 0$;

(v) for each $x \in V$, there exists $\frac{\partial\varphi}{\partial s}(s, x)$ and $\frac{\partial\varphi}{\partial s}(s, x) \leq 0$ for a.a. $s \in [\tau, T]$.

We will use the following result.

Theorem 4.4 ([36]). *If A satisfies hypotheses (i)–(v), then the generated process $\{V(t,\tau)\}_{t\geq\tau}$ by $A = \{A(t)\}_{t>\tau}$ is compact.*

Returning to our specific problem, i.e., if we consider $A(t) : H \to H$ given by $A(t)u = -D\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + C(t)|u|^{p(x)-2}u$, where $H = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ a bounded smooth domain, $p(\cdot) \in C(\bar{\Omega}, \mathbb{R})$, $p^+ := \max_{x\in\bar{\Omega}} p(x) \geq p^- := \min_{x\in\bar{\Omega}} p(x) > 2$. and $C : [\tau, T] \times \Omega \to \mathbb{R}$ is a function in $L^\infty([\tau, T] \times \Omega)$ such that $0 < \gamma \leq C(t, x)$ for almost all (t, x) $\in [\tau, T] \times \Omega$, for some positive constant γ, and $C(t, x) \geq C(s, x)$ for a.a. $x \in \Omega$ and $t \leq s$ in $[\tau, T]$. In particular, we have $D(A(t)) = V := W^{1,p(\cdot)}(\Omega) \subset\subset H$ for all $t \in [\tau, T]$, $\bar{V} = H$ and $A(t) = \partial\varphi^t$ where $\varphi^t : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\varphi^t(u) := \begin{cases} \left[\displaystyle\int_\Omega \frac{D}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{C(t,x)}{p(x)}|u|^{p(x)}dx\right], & \text{if } u \in W^{1,p(x)}(\Omega) \\ +\infty, & \text{otherwise} \end{cases} \tag{4.2}$$

is a convex, proper and lower semicontinuous map. It is easy to see that $A = \{A(t)\}_{t>\tau}$ satisfies all the abstract hypotheses (i)–(v) above. Moreover, we had already seen that **Condition A** is also satisfied.

Hence, considering $D(u(\tau), v(\tau))$ the set of the solutions of (S) with initial data (u_τ, v_τ) and defining $G(\tau) := \bigcup_{(u_\tau,v_\tau)\in H\times H} D(u(\tau), v(\tau))$ and $\mathbb{G} := \{G(\tau)\}_{\tau\in\mathbb{R}}$, we have

Theorem 4.5. *$\mathbb{G}$ is an exact generalized process.*

A multivalued process $\{U_{\mathcal{G}}(t,\tau)\}_{t\geq\tau}$ defined by a generalized process $\mathcal{G}$ is a family of multivalued operators $U_{\mathcal{G}}(t,\tau) : P(X) \to P(X)$ with $-\infty < \tau \leq t < +\infty$, such that for each τ $\in \mathbb{R}$

$$U_{\mathcal{G}}(t,\tau)E = \{\varphi(t); \varphi \in \mathcal{G}(\tau), \text{ with } \varphi(\tau) \in E\}, \qquad t \geq \tau.$$

Theorem 4.6 ([37]). *Let $\mathcal{G}$ be an exact generalized process. Suppose that $\{U_{\mathcal{G}}(t,\tau)\}_{t\geq\tau}$ is a multivalued process defined by $\mathcal{G}$, then we have that $\{U_{\mathcal{G}}(t,\tau)\}_{t\geq\tau}$ is an exact multivalued process on $P(X)$, i.e.,*

1. *$U_{\mathcal{G}}(t,t) = Id_{P(X)}$,*
2. *$U_{\mathcal{G}}(t,\tau) = U_{\mathcal{G}}(t,s)U_{\mathcal{G}}(s,\tau)$ for all $-\infty < \tau \leq s \leq t < +\infty$.*

A family of sets $K = \{K(t) \subset X : t \in \mathbb{R}\}$ will be called a nonautonomous set. The family K is closed (compact, bounded) if $K(t)$ is closed (compact, bounded) for all $t \in \mathbb{R}$. The ω-limit set $\omega(t, E)$ consists of the pullback limits of all converging sequences $\{\xi_n\}_{n\in\mathbb{N}}$ where $\xi_n \in U_{\mathcal{G}}(t, s_n)E, s_n \to -\infty$. Let $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ be a family of subsets of X. We have the following concepts of invariance:

- $\mathcal{A}$ is positively invariant if $U_{\mathcal{G}}(t,\tau)\mathcal{A}(\tau) \subset \mathcal{A}(t)$ for all $-\infty < \tau \le t < \infty$;
- $\mathcal{A}$ is negatively invariant if $\mathcal{A}(t) \subset U_{\mathcal{G}}(t,\tau)\mathcal{A}(\tau)$ for all $-\infty < \tau \le t < \infty$;
- $\mathcal{A}$ is invariant if $U_{\mathcal{G}}(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $-\infty < \tau \le t < \infty$.

Definition 4.7. Let $t \in \mathbb{R}$.

1. A set $\mathcal{A}(t) \subset X$ pullback attracts a set $B \in X$ at time t if
$$\operatorname{dist}(U_{\mathcal{G}}(t,s)B, \mathcal{A}(t)) \to 0 \quad \text{as } s \to -\infty.$$
2. A family $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ pullback attracts bounded sets of X if $\mathcal{A}(\tau) \subset X$ pullback attracts all bounded subsets at τ, for each $\tau \in \mathbb{R}$. In this case, we say that the nonautonomous set $\mathcal{A}$ is pullback attracting.
3. A set $\mathcal{A}(t) \subset X$ pullback absorbs bounded subsets of X at time t if, for each bounded set B in X, there exists $T = T(t, B) \le t$ such that $U_{\mathcal{G}}(t,\tau)B \subset \mathcal{A}(t)$ for all $\tau \le T$.
4. A family $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ pullback absorbs bounded subsets of X if for each $t \in \mathbb{R}$ $\mathcal{A}(t)$ pullback absorbs bounded sets at time t.

Following the ideas of [25] we obtain

Lemma 4.8. *Let (u_1, u_2) be a solution of problem (S). Then there exist a positive number r_0 and a constant T_0, which do not depend on the initial data, such that*
$$\|(u_1(t), u_2(t))\|_{H\times H} \le r_0, \qquad \forall\, t \ge T_0 + \tau.$$

Considering $Y_q := W^{1,q(\cdot)}(\Omega)$, we have

Lemma 4.9. *Let (u_1, u_2) be a solution of problem (S). Then there exist positive constants r_1 and $T_1 > T_0$, which do not depend on the initial data, such that*
$$\|(u_1(t), u_2(t))\|_{Y_p\times Y_q} \le r_1, \qquad \forall\, t \ge T_1 + \tau.$$

Let $U_{\mathbb{G}}$ be the multivalued process defined by the generalized process $\mathbb{G}$. We know from [33] that for all $t \ge s$ in $\mathbb{R}$ the map $x \mapsto U_{\mathbb{G}}(t,s)x \in P(H \times H)$ is closed, so we obtain from Theorem 18 in [10] the following result

Theorem 4.10. *If for any $t \in \mathbb{R}$ there exists a nonempty compact set $D(t)$ which pullback attracts all bounded sets of $H \times H$ at time t, then the set $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ with $\mathcal{A}(t) = \overline{\bigcup_{B\in\mathcal{B}(H\times H)} \omega_{pb}(t,B)}$, is the unique compact, negatively invariant pullback attracting set which is minimal in the class of closed pullback attracting nonautonomous sets. Moreover, the sets $\mathcal{A}(t)$ are compact.*

Here $\omega_{pb}(t, B)$ is the pullback omega limit set starting in the set B and ending at time t.

Theorem 4.11. *The multivalued evolution process $U_{\mathbb{G}}$ associated with system (S) has a compact, negatively invariant pullback attracting set $\mathfrak{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ which is minimal in the class of closed pullback attracting nonautonomous sets. Moreover, the sets $\mathcal{A}(t)$ are compact.*

Proof. By Lemma 4.9 we have that the family $D(t) := \overline{B_{Y_p\times Y_q}(0, r_1)}^{H\times H}$ of compact sets of $H \times H$ is attracting. The result thus follows from Theorem 4.10. □

5 Limit problems and convergence properties

In the remainder of the paper we restrict attention to the case that the coefficient functions $C_1(t)$ and $C_2(t)$ depend only on the time variable t and not on the spatial variable $x \in \Omega$.

Our main objective is to consider what happens when D_s increases to infinity and $p_s(\cdot) \to p > 2$, $q_s(\cdot) \to q > 2$ in $L^\infty(\Omega)$ as $s \to \infty$ in the system

$$\begin{cases} \frac{\partial u_s}{\partial t} - \operatorname{div}(D_s|\nabla u_s|^{p_s(x)-2}\nabla u_s) + C_1(t)|u_s|^{p_s(x)-2}u_s \in F(u_s,v_s),\ t > \tau, \\ \frac{\partial v_s}{\partial t} - \operatorname{div}(D_s|\nabla v_s|^{q_s(x)-2}\nabla v_s) + C_2(t)|v_s|^{q_s(x)-2}v_s \in G(u_s,v_s),\ t > \tau, \\ \dfrac{\partial u_s}{\partial n}(t,x) = \dfrac{\partial v_s}{\partial n}(t,x) = 0,\ t \geq \tau,\ x \in \partial\Omega, \\ u_s(\tau,x) = u_{\tau s}(x),\ v_s(\tau,x) = v_{\tau s}(x), x \in \Omega, \end{cases} \tag{5.1}$$

where $u_{\tau s}, v_{\tau s} \in H := L^2(\Omega)$, and to prove that the limit problem is described by an ordinary differential system.

Firstly, we observe that the gradients of the solutions of problem (5.1) converge in norm to zero as $s \to \infty$, which allows us to guess the limit problem

$$\begin{cases} \dot{u} + \phi_p^t(u) \in \widetilde{F}(u,v), \\ \dot{v} + \phi_q^t(v) \in \widetilde{G}(u,v), \\ u(\tau) = u_\tau, v(\tau) = v_\tau, \end{cases} \tag{5.2}$$

where $\phi_p^t(w) := C_1(t)|w|^{p-2}w$, $\phi_q^t(w) := C_2(t)|w|^{q-2}w$, $\widetilde{F} := F_{|\mathbb{R}\times\mathbb{R}}, \widetilde{G} := G_{|\mathbb{R}\times\mathbb{R}} : \mathbb{R}\times\mathbb{R} \to P(\mathbb{R})$ if we identify $\mathbb{R}$ with the constant functions which are in H, since Ω is a bounded set.

The next theorem confirms that the system (5.2) is a good candidate for the limit problem. The proof of the next result follows the ideas of [35] and will not present the proof here since the nonautonomous terms $C_{1,2}(t)$ do not present difficulties for the proof (see also [21] for the problem with only one equation).

Theorem 5.1. *If (u_s, v_s) is a solution of (5.1), then for each $t > T_1 + \tau$, the sequences of real numbers $\{\|\nabla u_s(t)\|_H\}_{s\in\mathbb{N}}$ and $\{\|\nabla v_s(t)\|_H\}_{s\in\mathbb{N}}$ both possess subsequences $\{\|\nabla u_{s_j}(t)\|_H\}$ and $\{\|\nabla v_{s_j}(t)\|_H\}$ that converge to zero as $j \to +\infty$, where T_1 is the positive constant in Lemma 4.9.*

In order to prove the existence of a global solution for the limit problem we consider the following abstract result of Barbu's book [6] for a Banach space X : Let $\tau \in \mathbb{R}$ and $T > \tau$ and consider a family of nonlinear operators $\mathcal{H}(t) : X \to X^*$, $t \in [\tau, T]$ satisfying:

(i) $\mathcal{H}(t)$ is monotone and hemicontinuous from X to X^* for almost every $t \in]\tau, T)$.

(ii) Function $\mathcal{H}(\cdot)u(\cdot) : [\tau, T] \to X^*$ is measurable for every $u \in L^p(\tau, T; X)$.

(iii) There is a constant C such that

$$\|\mathcal{H}(t)u\|_{X^*} \leq C(\|u\|_X^{p-1} + 1) \quad \text{for } u \in X \text{ and } t \in]\tau, T).$$

(iv) There are constants α, ω ($\omega > 0$) such that

$$\langle \mathcal{H}(t)u, u\rangle \geq \omega\|u\|_X^p + \alpha \quad \text{for } u \in X \text{ and } t \in]\tau, T).$$

Proposition 5.2 ([6, Theorem 4.2]). *Consider a Gelfand triple given by (X, H, X^*) and suppose that (i)–(iv) hold. If $u_\tau \in H$ and $f \in L^q(\tau, T; X^*)$ $(\frac{1}{p} + \frac{1}{q} = 1)$, then there exists a unique function $u(t)$ which is X^*-valued absolutely continuous on $[\tau, T]$ and satisfies*

$$u \in L^p(\tau, T; X) \cap C([\tau, T]; H), \qquad \frac{du}{dt} \in L^q(\tau, T, X^*),$$
$$\frac{du}{dt}(t) + \mathcal{H}(t)u(t) = f(t), \quad a.e. \text{ on } (\tau, T), \quad u(\tau) = u_\tau.$$

Lemma 5.3. *The problem (5.2) has a global solution.*

Proof. Considering $\mathcal{H}(t) : \mathbb{R} \to \mathbb{R}$, defined by $\mathcal{H}(t)u := C(t)|u|^{p-2}u$, it is trivial to check (i)–(iv) above for $\mathcal{H}(t)$ with $X = H = X^* = \mathbb{R}$. Thus, for a given $f \in L^2(\tau, T; \mathbb{R})$, we have from Proposition 5.2 that there exists a unique function $u \in C([\tau, T]; \mathbb{R})$ which is a strong solution to the problem

$$\frac{du}{dt}(t) + \mathcal{H}(t)u(t) = f(t), \qquad u(\tau) = u_\tau \in \mathbb{R}.$$

Hence, with the same argument as in the proof of Theorem 41 in [36] we conclude that the limit problem (5.2) has a global strong solution. □

Remark 5.4. In the proof of the previous theorem we only need that $C(\cdot)$ is measurable and $\gamma \leq C(t)$. The constant γ is taken uniform in τ and T in order to yield global solutions.

The next result guarantees that (5.2) is in fact the limit problem for (5.1), as $s \to \infty$. The proof is analogous to what was done in [35] for the autonomous case, so will not be give here since the nonautonomous terms $C_{1,2}(t)$ do not present any difficulties.

Theorem 5.5. *Let (u_s, v_s) be a solution of the problem (5.1). Suppose that $(u_s(\tau), v_s(\tau)) = (u_{\tau s}, v_{\tau s})$ $\to (u_\tau, v_\tau) \in \mathbb{R} \times \mathbb{R}$ in the topology of $H \times H$ as $s \to +\infty$. Then there exists a solution (u, v) of the problem (5.2) satisfying $(u(\tau), v(\tau)) = (u_\tau, v_\tau)$ and a subsequence $\{(u_{s_j}, v_{s_j})\}_j$ of $\{(u_s, v_s)\}_s$ such that, for each $T > \tau$, $u_{s_j} \to u$, $v_{s_j} \to v$ in $C([\tau, T]; H)$ as $j \to +\infty$.*

Remark 5.6. Theorem 5.5 remains valid without the hypothesis $(u_\tau, v_\tau) \in \mathbb{R} \times \mathbb{R}$, whenever $(u_{\tau s}, v_{\tau s}) \in \mathcal{A}_s(\tau)$, $\forall\, s \in \mathbb{N}$, because in this case we prove, analogously to Lemma 6.2 in [21], that u_τ and v_τ are independent of x.

5.1 Upper semicontinuity of the family of pullback attractors

We start this section proving the existence of the pullback attractor for the limit problem.

Theorem 5.7. *The limit problem (5.2) defines a generalized process $\mathbb{G}^\infty$ which has a pullback attractor $\mathcal{U}_\infty = \{\mathcal{A}_\infty(t);\ t \in \mathbb{R} \times \mathbb{R}\}$.*

Proof. That limit problem (5.2) defines a generalized process $\mathbb{G}^\infty$ follows in the same way as before for the system (S).

Let us focus on the existence of the pullback attractor. Multiplying the equation $\dot{u} + C_1(t)|u|^{p-2}u = f(t)$ by u and using the assumption that (F, G) is positively sublinear and Young's Inequality to estimate $f(t).u(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 \leq -\frac{\gamma}{2}|u(t)|^p + c, \qquad t \geq \tau$$

where $c > 0$ is a constant. Therefore, the map $y(t) := |u(t)|^2$ satisfies the inequality

$$\frac{d}{dt}y(t) \leq -\gamma(y(t))^{p/2} + 2c, \qquad t \geq \tau.$$

So, by Lemma 5.1 in [39],

$$|u(t)|^2 \leq \left(\frac{2c}{\gamma}\right)^{2/p} + \left(\gamma\left(\frac{p}{2}-1\right)(t-\tau)\right)^{-\frac{2}{p-2}}, \qquad \forall\, t \geq \tau. \tag{5.3}$$

Let $\xi_1 > 0$ such that $\left(\gamma(\frac{p}{2}-1)\xi_1\right)^{-\frac{2}{p-2}} \leq 1$, then

$$|u(t)| \leq \left[\left(\frac{2c}{\gamma}\right)^{2/p} + 1\right]^{1/2} =: \kappa_1, \qquad \forall\, t \geq \xi_1 + \tau.$$

Analogously, we can prove that

$$|v(t)| \leq \left[\left(\frac{2c}{\gamma}\right)^{2/q} + 1\right]^{1/2} =: \kappa_2, \qquad \forall\, t \geq \xi_2 + \tau.$$

Thus, considering $\kappa := \max\{\kappa_1, \kappa_2\}$, we have that the family $K(t) := B_{\mathbb{R}\times\mathbb{R}}[0,\kappa]$ of compact sets of $\mathbb{R}\times\mathbb{R}$ pullback attracts bounded sets of $\mathbb{R}\times\mathbb{R}$ at time t. Consequently, we have by Theorem 4.10 that the evolution process $\{S_\infty(t,s)\}_{t\geq s}$ defined by $\mathbb{G}^\infty$ has a pullback attractor $\mathcal{U}_\infty = \{\mathcal{A}_\infty(t);\ t \in \mathbb{R}\}$. □

Theorem 5.8. *The family of pullback attractors $\{\mathcal{U}_s;\ s \in \mathbb{N}\}$ associated with system (5.1) is upper semicontinuous on s at infinity, in the topology of H, i.e., for each $\tau \in \mathbb{R}$,*

$$\lim_{s\to+\infty} \operatorname{dist}(\mathcal{A}_s(\tau), \mathcal{A}_\infty(\tau)) = 0.$$

Proof. The proof follows the same ideas used in the autonomous version considered in [35], but instead of constructing a bounded complete orbit for a generalized process here we have to construct a complete bounded trajectory for a generalized process using Theorem 5.5 and working in an analogous way as in the proof of Theorem 6.1 in [34]. □

Remark 5.9. Note that if $p_s(\cdot) \equiv p$ and $q_s(\cdot) \equiv q$ the family of attractors is also lower semicontinuous since each solution of (5.2) is also a solution of (5.1) when we consider the constants C_1 and C_2 depend only on time in (5.1). For the general case of a variable exponent, lower semicontinuity is an open problem.

Remark 5.10. The assumption on the nonincreasing nature of $C_i(t)$ implies that the pointwise limit C_i^* as $t \to \infty$ exists and satisfies $0 < \gamma \leq C_i^*$. Then the limit problem with C_i^* is autonomous and has an autonomous attractor $\mathcal{A}_\infty$ as a particular case of the results in this paper. This means that the original problem is asymptotic autonomous. It would be interesting to compare the asymptotic behaviour as $t \to \infty$ of its pullback attractor with this autonomous attractor. Applying Theorem 5.3 in [25] we obtain $\lim_{t\to+\infty} \operatorname{dist}(\mathcal{A}_\infty(t), \mathcal{A}_\infty) = 0$.

5.2 Forward attraction and omega limit sets

Pullback attractors describe the behaviour of a system from the past and, in general, have little to say about the future behaviour of the system. There is a corresponding concept of forward attractor involving the usual forward attraction instead of pullback attraction, but such forward attractors rarely exist and, when they do, need not be unique. See Kloeden & Yang [26], where an alternative characterization of forward attraction is developed using omega limit sets.

By (5.3) the closed and bounded (hence compact) absorbing set $B_{\mathbb{R}\times\mathbb{R}}[0,\kappa]$ is forward absorbing for the generalized process $\mathbb{G}^{\infty}$ on $\mathbb{R}^2$ generated by the limit problem (5.2). Moreover, the set $B := \cup_{0\le t\le T_\kappa}\mathbb{G}^{\infty}(t, B_{\mathbb{R}\times\mathbb{R}}[0,\kappa])$, where T_κ is the time for the set $B_{\mathbb{R}\times\mathbb{R}}[0,\kappa]$ to absorb itself under $\mathbb{G}^{\infty}$, is also positive invariant under $\mathbb{G}^{\infty}$. In addition, its absorbing property here is uniform in the sense that for any bounded subset D of $\mathbb{R}^2$ and every τ there exists a $T_D \ge 0$ such that

$$\mathbb{G}^{\infty}(t,\tau,x_0) \subset B \qquad \forall t \ge \tau + T_D,\ x_0 \in D,$$

since the estimate (5.3) depends just on the elapsed time and not the actual times.

ω-limit sets were defined and investigated in [26, Chapter 12] for single valued processes, but analogous definitions hold for a generalized process $\mathbb{G}^{\infty}$. Specifically, the ω-limit set is defined by

$$\omega_{B,\tau} := \bigcap_{t\ge\tau}\overline{\bigcup_{s\ge t}\mathbb{G}^{\infty}(s,\tau,B)}.$$

It is a nonempty compact set of B for each τ. Note that

$$\lim_{t\to\infty}\operatorname{dist}_{\mathbb{R}^2}\left(\mathbb{G}^{\infty}(t,\tau,B),\omega_{B,\tau}\right) = 0 \tag{5.4}$$

for each τ and that $\omega_{B,\tau} \subset \omega_{B,\tau'} \subset B$ for $\tau \le \tau'$. Hence, the set

$$\omega_B := \overline{\bigcup_{\tau\in\mathbb{R}}\omega_{B,\tau}} \subset B$$

is nonempty and compact. It contains all of the future limit points of the generalized process $\mathbb{G}^{\infty}$ starting in the set B at some time $\tau \ge T^*$. In particular, it contains the omega limit points of the pullback attractor, i.e.,

$$\bigcap_{t\ge\tau}\overline{\bigcup_{s\ge t}\mathcal{A}_\infty(s)} = \bigcap_{t\ge\tau}\overline{\bigcup_{s\ge t}\mathbb{G}^{\infty}(s,\tau,\mathcal{A}_\infty(\tau))} \subset \omega_{B,\tau} \subset \omega_B$$

for each $\tau \in \mathbb{R}$.

The set ω_B characterises the forward asymptotic behaviour of the nonautonomous system $\mathbb{G}^{\infty}$. It was called the *forward attracting set* of the nonautonomous system in [26] and is closely related to the Haraux–Vishik uniform attractor, but it may be smaller and does not require the generating system to be defined for all time or the attraction to be uniform in the initial time.

The forward attracting set ω_B need not be invariant for the generalized process $\mathbb{G}^{\infty}$, but in view of the above uniform absorbing property it is *asymptotically positive invariant* [26, Chapter 12], i.e., if for every $\varepsilon > 0$ here exists a $T(\varepsilon)$ such that

$$\mathbb{G}^{\infty}(t,\tau,\omega_B) \subset B_\varepsilon(\omega_B), \qquad t \ge \tau,$$

for each $\tau \ge T(\varepsilon)$.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **11**, 1–14; https://doi.org/10.14232/ejqtde.2021.1.11 www.math.u-szeged.hu/ejqtde/

Nonautonomous equations and almost reducibility sets

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Received 12 May 2020, appeared 23 February 2021
Communicated by Mihály Pituk

Abstract. For a nonautonomous differential equation, we consider the almost reducibility property that corresponds to the reduction of the original equation to an autonomous equation via a coordinate change preserving the Lyapunov exponents. In particular, we characterize the class of equations to which a given equation is almost reducible. The proof is based on a characterization of the almost reducibility to an autonomous equation with a diagonal coefficient matrix. We also characterize the notion of almost reducibility for an equation $x' = A(t, \theta)x$ depending continuously on a real parameter θ. In particular, we show that the almost reducibility set is always an $F_{\sigma\delta}$-set and for any $F_{\sigma\delta}$-set containing zero we construct a differential equation with that set as its almost reducibility set.

Keywords: almost reducibility, nonautonomous equations.

2020 Mathematics Subject Classification: 37D99.

1 Introduction

We first describe the reducibility property and the type of problems considered in the paper. Let $A(t)$ and $B(t)$ be $q \times q$ matrices varying continuously with $t \geq 0$ and consider the linear equations

$$x' = A(t)x \quad \text{and} \quad y' = B(t)y. \tag{1.1}$$

Let $T(t,s)$ and $S(t,s)$ be the corresponding evolution families such that

$$T(t,s)x(s) = x(t) \quad \text{and} \quad S(t,s)y(s) = y(t)$$

for any solutions x and y of the equations in (1.1) and for any $t,s \geq 0$. We say that the equations are *equivalent via a coordinate change* $U(t)$ given by invertible $q \times q$ matrices if

$$U(t)^{-1}T(t,s)U(s) = S(t,s) \quad \text{for all } t,s \geq 0. \tag{1.2}$$

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More generally, one can also consider piecewise continuous functions $A(t)$ and $B(t)$ (see Section 2), in which case the evolution families $T(t,s)$ and $S(t,s)$ are still continuous in (t,s).

In this paper we consider the class of equations that are equivalent to an autonomous equation. Namely, we say that the equation $x' = A(t)x$ is *reducible via a coordinate change* $U(t)$ if it is equivalent to some autonomous equation $y' = By$. Moreover, we say that the equation $x' = A(t)x$ is *almost reducible* if it is equivalent to some autonomous equation via a Lyapunov coordinate change $U(t)$, that is, a coordinate change satisfying

$$\lim_{t\to\infty} \frac{1}{t} \log\|U(t)\| = \lim_{t\to\infty} \frac{1}{t} \log\|U(t)^{-1}\| = 0. \tag{1.3}$$

The Lyapunov coordinate changes are the only coordinate changes that preserve simultaneously the Lyapunov exponents of all sequences of invertible matrices with a finite Lyapunov exponent. More precisely, for each $v \in \mathbb{R}^q$ let

$$\lambda_A(v) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log\|T(t,0)v\|$$

be the Lyapunov exponent associated with the equation $x' = A(t)x$, with the convention that $\log 0 = -\infty$. The Lyapunov exponent $\lambda_B(v)$ for the equation $y' = B(t)y$ is defined similarly. The former statement on the preservation of the Lyapunov exponents means that a coordinate change $U(t)$ is a Lyapunov coordinate change if and only if the evolution families of any two equivalent equations as in (1.1) that satisfy (1.2) also satisfy

$$\lambda_A(U(0)v) = \lambda_B(v) \quad \text{for all } v \in \mathbb{R}^q.$$

This causes that the almost reducibility property occurs naturally whenever we want to reduce the original dynamics to a simpler one without changing the asymptotic behavior given by the Lyapunov exponents.

A first notion of reducibility is due to Lyapunov [5] (see [7] for an English translation). He considered instead bounded coordinate changes with bounded inverses, that is, transformations satisfying

$$\sup_{t\ge 0}\|U(t)\| < +\infty \quad \text{and} \quad \sup_{t\ge 0}\|U(t)^{-1}\| < +\infty. \tag{1.4}$$

We refer the reader to [4,6,8,9] and the references therein for some early results as well as to the book [3] for a global panorama of the area in 1980. While the coordinate changes satisfying (1.4) are appropriate to study uniform Lyapunov stability (because bounded coordinate changes preserve this type of stability), in order to study nonuniform Lyapunov stability it is crucial to consider Lyapunov coordinate changes as in (1.3).

We first give a characterization of the almost reducibility of an equation to an autonomous equation with a diagonal coefficient matrix (see Theorem 2.1).

Theorem 1.1. *For an equation $x' = A(t)x$ on $\mathbb{R}^q$ such that the Lyapunov exponent λ_A is finite on $\mathbb{R}^q \setminus \{0\}$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} A(s)\, ds = \inf \sum_{j=1}^q \lambda_A(v_j)$$

with the infimum taken over all bases $v_1, \dots, v_q$ for $\mathbb{R}^q$ if and only if the equation is almost reducible to an equation $y' = By$ with B diagonal.

We shall use this result to characterize the autonomous equations to which a given equation is almost reducible (see Theorem 2.2).

Theorem 1.2. *$x' = A(t)x$ is almost reducible to $y' = By$ and $y' = Cy$ if and only if the eigenvalues of B and C, counted with multiplicities and eventually up to a permutation, have the same real parts.*

We also characterize completely the notion of almost reducibility for continuous 1-parameter families of linear differential equations. Namely, we consider equations $x' = A(t,\theta)x$ depending continuously on a real parameter θ. The *almost reducibility set* of this equation is the set of all $\theta \in \mathbb{R}$ for which the equation is almost reducible. We have the following result (see Theorem 3.1).

Theorem 1.3. *The almost reducibility set of $x' = A(t,\theta)x$ is an $F_{\sigma\delta}$-set.*

Finally, we establish a partial converse of Theorem 1.3. Namely, we construct a differential equation with given $F_{\sigma\delta}$-set containing zero as its almost reducibility set (see Theorem 4.1).

Theorem 1.4. *Given an integer $q \geq 2$ and an $F_{\sigma\delta}$-set M containing zero, there exists an equation $x' = A(t,\theta)x$ whose almost reducibility set is equal to M. Moreover, given an unbounded nondecreasing function $\rho(t) \geq 0$, we may require that*

$$\|A(t,\theta)\| \leq \rho(t)(1+|\theta|) \quad \text{for all } t \geq 0 \text{ and } \theta \in \mathbb{R}.$$

The proof of Theorem 1.4 is partly inspired by arguments in [1].

2 The notion of almost reducibility

We introduce the notion of almost reducibility for the class of nonautonomous linear equations and we establish some of its basic properties. In particular, we characterize completely the class of autonomous equations to which a given nonautonomous equation is almost reducible.

Let M_q be the set of all $q \times q$ matrices with real entries and let $GL_q \subset M_q$ be the subset of all invertible matrices. Consider a piecewise continuous function $A \colon \mathbb{R}_0^+ \to M_q$. We say that the equation

$$x' = A(t)x \tag{2.1}$$

is *almost reducible* to an equation $x' = Bx$ for some matrix $B \in M_q$ if there exist matrices $U(t) \in GL_q$ for $t \geq 0$ satisfying

$$\lim_{t\to\infty} \frac{1}{t} \log\|U(t)\| = \lim_{t\to\infty} \frac{1}{t} \log\|U(t)^{-1}\| = 0 \tag{2.2}$$

such that

$$U(t)^{-1}T(t,s)U(s) = e^{B(t-s)} \quad \text{for } t,s \geq 0, \tag{2.3}$$

where $T(t,s)$ is the evolution family associated with equation (2.1). This means that we have $T(t,s)x(s) = x(t)$ for any solution $x = x(t)$ of the equation $x' = A(t)x$ and all $t,s \geq 0$. Then we also say that equation (2.1) is *almost reducible*. The family $(U(t))_{t\geq 0}$ is called a *Lyapunov coordinate change*.

We start by describing when a nonautonomous equation is almost reducible to an autonomous equation with a diagonal coefficient matrix. The *Lyapunov exponent* $\lambda \colon \mathbb{R}^q \to [-\infty, +\infty]$ associated with equation (2.1) is defined by

$$\lambda(v) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log\|T(t,0)v\|,$$

with the convention that $\log 0 = -\infty$. We shall always assume that λ takes only finite values on $\mathbb{R}^q \setminus \{0\}$. It follows from the theory of Lyapunov exponents that these finite values are say $\lambda_1 < \cdots < \lambda_p$ for some positive integer $p \le q$ and that the sets

$$E_i = \{v \in \mathbb{R}^q : \lambda(v) \le \lambda_i\}$$

are linear subspaces for $i = 1, \ldots, p$. A basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$ is said to be *normal* (with respect to equation (2.1)) if for each $i = 1, \ldots, p$ some elements of $\{v_1, \ldots, v_q\}$ form a basis for E_i.

Theorem 2.1. *Let $x' = A(t)x$ be an equation on $\mathbb{R}^q$ whose Lyapunov λ takes only finite values on $\mathbb{R}^q \setminus \{0\}$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} A(s)\, ds = \sum_{j=1}^q \lambda(v_j) \tag{2.4}$$

for some normal basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$ if and only if the equation $x' = A(t)x$ is almost reducible to an autonomous equation with a diagonal coefficient matrix, whose entries on the diagonal are then necessarily $\lambda(v_1), \ldots, \lambda(v_q)$, up to a permutation.

Proof. Assume first that (2.4) holds for some normal basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$. Let $U(0)$ be the matrix with columns $v_1, \ldots, v_q$ and for each $t > 0$, let

$$U(t) = T(t,0)U(0) \operatorname{diag}\big(e^{-\lambda(v_1)t}, \ldots, e^{-\lambda(v_q)t}\big).$$

Then

$$U(t)^{-1}T(t,s)U(s) = \operatorname{diag}\big(e^{\lambda(v_1)(t-s)}, \ldots, e^{\lambda(v_q)(t-s)}\big),$$

that is, property (2.3) holds taking

$$B = \operatorname{diag}\big(\lambda(v_1), \ldots, \lambda(v_q)\big).$$

In order to show that $(U(t))_{t\ge 0}$ is a Lyapunov coordinate change, notice that the columns of $U(t)$ are the vectors

$$T(t,0)v_1 e^{-\lambda(v_1)t}, \ldots, T(t,0)v_q e^{-\lambda(v_q)t}.$$

Since

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log\big(\|T(t,0)v_i\| e^{-\lambda(v_i)t}\big) = 0, \tag{2.5}$$

we obtain

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \|U(t)\| \le 0.$$

Now we consider the matrices

$$U(t)^{-1} = \operatorname{diag}\big(e^{\lambda(v_1)t}, \ldots, e^{\lambda(v_q)t}\big)(T(t,0)U(0))^{-1}.$$

We have

$$(T(t,0)U(0))^{-1} = C(t)/\det(T(t,0)U(0))$$

for some matrices $C(t)$ with (i,j) entry given by $(-1)^{i+j}\Delta^{ji}(t)$, where $\Delta^{ji}(t)$ is the determinant of the matrix obtained from $T(t,0)U(0)$ erasing its jth line and ith column. Then

$$U(t)^{-1} = D(t) \frac{\exp \sum_{j=1}^q \lambda(v_j)t}{\det(T(t,0)U(0))}, \tag{2.6}$$

where

$$D(t) = \operatorname{diag}\big(e^{-\sum_{j\neq 1}\lambda(v_j)t}, \dots, e^{-\sum_{j\neq q}\lambda(v_j)(m-1)}\big)C(t).$$

By Liouville's theorem we have

$$\det T(t,0) = \exp\int_0^t \operatorname{tr} A(s)\,ds \tag{2.7}$$

and so it follows from (2.4) that

$$\lim_{t\to\infty}\frac{1}{t}\log\det T(t,0) = \sum_{j=1}^q \lambda(v_j).$$

Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\log\frac{\exp\sum_{j=1}^q\lambda(v_j)t}{|\det(T(t,0)U(0))|} = 0. \tag{2.8}$$

The (i,j) entry of $D(t)$ is given by $(-1)^{i+j}\bar{\Delta}^{ji}(t)$, where $\bar{\Delta}^{ji}(t)$ is the determinant of the matrix obtained from $T(t,0)U(0)$ dividing each kth column by $e^{\lambda(v_k)t}$ and then erasing the jth line and the ith column. It follows from (2.5) that

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log|\bar{\Delta}^{ji}(t)| \leq 0 \quad\text{and so}\quad \overline{\lim_{t\to\infty}}\frac{1}{t}\log\|D(t)\| \leq 0.$$

Therefore, by (2.6) and (2.8), we obtain

$$\underline{\lim_{t\to\infty}}\frac{1}{t}\log\big(\|U(t)^{-1}\|^{-1}\big) = -\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|U(t)^{-1}\| \geq 0,$$

which shows that $(U(t))_{t\geq 0}$ is a Lyapunov coordinate change.

Now assume that the equation $x' = A(t)x$ is almost reducible to an autonomous equation with a diagonal coefficient matrix, that is,

$$U(t)^{-1}T(t,s)U(s) = \operatorname{diag}\big(e^{a_1(t-s)}, \dots, e^{a_q(t-s)}\big) \tag{2.9}$$

for some matrices $U(t) \in GL_q$, for $t \geq 0$, satisfying (2.2) and some numbers $a_1, \dots, a_q \in \mathbb{R}$. Let $v_1, \dots, v_q$ be the columns of $U(0)$. Then

$$\|U(t)^{-1}T(t,0)v_i\| = e^{a_i t}.$$

By (2.2), this implies that the basis $v_1, \dots, v_q$ is normal with $\lambda(v_i) = a_i$ for $i = 1, \dots, q$. Moreover, again by (2.9), we have

$$\det(U(t)^{-1})\det T(t,0)\det U(0) = e^{\sum_{j=1}^q\lambda(v_i)t}. \tag{2.10}$$

Since $\det U(t)$ is a sum of products of the entries of $U(t)$, by (2.2) we have

$$\lim_{t\to\infty}\frac{1}{t}\log|\det U(t)| = 0$$

and so it follows from (2.7) and (2.10), that identity (2.4) holds. □

We use Theorem 2.1 to characterize the class of autonomous equations to which an equation $x' = A(t)x$ is almost reducible.

Theorem 2.2. *Let $x' = A(t)x$ be an equation on $\mathbb{R}^q$ that is almost reducible to an equation $x' = Bx$. Then the equation $x' = A(t)x$ is almost reducible to an equation $x' = Cx$ if and only if the eigenvalues $\lambda_i(B)$ and $\lambda_i(C)$, respectively, of B and C counted with multiplicities, satisfy*

$$\operatorname{Re}\lambda_i(B) = \operatorname{Re}\lambda_i(C) \quad \textit{for } i = 1, \dots, q,$$

eventually up to a permutation.

Proof. First assume that the equation $x' = A(t)x$ is almost reducible to both $x' = Bx$ and $x' = Cx$. Consider Lyapunov coordinate changes $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ such that

$$U(t)^{-1}T(t,s)U(s) = e^{B(t-s)} \quad \text{and} \quad V(t)^{-1}T(t,s)V(s) = e^{C(t-s)}$$

for $t, s \geq 0$. Then

$$W(t)^{-1}e^{B(t-s)}W(s) = e^{C(t-s)}$$

for $t, s \geq 0$, where the matrices $W(t) = U(t)^{-1}V(t)$ form again a Lyapunov coordinate change. It follows readily from the identity

$$W(t)^{-1}e^{Bt}W(0) = e^{Ct}$$

that the Lyapunov exponents λ^B and λ^C associated, respectively, with the equations $x' = Bx$ and $x' = Cx$ satisfy

$$\lambda^B(W(0)v) = \lambda^C(v) \quad \text{for all } v \in \mathbb{R}^q. \tag{2.11}$$

The values of λ^B and λ^C are, respectively, $\operatorname{Re}\lambda_i(B)$ and $\operatorname{Re}\lambda_i(C)$ for $i = 1, \dots, q$, counted with their multiplicities and so it follows readily from (2.11) that

$$\operatorname{Re}\lambda_i(B) = \operatorname{Re}\lambda_i(C) \quad \text{for } i = 1, \dots, q, \tag{2.12}$$

eventually up to a permutation.

Now assume that property (2.12) holds, eventually up to a permutation. Again, the values of the Lyapunov exponents λ^B and λ^C are, respectively, $\operatorname{Re}\lambda_i(B)$ and $\operatorname{Re}\lambda_i(C)$ for $i = 1, \dots, q$, counted with their multiplicities. Therefore, condition (2.4) holds for the differential equations $x' = Bx$ and $x' = Cx$. By Theorem 2.1, there exist Lyapunov coordinate changes $(\bar{U}(t))_{t\geq 0}$ and $(\bar{V}(t))_{t\geq 0}$ such that

$$\bar{U}(t)^{-1}e^{B(t-s)}\bar{U}(s) = \operatorname{diag}\big(\operatorname{Re}\lambda_1(B), \dots, \operatorname{Re}\lambda_q(B)\big)^{t-s}$$

and

$$\bar{V}(t)^{-1}e^{C(t-s)}\bar{V}(s) = \operatorname{diag}\big(\operatorname{Re}\lambda_1(C), \dots, \operatorname{Re}\lambda_q(C)\big)^{t-s}$$

for $t \geq 0$. By (2.12), we obtain

$$\bar{U}(t)^{-1}e^{B(t-s)}\bar{U}(s) = \bar{V}(t)^{-1}e^{C(t-s)}\bar{V}(s)$$

for $t \geq 0$ and so

$$W(t)^{-1}T(t,s)W(s) = e^{C(t-s)}$$

for $t, s \geq 0$, where

$$W(t) = U(t)\bar{U}(t)\bar{V}(t)^{-1}$$

for each $t \geq 0$. Since $(W(t))_{t\geq 0}$ is a Lyapunov coordinate change, we conclude that $x' = A(t)x$ is almost reducible to the equation $x' = Cx$. $\square$

3 Characterization of almost reducibility sets

In this section we give a characterization of the almost reducibility sets of a differential equation $x' = A(t,\theta)x$ depending on a real parameter θ. Namely, we show that any such set is an $F_{\sigma\delta}$-set. More precisely, let $\mathcal{M}$ be the set of all equations $x' = A(t,\theta)x$ such that the map

$$\mathbb{R}_0^+ \times \mathbb{R} \ni (t,\theta) \mapsto A(t,\theta) \in M_q$$

is piecewise continuous in t and continuous in θ. We denote by $T_\theta(t,s)$ the corresponding evolution family. The *almost reducibility set* of an equation $x' = A(t,\theta)x$ is the set of all $\theta \in \mathbb{R}$ for which the equation is almost reducible.

Theorem 3.1. *The almost reducibility set of any equation $x' = A(t,\theta)x$ in $\mathcal{M}$ is an $F_{\sigma\delta}$-set.*

Proof. Let M be the almost reducibility set of the equation. For each $n \in \mathbb{N}$ and $\varepsilon > 0$, we define a function $g_{n,\varepsilon} \colon M_q \times GL_q \times \mathbb{R} \to [0,n]$ by

$$g_{n,\varepsilon}(B,C,\theta) = \sup_{t\geq 0} \min\{n, h_t(B,C,\theta)\},$$

where

$$f_t(B,C,\theta) = \max\bigl\{\|e^{Bt}CT_\theta(0,t)\|e^{-\varepsilon t}, \|T_\theta(t,0)C^{-1}e^{-Bt}\|e^{-\varepsilon t}\bigr\}.$$

The function $g_{n,\varepsilon}$ is lower semicontinuous in (B,C,θ) since the functions

$$\|e^{Bt}CT_\theta(0,t)\|e^{-\varepsilon t} \quad \text{and} \quad \|T_\theta(t,0)C^{-1}e^{-Bt}\|e^{-\varepsilon t}$$

are continuous (in view of the continuous dependence of a solution on a parameter) and the supremum of any number of continuous functions is lower semicontinuous. Therefore, the set

$$D_{n,\varepsilon} = g_{n,\varepsilon}^{-1}(-\infty, n/2]$$

is closed for each $n \in \mathbb{N}$ and $\varepsilon > 0$.

Lemma 3.2. *The equation $x' = A(t,\theta)x$ is almost reducible to the equation $x' = Bx$ if and only if there exists $C \in GL_q$ such that for each $\varepsilon > 0$ we have*

$$g_{n,\varepsilon}(B,C,\theta) \leq n/2 \quad \textit{for some } n \in \mathbb{N}. \tag{3.1}$$

Proof of the lemma. First assume that the equation $x' = A(t)x$ is almost reducible to the equation $x' = Bx$. Then there exists a Lyapunov coordinate change $(U(t))_{t\geq 0}$ satisfying (2.3). By property (2.2), for each $\varepsilon > 0$ we have

$$-\varepsilon < -\frac{1}{t}\log\|U(t)^{-1}\| \leq \frac{1}{t}\log\|U(t)\| < \varepsilon$$

for any sufficiently large t and so there exists $c = c(\varepsilon) > 0$ such that

$$c^{-1}e^{-\varepsilon t} < \|U(t)^{-1}\|^{-1} \leq \|U(t)\| < ce^{\varepsilon t} \tag{3.2}$$

for all $t \geq 0$. Now take $C = U(0)^{-1}$. By (2.3) with $s = 0$ we have

$$U(t) = T_\theta(t,0)C^{-1}e^{-Bt} \quad \text{and} \quad U(t)^{-1} = e^{Bt}CT_\theta(0,t).$$

Hence, it follows readily from (3.2) that

$$\sup_{t\geq 0}\left(\|e^{Bt}CT_\theta(0,t)\|e^{-\varepsilon t}+\|T_\theta(t,0)C^{-1}e^{-Bt}\|e^{-\varepsilon t}\right)<\infty$$

and so property (3.1) holds.

Now assume that there exists $C\in GL_q$ satisfying (3.1) for each $\varepsilon>0$. Then there exists $n\in\mathbb{N}$ such that

$$\max\left\{\|e^{Bt}CT_\theta(0,t)\|e^{-\varepsilon t},\|T_\theta(t,0)C^{-1}e^{-Bt}\|e^{-\varepsilon t}\right\}\leq n/2$$

for all $t\geq 0$ and so

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|e^{Bt}CT_\theta(0,t)\|\leq 0 \tag{3.3}$$

and

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|T_\theta(t,0)C^{-1}e^{-Bt}\|\leq 0. \tag{3.4}$$

Finally, let

$$U(t)=T_\theta(t,0)C^{-1}e^{-Bt}\quad\text{for } t\geq 0.$$

Note that $U(0)=C^{-1}$. Therefore,

$$\begin{aligned}e^{B(t-s)}&=e^{Bt}e^{-Bs}\\&=U(t)^{-1}T_\theta(t,0)U(0)\left(U(0)^{-1}T_\theta(0,s)U(s)\right)\\&=U(t)^{-1}T_\theta(t,s)U(s).\end{aligned}$$

Moreover, since

$$U(t)^{-1}=e^{Bt}CT_\theta(0,t),$$

it follows readily from (3.3) and (3.4) that

$$0\leq\underline{\lim_{t\to\infty}}\frac{1}{t}\log(\|U(t)^{-1}\|^{-1})\leq\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|U(t)\|\leq 0$$

and so condition (2.2) also holds. □

By Lemma 3.2, the equation $x'=A(t,\theta)x$ is almost reducible if and only if there exist $B\in M_q$ and $C\in GL_q$ such that

$$(B,C,\theta)\in D_\varepsilon:=\bigcup_{n\in\mathbb{N}}D_{n,\varepsilon}$$

for each $\varepsilon>0$. Therefore, the almost reducibility set is

$$M=\bigcap_{\varepsilon>0}\pi(D_\varepsilon),$$

where $\pi\colon M_q\times GL_q\times\mathbb{R}\to\mathbb{R}$ is the projection onto the third component. For $k\in\mathbb{N}$ let

$$E_k=\left\{(B,C,\theta)\in M_q\times GL_q\times\mathbb{R}:\|B\|\leq k,\ k^{-1}\leq|\det C|\leq k,\ |\theta|\leq k\right\}.$$

Then each set $D_{n,\varepsilon}\cap E_k$ is compact and

$$D_\varepsilon=\bigcup_{n\in\mathbb{N}}D_{n,\varepsilon}=\bigcup_{n,k\in\mathbb{N}}(D_{n,\varepsilon}\cap E_k).$$

Therefore,

$$M=\bigcap_{\varepsilon>0}\pi(D_\varepsilon)=\bigcap_{p\in\mathbb{N}}\bigcup_{n,k\in\mathbb{N}}\pi(D_{n,1/p}\cap E_k)$$

and since the map π is continuous, each set $\pi(D_{n,1/p}\cap E_k)$ is compact. This shows that the almost reducibility set M is an $F_{\sigma\delta}$-set. □

4 Construction of families of equations

We also construct (as explicitly as possible) a differential equation in $\mathcal{M}$ with a given $F_{\sigma\delta}$-set containing zero as its almost reducibility set.

Theorem 4.1. *Given an integer $q \geq 2$ and an $F_{\sigma\delta}$-set M containing zero, there exists an equation $x' = A(t,\theta)x$ in $\mathcal{M}$ whose almost reducibility set is equal to M. Moreover, given an unbounded nondecreasing function $\rho(t) \geq 0$, we may require that*

$$\|A(t,\theta)\| \leq \rho(t)(1+|\theta|) \quad \text{for all } t \geq 0 \text{ and } \theta \in \mathbb{R}.$$

Proof. We start by describing some auxiliary notions that will be used in the proof. Given $a, b, c, \theta \in \mathbb{R}$, we consider the 2×2 matrices

$$B(u,\theta) = \begin{pmatrix} a\theta & c(1-\theta)+b\theta \\ -c(1-\theta)-b\theta & -a\theta \end{pmatrix}, \tag{4.1}$$

where $u = (a,b,c)$ and

$$\nu = \nu(u,\theta) = \sqrt{(a^2-(b-c)^2)\theta^2 - 2c(b-c)\theta - c^2}. \tag{4.2}$$

Then $B(u,\theta)$ has eigenvalues $\pm\nu$. Given $r, s \in \mathbb{R}$ with $rs > 0$ and $d \in \mathbb{R}^+$, we define

$$a = d(s-r), \quad b = d(2rs - r - s), \quad c = 2drs. \tag{4.3}$$

Then

$$a^2 - (b-c)^2 = -4d^2rs < 0$$

and one can show that $\theta \in [r,s]$ if and only if

$$P(u,\theta) := (a^2-(b-c)^2)\theta^2 - 2c(b-c)\theta - c^2 \geq 0. \tag{4.4}$$

Since M is an $F_{\sigma\delta}$-set containing zero, one can write

$$\mathbb{R} \setminus M = \bigcup_{w\in\mathbb{N}} H^w, \quad \text{where } H^w = \bigcap_{i\in\mathbb{N}} U_i^w$$

for some nonempty open sets $U_i^w \subset \mathbb{R}\setminus\{0\}$ satisfying $U_{i+1}^w \subset U_i^w$ for each $w, i \in \mathbb{N}$. Moreover, $U_i^w = \bigcup_{m\in\mathbb{N}} I_{im}^w$ for some nonempty open finite intervals $I_{im}^w \subset \mathbb{R}\setminus\{0\}$ with the property that each $\theta \in U_i^w$ belongs to at most two intervals I_{im}^w (for each $w, i \in \mathbb{N}$).

We still need an additional decomposition. For each interval $I_{im}^w = (\alpha,\beta)$, we consider the sequence $(c_l)_{l\in\mathbb{Z}}$ defined recursively as follows. Take $c_0 = (\alpha+\beta)/2$. For each $l \in \mathbb{N}$, let

$$c_{2l} = \frac{c_{2l-2}+\beta}{2}, \quad c_{-2l} = \frac{c_{-2l+2}+\alpha}{2}$$

and

$$c_{2l-1} = \frac{c_{2l-2}+c_{2l}}{2}, \quad c_{-2l+1} = \frac{c_{-2l+2}+c_{-2l}}{2}.$$

We define $J_{iml}^w = [c_l, c_{l+2}]$ for $l \in \mathbb{Z}$ and so

$$I_{im}^w = \bigcup_{l\in\mathbb{Z}} J_{iml}^w.$$

Note that each point $\theta \in U_i^w$ belongs to at most three intervals J_{iml}^w (for each $w, i, m \in \mathbb{N}$). Moreover, given $\theta \in I_{im}^w$, there exists $l = l(\theta) \in \mathbb{Z}$ with $\theta \in J_{iml}^w$ such that θ is at least at a distance $|J_{iml}^w|/6$ from each endpoint of J_{iml}^w (where $|I|$ denotes the length of the interval I).

Now let $\iota\colon \mathbb{N} \to \mathbb{N}^3 \times \mathbb{Z}$ be a bijection. Writing $J_{iml}^w = [r,s]$ and $\eta = \iota^{-1}(w,i,m,l)$, we consider the unique $d = d(\eta) \in \mathbb{R}^+$ such that

$$\max_{\theta\in\mathbb{R}} P(u(\eta),\theta) = d^2 rs(r-s)^2 = \frac{1}{w}, \tag{4.5}$$

with $u(\eta) = (a,b,c)$ given by (4.3). Then

$$P(u(\eta),\theta) \geq \frac{5}{9} d^2 rs(r-s)^2 = \frac{5}{9w} \quad \text{for } \theta \in \left[r + \frac{s-r}{6}, s - \frac{s-r}{6}\right]. \tag{4.6}$$

Consider the function $\sigma(t) = \min\{\rho(t), t\}$ for $t \geq 0$. Moreover, consider a strictly increasing sequence of positive integers $(\ell_j)_{j\in\mathbb{N}}$ such that $\ell_1 = 1$,

$$\frac{\ell_{3j-2}}{\ell_{3j-1}} \sum_{i=1}^{j-1} \sigma(\ell_{3i-1}) < \frac{1}{j}, \quad \frac{\ell_{3j-1}}{\ell_{3j}} < \frac{1}{j}, \quad \ell_{3j+1} = 2\ell_{3j} - \ell_{3j-1} \tag{4.7}$$

and

$$\sigma(\ell_{3j-1}) \geq 2\kappa \|u(j)\| \tag{4.8}$$

for all $j \in \mathbb{N}$, where $\kappa > 0$ is fixed constant such that

$$\left\| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \right\| \leq \kappa \|(a,b,c,d)\| \quad \text{for any } a,b,c,d \in \mathbb{R}.$$

Finally, let $\Delta_j = [\ell_j, \ell_{j+1})$ for each $j \in \mathbb{N}$ and define

$$A(t,\theta) = \begin{cases} B(u(j),\theta) & \text{if } t \in \Delta_{3j} \text{ for some } j \in \mathbb{N}, \\ -B(u(j),\theta) & \text{if } t \in \Delta_{3j-1} \text{ for some } j \in \mathbb{N}, \\ \mathrm{Id} & \text{if } t \in \Delta_{3j-2} \text{ for some } j \in \mathbb{N}. \end{cases}$$

By (4.1) together with (4.8), we obtain

$$\begin{aligned} \|A(t,\theta)\| &\leq \|B(u(j),\theta)\| \leq \kappa \|u(j)\| \\ &\leq \sigma(\ell_{3j-1})(1+|\theta|) \\ &\leq \sigma(t)(1+|\theta|) \leq \rho(t)(1+|\theta|). \end{aligned}$$

Lemma 4.2. *$x' = A(t,\theta)x$ is not almost irreducible for $\theta \in \mathbb{R} \setminus M$.*

Proof of the lemma. Take $w \in \mathbb{N}$ such that $\theta \in U_i^w$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\theta \in I_{im}^w$. Moreover, let $l = l(\theta) \in \mathbb{Z}$ be the integer introduced before (4.5) and write $\iota^{-1}(w,i,m,l) = r_i$. For each $j \in \Delta_{3r_i-1} \cup \Delta_{3r_i}$ the matrices $\pm B(u(3r_i),\theta)$ have real eigenvalues. Denoting their (common) top eigenvalue by ν_i, it follows readily from (4.2) and (4.4) together with (4.5) and (4.6) that

$$\frac{1}{2\sqrt{w}} \leq \nu_i \leq \frac{1}{\sqrt{w}}.$$

Denoting by $T_\theta(t,s)$ the evolution family associated with the equation $x' = A(t,\theta)x$, we have $T_\theta(\ell_{3r_i-1}, 0) = \mathrm{Id}$ and so

$$T_\theta(\ell_{3r_i}, 0) = T_\theta(\ell_{3r_i}, \ell_{3r_i-1}).$$

Therefore,

$$\begin{aligned}\|T_\theta(\ell_{3r_i},0)\| &= \|T_\theta(\ell_{3r_i},\ell_{3r_i-1})\| \\ &= \left\|e^{(\ell_{3r_i}-\ell_{3r_i-1})B(u(3r_i),\theta)}\right\| \\ &\geq e^{\nu_i(\ell_{3r_i}-\ell_{3r_i-1})} \\ &\geq \exp\left(\frac{\ell_{3r_i}-\ell_{3r_i-1}}{2\sqrt{w}}\right)\end{aligned}$$

which in view of (4.7) gives

$$\varlimsup_{i\to\infty}\frac{1}{\ell_{3r_i}}\log\|T_\theta(\ell_{3r_i},0)\| \geq \varlimsup_{i\to\infty}\frac{1}{2\sqrt{w}}\left(1-\frac{\ell_{3r_i-1}}{\ell_{3r_i}}\right) = \frac{1}{2\sqrt{w}} > 0. \tag{4.9}$$

Now we assume that the equation $x' = A(t,\theta)x$ is almost reducible to an equation $x' = Bx$. Then there exist matrices $U(t)$ satisfying (2.2) and (2.3). Since $T_\theta(\ell_{3r_i-1},0) = \mathrm{Id}$, we have

$$e^{B\ell_{3r_i-1}} = U(\ell_{3r_i-1})^{-1}U(0)$$

and

$$e^{-B\ell_{3r_i-1}} = U(0)^{-1}U(\ell_{3r_i-1})$$

for all $i \in \mathbb{N}$. Therefore, for each $\varepsilon > 0$ there exists $c_0 = c_0(\varepsilon) > 0$ such that

$$\max\left\{\|e^{B\ell_{3r_i-1}}\|, \|e^{-B\ell_{3r_i-1}}\|\right\} \leq c_0 e^{\varepsilon\ell_{3r_i-1}}$$

for all $i \in \mathbb{N}$. Since $\ell_{3r_i-1} \to \infty$ when $i \to \infty$ and ε is arbitrary, all eigenvalues of B have real part equal to 0 and so

$$\|e^{Bt}\| \leq c_1(1+|t|) \quad \text{for some } c_1 > 0 \text{ and any } t \geq 0.$$

On the other hand, by (2.3) we have

$$T_\theta(t,0) = U(t)e^{Bt}U(0)^{-1}$$

and so

$$\begin{aligned}\|T_\theta(\ell_{3r_i},0)\| &\leq \|U(0)^{-1}\| \cdot \|U(\ell_{3r_i})\| \cdot \|e^{B\ell_{3r_i}}\| \\ &\leq c_1(1+|\ell_{3r_i}|)\|U(0)^{-1}\| \cdot \|U(\ell_{3r_i})\|.\end{aligned}$$

Finally, taking into account that $U(t)$ satisfies (2.2) we obtain

$$\varlimsup_{i\to\infty}\frac{1}{\ell_{3r_i}}\log\|T_\theta(\ell_{3r_i},0)\| \leq 0,$$

which contradicts (4.9). This shows that the equation $x' = A(t,\theta)x$ is not almost reducible. □

Lemma 4.3. *$x' = A(t,\theta)x$ is almost reducible for $\theta \in M$.*

Proof of the lemma. Since $\theta \notin H^w$ for every $w \in \mathbb{N}$, θ belongs to at most finitely many sets U_i^w, $i \in \mathbb{N}$ (because $U_{i+1}^w \subset U_i^w$ for each $w, i \in \mathbb{N}$) and since each element of U_i^w belongs to at most two intervals I_{im}^w with $m \in \mathbb{N}$ and to at most three closed intervals J_{iml}^w with $l \in \mathbb{Z}$, we conclude that θ belongs to finitely many closed intervals J_{iml}^w with $i, m \in \mathbb{N}$ and $l \in \mathbb{Z}$ for each $w \in \mathbb{N}$. This implies that for each $w \in \mathbb{N}$ there exists $N = N_w \in \mathbb{N}$ such that for $\eta \geq N$

with $\iota(\eta) = (w_\eta, i_\eta, m_\eta, l_\eta)$ we have $\theta \notin J^{w_\eta}_{i_\eta m_\eta l_\eta}$ and so also $P(u(\eta), \theta) < 0$ whenever $w_\eta \le w$. In particular, for $\eta \ge N$ with $w_\eta \le w$ we have $\nu = i\bar{\nu}$ with $\bar{\nu} \in \mathbb{R}$ and so

$$\|e^{B(u(\eta),\theta)t}\| \le 1 + 2\|B(u(\eta), \theta)\| \cdot |t|$$

(see for example [2, p. 65]). For the values of $\eta \ge N$ with $w_\eta > w$, in view of (4.5) we have

$$P(u(\eta), \theta) \le \frac{1}{w_\eta} \le \frac{1}{w+1}.$$

Take $w \in \mathbb{N}$. If $\eta \ge N$, then

$$\begin{aligned}
\|T_\theta(t, \ell_{3\eta-1})\| &\le \left\|e^{B(u(\eta),\theta)(t-\ell_{3\eta-1})}\right\| \\
&\le \big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)(t-\ell_{3\eta-1})\big) \exp\left(\frac{t-\ell_{3\eta-1}}{\sqrt{w+1}}\right) \\
&\le \big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) \exp\left(\frac{t-\ell_{3\eta-1}}{\sqrt{w+1}}\right)
\end{aligned} \tag{4.10}$$

for $t \in \Delta_{3\eta-1} \cup \Delta_{3\eta}$. Now take

$$t \in \Delta_{3\eta-1} \cup \Delta_{3\eta} \cup \Delta_{3\eta+1} \quad \text{with } \eta \in \mathbb{N}.$$

Since $A(t, \theta) = \mathrm{Id}$ for $t \in \Delta_{3\eta-2}$ and

$$T_\theta(t, \ell_{3N-1}) = T_\theta(t, \ell_{3\eta-1}) \prod_{i=N}^{\eta-1} T_\theta(\ell_{3i+1}, \ell_{3i-1}),$$

using (4.10) we obtain

$$\begin{aligned}
\|T_\theta(t, \ell_{3N-1})\| &\le \|T_\theta(t, \ell_{3\eta-1})\| \prod_{i=N}^{\eta-1} \|T_\theta(\ell_{3i+1}, \ell_{3i-1})\| \\
&\le \big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) \prod_{i=N}^{\eta-1} \big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) \\
&\quad \times \exp\left(\frac{1}{\sqrt{w+1}}\left((t-\ell_{3\eta-1}) + \sum_{i=N}^{\eta-1}(\ell_{3i+1} - \ell_{3i-1})\right)\right) \\
&\le \big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) \prod_{i=1}^{\eta-1} \big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) \\
&\quad \times \exp\left(\frac{t - \ell_{3\eta-1} + \ell_{3\eta-2} - \ell_{3N-1}}{\sqrt{w+1}}\right) \\
&\le \big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) \prod_{i=1}^{\eta-1} \big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) \exp\frac{t}{\sqrt{w+1}}.
\end{aligned}$$

Then

$$\begin{aligned}
\overline{\lim_{t\to\infty}} \frac{1}{t} \log\|T_\theta(t, \ell_{3N-1})\| &\le \frac{1}{\sqrt{w+1}} + \overline{\lim_{t\to\infty}} \frac{1}{t} \log\big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) \\
&\quad + \overline{\lim_{t\to\infty}} \frac{1}{t} \sum_{i=1}^{\eta-1} \log\big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big).
\end{aligned} \tag{4.11}$$

Since

$$\sigma(\ell_{3\eta-1}) = \min\{\rho(\ell_{3\eta-1}), \ell_{3\eta-1}\} \le \ell_{3\eta-1} \le t,$$

we obtain

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log\big(1 + 2\sigma(\ell_{3\eta-1})(1+|\theta|)t\big) = 0. \tag{4.12}$$

Moreover, since $\log(1+x) \le x$ for all $x \ge 0$, it follows from (4.7) that

$$\begin{aligned}
\frac{1}{t}\sum_{i=1}^{\eta-1} \log\big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) &\le \frac{2}{\ell_{3\eta-1}} \sum_{i=1}^{\eta-1} \big(\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) \\
&\le \frac{2(1+|\theta|)\ell_{3\eta-2}}{\ell_{3\eta-1}} \sum_{i=1}^{\eta-1} \sigma(\ell_{3i-1}) \\
&\le \frac{2(1+|\theta|)}{\eta}.
\end{aligned}$$

Therefore,

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \sum_{i=1}^{\eta-1} \log\big(1 + 2\sigma(\ell_{3i-1})(1+|\theta|)\ell_{3i+1}\big) = 0 \tag{4.13}$$

since $\eta \to \infty$ when $t \to \infty$. By (4.12) and (4.13), it follows from (4.11) that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log\|T_\theta(t, \ell_{3N-1})\| \le \frac{1}{\sqrt{w+1}}$$

for any $w \in \mathbb{N}$ and so

$$\overline{\lim_{t\to\infty}} \frac{1}{n} \log\|T_\theta(t,0)\| \le 0.$$

One can also show that

$$\overline{\lim_{t\to\infty}} \frac{1}{n} \log\|T_\theta(0,t)\| \le 0$$

interchanging $B(u,\theta)$ with $-B(u,\theta)$ in the definition of $A(t,\theta)$. This implies that

$$\begin{aligned}
\underline{\lim_{t\to\infty}} \frac{1}{t} \log\|T_\theta(t,0)\| &\ge \underline{\lim_{t\to\infty}} \frac{1}{t} \log\big(\|T_\theta(0,t)\|^{-1}\big) \\
&= -\overline{\lim_{t\to\infty}} \frac{1}{t} \log\|T_\theta(0,t)\| \ge 0
\end{aligned}$$

and so

$$\lim_{t\to\infty} \frac{1}{t} \log\|T(t,0)^{\pm 1}\| = 0.$$

For $U_\theta(t) = T_\theta(t,0)$ we have

$$U_\theta(t)^{-1} T_\theta(t,0) U_\theta(t) = T_\theta(0,t) T_\theta(t,0) = \mathrm{Id}.$$

So, identity (2.3) holds with $B = 0$. This shows that the differential equation $x' = A(t,\theta)x$ is almost reducible. $\square$

In order to construct an equation $x' = \tilde{A}(t,\theta)x$ on $\mathbb{R}^q$ with almost reducibility set M for $q > 2$, it suffices to take

$$\tilde{A}(t,\theta) = \mathrm{diag}(A(t,\theta), 0).$$

This concludes the proof of the theorem. $\square$

Acknowledgment

This research was partially supported by FCT/Portugal through UID/MAT/04459/2019.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **12**, 1–27; https://doi.org/10.14232/ejqtde.2021.1.12 www.math.u-szeged.hu/ejqtde/

General solutions to subclasses of a two-dimensional class of systems of difference equations

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Received 29 September 2020, appeared 26 February 2021
Communicated by Armengol Gasull

Abstract. We show practical solvability of the following two-dimensional systems of difference equations

$$x_{n+1} = \frac{u_{n-2}v_{n-3} + a}{u_{n-2} + v_{n-3}}, \quad y_{n+1} = \frac{w_{n-2}s_{n-3} + a}{w_{n-2} + s_{n-3}}, \quad n \in \mathbb{N}_0,$$

where u_n, v_n, w_n and s_n are x_n or y_n, by presenting closed-form formulas for their solutions in terms of parameter a, initial values, and some sequences for which there are closed-form formulas in terms of index n. This shows that a recently introduced class of systems of difference equations, contains a subclass such that one of the delays in the systems is equal to four, and that they all are practically solvable, which is a bit unexpected fact.

Keywords: system of difference equations, solvable systems, practical solvability.

2020 Mathematics Subject Classification: 39A45.

1 Introduction

Solvability of difference equations is one of the basic topics studied in the area. Presentations of some results in the topic can be found in any book on the equations, for instance, in: [4, 5, 9, 10, 12, 13]. The equations frequently appear in various applications (see, e.g., [4, 5, 7, 8, 11, 12, 23, 25, 41]). There has been also some recent interest in solvability (see, e.g., [2, 22, 28–32, 35, 37–40]). If it is not easy to find solutions to the equations, researchers try to find their invariants, as it was the case in [15–17, 21, 26, 27, 33, 34]. In some cases they can be used also for solving the equations and systems, as it was the case in [33, 34].

Each difference equation can be used for forming systems of difference equations possessing some types of symmetry. A way for forming such systems can be found in [28].

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Papaschinopoulos, Schinas and some of their colleagues proposed studying some systems of this and other types (see, e.g., [6, 14–21, 26, 27]). We have devoted a part of our research to solvable systems of difference equations, unifying the two topics (see, e.g., [2,28–32,35,38–40]).

During the last several years, we have studied, among other things, practical solvability of product-type systems of difference equations. For some of our previous results in the topic see, for instance, [29,30], as well as the related references therein. The systems are theoretically solvable, but only several subclasses are practically solvable, which has been one of the main reasons for our study of the systems.

Quite recently, we have started studying solvability of the, so called, hyperbolic-cotangent-type systems of difference equations. They are given by

$$x_{n+1} = \frac{u_{n-k}v_{n-l} + a}{u_{n-k} + v_{n-l}}, \quad y_{n+1} = \frac{w_{n-k}s_{n-l} + a}{w_{n-k} + s_{n-l}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where delays k and l are nonnegative integers, parameter a and initial values are complex numbers, whereas each of the four sequences u_n, v_n, w_n and s_n is one of the sequences x_n or y_n for all possible values of index n.

Note that this is a class of nonlinear systems of difference equations which is formed by using the method for forming symmetric types of systems of difference equations described in [28]. For the case of one-dimensional difference equation corresponding to the systems in (1.1) see [24] and [37].

What is interesting is that the systems in (1.1) are connected to product-type ones. As we have mentioned the product-type systems are theoretically solvable, but only few of them are practically solvable. The reason for this lies in impossibility to solve all polynomial equations, as well as the fact that with each product-type system of difference equations is associated a polynomial. The mentioned connection between the systems in (1.1) and product-type ones implies that also only several subclasses of the systems in (1.1) are practically solvable. Moreover, the connection shows that for guaranteeing practical solvability of all the systems in (1.1) values of k and l seems should be small. Note that we may assume $k \le l$. The case $k = 0$ and $l = 1$ was studied in [39] and [40], whereas in [32] was presented another solution to the problem. The case $k = 1$ and $l = 2$ was studied in [31], whereas the case $k = 0$ and $l = 2$ was studied in [35], which finished the study of practical solvability in the case when $\max\{k, l\} \le 2$ and $k \ne l$. The case $k = l \in \mathbb{N}_0$ was solved in [36].

Thus, it is of some interest to see if all the systems in (1.1) are solvable when $l = 3$ and k is such that $0 \le k \le 2$.

One of the cases is obvious. Namely, if $k = 1$, then the systems in (1.1) are with interlacing indices (the notion and some examples can be found in [38]), since each of the systems in (1.1) in this case, reduces to two systems of the exactly same form with $k = 0$ and $l = 1$. Thus, it is of some interest to study the other cases.

Here, we show that the systems of difference equations

$$x_{n+1} = \frac{u_{n-2}v_{n-3} + a}{u_{n-2} + v_{n-3}}, \quad y_{n+1} = \frac{w_{n-2}s_{n-3} + a}{w_{n-2} + s_{n-3}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

are practically solvable, that is, we show the solvability of all sixteen systems in (1.1), in the case $k = 2$ and $l = 3$, which is a bit surprising result. Namely, as we have said, to each system in (1.2) is associated a polynomial, several of which have degree bigger than four (some of them have degree eight). By a well-known theorem of Abel [1], polynomials of degree bigger than four need not be solvable by radicals. However, it turns out that all the associate polynomials to the systems in (1.2) are solvable by radicals, implying practical solvability of the corresponding systems. Using the fact that there is no universal method for showing practical solvability of such systems, as well as the fact that the situation in the case $\max\{k,l\} \geq 5$ is different, shows the importance of studying solvability of the systems in (1.2).

The case $a = 0$ was considered in [32] where it was shown its theoretical solvability. Namely, by using the changes of variables

$$x_n = \frac{1}{\widehat{x}_n}, \quad y_n = \frac{1}{\widehat{y}_n},$$

system (1.2) becomes linear, from which together with a known theorem from the theory of homogeneous linear difference equations with constant coefficients the theoretical solvability of the system follows. Hence, from now on we will consider only the case $a \neq 0$.

2 Connection of (1.2) to product-type systems and a lemma

First, we present above mentioned connection of the systems in (1.2) to some product-type systems.

Some simple calculations yield

$$x_{n+1} \pm \sqrt{a} = \frac{(u_{n-2} \pm \sqrt{a})(v_{n-3} \pm \sqrt{a})}{u_{n-2} + v_{n-3}} \quad \text{and} \quad y_{n+1} \pm \sqrt{a} = \frac{(w_{n-2} \pm \sqrt{a})(s_{n-3} \pm \sqrt{a})}{w_{n-2} + s_{n-3}},$$

for $n \in \mathbb{N}_0$, implying

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{u_{n-2}+\sqrt{a}}{u_{n-2}-\sqrt{a}} \cdot \frac{v_{n-3}+\sqrt{a}}{v_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{w_{n-2}+\sqrt{a}}{w_{n-2}-\sqrt{a}} \cdot \frac{s_{n-3}+\sqrt{a}}{s_{n-3}-\sqrt{a}}, \tag{2.1}$$

for $n \in \mathbb{N}_0$.

System (2.1) written in a compact form, can be written as follows

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.2}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.3}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.4}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}} \cdot \frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.5}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.6}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.7}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.8}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.9}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.10}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.11}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.12}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.13}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.14}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.15}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.16}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.17}$$

for $n\in\mathbb{N}_0$.

Let

$$\zeta_n=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\quad\text{and}\quad \eta_n=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}},$$

then

$$x_n=\sqrt{a}\,\frac{\zeta_n+1}{\zeta_n-1}\quad\text{and}\quad y_n=\sqrt{a}\,\frac{\eta_n+1}{\eta_n-1}, \tag{2.18}$$

so (2.2)–(2.17) become

$$\zeta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \tag{2.19}$$

$$\zeta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\zeta_{n-3}, \tag{2.20}$$

$$\zeta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\eta_{n-3}, \tag{2.21}$$

$$\zeta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\eta_{n-3}, \tag{2.22}$$

$$\zeta_{n+1} = \eta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \tag{2.23}$$

$$\zeta_{n+1} = \eta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\zeta_{n-3}, \tag{2.24}$$

$$\zeta_{n+1} = \eta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\eta_{n-3}, \tag{2.25}$$

$$\zeta_{n+1} = \eta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\eta_{n-3}, \tag{2.26}$$

$$\zeta_{n+1} = \zeta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \tag{2.27}$$

$$\zeta_{n+1} = \zeta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\zeta_{n-3}, \tag{2.28}$$

$$\zeta_{n+1} = \zeta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\eta_{n-3}, \tag{2.29}$$

$$\zeta_{n+1} = \zeta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\eta_{n-3}, \tag{2.30}$$

$$\zeta_{n+1} = \eta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \tag{2.31}$$

$$\zeta_{n+1} = \eta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\zeta_{n-3}, \tag{2.32}$$

$$\zeta_{n+1} = \eta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\eta_{n-3}, \tag{2.33}$$

$$\zeta_{n+1} = \eta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\eta_{n-3}, \tag{2.34}$$

for $n \in \mathbb{N}_0$.

So, if systems (2.19)–(2.34) are practically solvable, then by using (2.18) the systems (2.2)–(2.17) will be also such. Hence, it should be first proved practical solvability of systems (2.19)–(2.34).

The following auxiliary result is used for several times in the rest of the article. The proof is omitted since it can be found, for example, in [31].

Lemma 2.1. *Assume $R_k(s) = s^k - b_{k-1}s^{k-1} - b_{k-2}s^{k-2} - \cdots - b_0$, $b_0 \neq 0$, is a real polynomial with simple roots s_i, $i = \overline{1,k}$, and a_n, $n \geq l-k$, is defined by*

$$a_n = b_{k-1}a_{n-1} + b_{k-2}a_{n-2} + \cdots + b_0 a_{n-k}, \quad n \geq l,$$

with $a_{j-k} = 0$, $j = \overline{l, l+k-2}$, $a_{l-1} = 1$, and $l \in \mathbb{Z}$. Then

$$a_n = \sum_{i=1}^{k} \frac{s_i^{n+k-l}}{R_k'(s_i)}, \quad n \geq l-k.$$

3 Main results

Here we show that each of the product-type systems of difference equations in (2.19)–(2.34) is practically solvable, and following the analysis of each of the systems, by using the relations in (2.18), we present closed-form formulas for general solutions to systems (2.2)–(2.17).

3.1 System (2.19)

The equations in (2.19) immediately imply the following relation

$$\zeta_n = \eta_n, \quad n \in \mathbb{N}. \tag{3.1}$$

The first equation in (2.19) can be written as follows

$$\zeta_n = \zeta_{n-3}\zeta_{n-4} = \zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}, \tag{3.2}$$

for $n \in \mathbb{N}$, where, of course, the exponents are defined as follows

$$c_1 = d_1 = 1, \quad e_1 = f_1 = 0. \tag{3.3}$$

An application of the first equality in (3.2) into the second one yields

$$\zeta_n = (\zeta_{n-6}\zeta_{n-7})^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1} = \zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{c_1+f_1}\zeta_{n-7}^{c_1} = \zeta_{n-4}^{c_2}\zeta_{n-5}^{d_2}\zeta_{n-6}^{e_2}\zeta_{n-7}^{f_2},$$

for $n \geq 4$, where $c_2 := d_1$, $d_2 := e_1$, $e_2 := c_1 + f_1$ and $f_2 := c_1$.

It is natural to assume that the following relations hold

$$\zeta_n = \zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k}, \tag{3.4}$$

$$c_k = d_{k-1}, \quad d_k = e_{k-1}, \quad e_k = c_{k-1} + f_{k-1}, \quad f_k = c_{k-1} \tag{3.5}$$

for a $k \geq 2$ and $n \geq k + 2$.

Relations (3.2), (3.4) and (3.5) yield

$$\begin{aligned}\zeta_n &= (\zeta_{n-k-5}\zeta_{n-k-6})^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k},\\ &= \zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{c_k+f_k}\zeta_{n-k-6}^{c_k}\\ &= \zeta_{n-k-3}^{c_{k+1}}\zeta_{n-k-4}^{d_{k+1}}\zeta_{n-k-5}^{e_{k+1}}\zeta_{n-k-6}^{f_{k+1}},\end{aligned}$$

where

$$c_{k+1} := d_k, \quad d_{k+1} := e_k, \quad e_{k+1} := c_k + f_k. \quad f_{k+1} := c_k.$$

The inductive argument proves that (3.4) and (3.5) really hold for $2 \leq k \leq n - 2$.

It is easy to see that from (3.3) and (3.5), we get

$$c_n = c_{n-3} + c_{n-4}, \tag{3.6}$$

for $n \geq 5$ (in fact, for $n \in \mathbb{Z}$), and

$$c_0 = c_{-1} = 0, \quad c_{-2} = 1, \quad c_{-3} = c_{-4} = c_{-5} = 0, \quad c_{-6} = 1, \quad c_{-7} = -1. \tag{3.7}$$

Choose $k = n - 2$ in relation (3.4). Then (3.5) and (3.6) yield

$$\zeta_n = \zeta_0^{c_{n-2}}\zeta_{-1}^{d_{n-2}}\zeta_{-2}^{e_{n-2}}\zeta_{-3}^{f_{n-2}} = \zeta_0^{c_{n-2}}\zeta_{-1}^{c_{n-1}}\zeta_{-2}^{c_n}\zeta_{-3}^{c_{n-3}}, \tag{3.8}$$

for $n \in \mathbb{N}$. A simple verification shows that (3.8) holds also for $n \geq -3$.

Thus, (3.1) and (3.8) imply

$$\eta_n = \zeta_0^{c_{n-2}}\zeta_{-1}^{c_{n-1}}\zeta_{-2}^{c_n}\zeta_{-3}^{c_{n-3}}, \quad n \in \mathbb{N}. \tag{3.9}$$

Let

$$P_4(\lambda) = \lambda^4 - \lambda - 1 = 0. \tag{3.10}$$

It is the characteristic polynomial associated with (3.6). Its roots λ_j, $j=\overline{1,4}$, are simple and can be found by radicals [3].

Lemma 2.1 shows that the solution to (3.6) satisfying the initial conditions $c_{-5}=c_{-4}=c_{-3}=0$, $c_{-2}=1$, is given by

$$c_n=\sum_{j=1}^{4}\frac{\lambda_j^{n+5}}{P_4'(\lambda_j)},\quad n\in\mathbb{Z}. \tag{3.11}$$

The following theorem follows from (2.18), (3.8) and (3.9).

Theorem 3.1. *If $a\neq 0$, then the general solution to (2.2) is*

$$x_n=\sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},\quad n\geq -3,$$

$$y_n=\sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},\quad n\in\mathbb{N},$$

where c_n is given by (3.11).

3.2 System (2.20)

Since the first equation in (2.20) is the same as in (2.19), formula (3.8) must hold. Further, we have $\eta_n=\eta_{n-3}\zeta_{n-4}$, $n\in\mathbb{N}$, or equivalently

$$\eta_{3n+i}=\eta_{3(n-1)+i}\zeta_{3(n-1)+i-1},\quad n\in\mathbb{N}, \tag{3.12}$$

for $i=-2,-1,0$, and $n\in\mathbb{N}$.

Relations (3.8) and (3.12), for $i=-2$, yield

$$\begin{aligned}\eta_{3n-2}&=\eta_{-2}\prod_{j=1}^{n}\zeta_{3j-6}\\&=\eta_{-2}\prod_{j=1}^{n}\zeta_0^{c_{3j-8}}\zeta_{-1}^{c_{3j-7}}\zeta_{-2}^{c_{3j-6}}\zeta_{-3}^{c_{3j-9}}\\&=\eta_{-2}\zeta_0^{\sum_{j=1}^{n}c_{3j-8}}\zeta_{-1}^{\sum_{j=1}^{n}c_{3j-7}}\zeta_{-2}^{\sum_{j=1}^{n}c_{3j-6}}\zeta_{-3}^{\sum_{j=1}^{n}c_{3j-9}},\end{aligned} \tag{3.13}$$

for $n\in\mathbb{N}_0$.

From (3.8) and (3.12), for $i=-1$, we obtain

$$\begin{aligned}\eta_{3n-1}&=\eta_{-1}\prod_{j=1}^{n}\zeta_{3j-5}\\&=\eta_{-1}\prod_{j=1}^{n}\zeta_0^{c_{3j-7}}\zeta_{-1}^{c_{3j-6}}\zeta_{-2}^{c_{3j-5}}\zeta_{-3}^{c_{3j-8}}\\&=\eta_{-1}\zeta_0^{\sum_{j=1}^{n}c_{3j-7}}\zeta_{-1}^{\sum_{j=1}^{n}c_{3j-6}}\zeta_{-2}^{\sum_{j=1}^{n}c_{3j-5}}\zeta_{-3}^{\sum_{j=1}^{n}c_{3j-8}},\end{aligned} \tag{3.14}$$

for $n\in\mathbb{N}_0$.

From (3.8) and (3.12), for $i=0$, it follows that

$$\begin{aligned}\eta_{3n} &= \eta_0\prod_{j=1}^{n}\zeta_{3j-4}\\ &= \eta_0\prod_{j=1}^{n}\zeta_0^{c_{3j-6}}\zeta_{-1}^{c_{3j-5}}\zeta_{-2}^{c_{3j-4}}\zeta_{-3}^{c_{3j-7}}\\ &= \eta_0\zeta_0^{\sum_{j=1}^{n}c_{3j-6}}\zeta_{-1}^{\sum_{j=1}^{n}c_{3j-5}}\zeta_{-2}^{\sum_{j=1}^{n}c_{3j-4}}\zeta_{-3}^{\sum_{j=1}^{n}c_{3j-7}},\end{aligned}\tag{3.15}$$

for $n\in\mathbb{N}_0$.

From (3.6) and (3.7), we have

$$\sum_{j=1}^{n}c_{3j-9}=\sum_{j=1}^{n}(c_{3j-5}-c_{3j-8})=c_{3n-5},\tag{3.16}$$

$$\sum_{j=1}^{n}c_{3j-8}=\sum_{j=1}^{n}(c_{3j-4}-c_{3j-7})=c_{3n-4},\tag{3.17}$$

$$\sum_{j=1}^{n}c_{3j-7}=\sum_{j=1}^{n}(c_{3j-3}-c_{3j-6})=c_{3n-3}\tag{3.18}$$

$$\sum_{j=1}^{n}c_{3j-6}=\sum_{j=1}^{n}(c_{3j-2}-c_{3j-5})=c_{3n-2}-1,\tag{3.19}$$

$$\sum_{j=1}^{n}c_{3j-5}=\sum_{j=1}^{n}(c_{3j-1}-c_{3j-4})=c_{3n-1},\tag{3.20}$$

$$\sum_{j=1}^{n}c_{3j-4}=\sum_{j=1}^{n}(c_{3j}-c_{3j-3})=c_{3n},\tag{3.21}$$

for $n\in\mathbb{N}_0$.

From (3.13)–(3.21), we have

$$\eta_{3n-2}=\eta_{-2}\zeta_0^{c_{3n-4}}\zeta_{-1}^{c_{3n-3}}\zeta_{-2}^{c_{3n-2}-1}\zeta_{-3}^{c_{3n-5}},\tag{3.22}$$

$$\eta_{3n-1}=\eta_{-1}\zeta_0^{c_{3n-3}}\zeta_{-1}^{c_{3n-2}-1}\zeta_{-2}^{c_{3n-1}}\zeta_{-3}^{c_{3n-4}},\tag{3.23}$$

$$\eta_{3n}=\eta_0\zeta_0^{c_{3n-2}-1}\zeta_{-1}^{c_{3n-1}}\zeta_{-2}^{c_{3n}}\zeta_{-3}^{c_{3n-3}},\tag{3.24}$$

for $n\in\mathbb{N}_0$.

The following theorem follows from (2.18), (3.8), (3.22), (3.23) and (3.24).

Theorem 3.2. *If $a\neq 0$, then the general solution to (2.3) is*

$$x_n=\sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},\quad n\geq -3,$$

$$y_{3n-2}=\sqrt{a}\,\frac{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-5}}+1}{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-5}}-1}$$

$$y_{3n-1}=\sqrt{a}\,\frac{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-4}}+1}{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-4}}-1}$$

$$y_{3n} = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-3}}-1},$$

for $n \in \mathbb{N}_0$, where c_n is given by (3.11).

3.3 System (2.21)

Since the first equation in (2.21) is the same as in (2.19), formula (3.8) must hold. Further, we have $\eta_n = \zeta_{n-3}\eta_{n-4}$, for $n \in \mathbb{N}$, or equivalently

$$\eta_{4n+i} = \zeta_{4n-3+i}\eta_{4(n-1)+i}, \tag{3.25}$$

for $n \in \mathbb{N}$, $i = -3, -2, -1, 0$.

From (3.8) and (3.25), we have

$$\begin{aligned}\eta_{4n-3} &= \eta_{-3}\prod_{j=1}^{n}\zeta_{4j-6}\\ &= \eta_{-3}\prod_{j=1}^{n}\zeta_0^{c_{4j-8}}\zeta_{-1}^{c_{4j-7}}\zeta_{-2}^{c_{4j-6}}\zeta_{-3}^{c_{4j-9}}\\ &= \eta_{-3}\zeta_0^{\sum_{j=1}^{n}c_{4j-8}}\zeta_{-1}^{\sum_{j=1}^{n}c_{4j-7}}\zeta_{-2}^{\sum_{j=1}^{n}c_{4j-6}}\zeta_{-3}^{\sum_{j=1}^{n}c_{4j-9}},\end{aligned} \tag{3.26}$$

for $n \in \mathbb{N}_0$,

$$\begin{aligned}\eta_{4n-2} &= \eta_{-2}\prod_{j=1}^{n}\zeta_{4j-5}\\ &= \eta_{-2}\prod_{j=1}^{n}\zeta_0^{c_{4j-7}}\zeta_{-1}^{c_{4j-6}}\zeta_{-2}^{c_{4j-5}}\zeta_{-3}^{c_{4j-8}}\\ &= \eta_{-2}\zeta_0^{\sum_{j=1}^{n}c_{4j-7}}\zeta_{-1}^{\sum_{j=1}^{n}c_{4j-6}}\zeta_{-2}^{\sum_{j=1}^{n}c_{4j-5}}\zeta_{-3}^{\sum_{j=1}^{n}c_{4j-8}},\end{aligned} \tag{3.27}$$

for $n \in \mathbb{N}_0$, and

$$\begin{aligned}\eta_{4n-1} &= \eta_{-1}\prod_{j=1}^{n}\zeta_{4j-4}\\ &= \eta_{-1}\prod_{j=1}^{n}\zeta_0^{c_{4j-6}}\zeta_{-1}^{c_{4j-5}}\zeta_{-2}^{c_{4j-4}}\zeta_{-3}^{c_{4j-7}}\\ &= \eta_{-1}\zeta_0^{\sum_{j=1}^{n}c_{4j-6}}\zeta_{-1}^{\sum_{j=1}^{n}c_{4j-5}}\zeta_{-2}^{\sum_{j=1}^{n}c_{4j-4}}\zeta_{-3}^{\sum_{j=1}^{n}c_{4j-7}},\end{aligned} \tag{3.28}$$

for $n \in \mathbb{N}_0$,

$$\begin{aligned}\eta_{4n} &= \eta_{0}\prod_{j=1}^{n}\zeta_{4j-3}\\ &= \eta_{0}\prod_{j=1}^{n}\zeta_0^{c_{4j-5}}\zeta_{-1}^{c_{4j-4}}\zeta_{-2}^{c_{4j-3}}\zeta_{-3}^{c_{4j-6}}\\ &= \eta_{0}\zeta_0^{\sum_{j=1}^{n}c_{4j-5}}\zeta_{-1}^{\sum_{j=1}^{n}c_{4j-4}}\zeta_{-2}^{\sum_{j=1}^{n}c_{4j-3}}\zeta_{-3}^{\sum_{j=1}^{n}c_{4j-6}},\end{aligned} \tag{3.29}$$

for $n \in \mathbb{N}_0$.

Relations (3.6) and (3.7) yield

$$\sum_{j=1}^{n} c_{4j-9} = \sum_{j=1}^{n}(c_{4j-6} - c_{4j-10}) = c_{4n-6} - 1, \tag{3.30}$$

$$\sum_{j=1}^{n} c_{4j-8} = \sum_{j=1}^{n}(c_{4j-5} - c_{4j-9}) = c_{4n-5}, \tag{3.31}$$

$$\sum_{j=1}^{n} c_{4j-7} = \sum_{j=1}^{n}(c_{4j-4} - c_{4j-8}) = c_{4n-4}, \tag{3.32}$$

$$\sum_{j=1}^{n} c_{4j-6} = \sum_{j=1}^{n}(c_{4j-3} - c_{4j-7}) = c_{4n-3}, \tag{3.33}$$

$$\sum_{j=1}^{n} c_{4j-5} = \sum_{j=1}^{n}(c_{4j-2} - c_{4j-6}) = c_{4n-2} - 1, \tag{3.34}$$

$$\sum_{j=1}^{n} c_{4j-4} = \sum_{j=1}^{n}(c_{4j-1} - c_{4j-5}) = c_{4n-1}, \tag{3.35}$$

$$\sum_{j=1}^{n} c_{4j-3} = \sum_{j=1}^{n}(c_{4j} - c_{4j-4}) = c_{4n}, \tag{3.36}$$

for $n \in \mathbb{N}$.

From (3.26)–(3.36), we have

$$\eta_{4n-3} = \eta_{-3}\zeta_0^{c_{4n-5}}\zeta_{-1}^{c_{4n-4}}\zeta_{-2}^{c_{4n-3}}\zeta_{-3}^{c_{4n-6}-1}, \tag{3.37}$$

$$\eta_{4n-2} = \eta_{-2}\zeta_0^{c_{4n-4}}\zeta_{-1}^{c_{4n-3}}\zeta_{-2}^{c_{4n-2}-1}\zeta_{-3}^{c_{4n-5}}, \tag{3.38}$$

$$\eta_{4n-1} = \eta_{-1}\zeta_0^{c_{4n-3}}\zeta_{-1}^{c_{4n-2}-1}\zeta_{-2}^{c_{4n-1}}\zeta_{-3}^{c_{4n-4}}, \tag{3.39}$$

$$\eta_{4n} = \eta_0\zeta_0^{c_{4n-2}-1}\zeta_{-1}^{c_{4n-1}}\zeta_{-2}^{c_{4n}}\zeta_{-3}^{c_{4n-3}}, \tag{3.40}$$

for $n \in \mathbb{N}_0$.

The following theorem follows from (2.18), (3.8), (3.37)–(3.40).

Theorem 3.3. *If $a \neq 0$, then the general solution to (2.4) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1}, \quad n \geq -3,$$

$$y_{4n-3} = \sqrt{a}\,\frac{\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-5}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-6}-1}+1}{\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-5}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-6}-1}-1},$$

$$y_{4n-2} = \sqrt{a}\,\frac{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-5}}+1}{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-5}}-1},$$

$$y_{4n-1} = \sqrt{a}\,\frac{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-4}}+1}{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-4}}-1},$$

$$y_{4n} = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-3}}-1},$$

for $n \in \mathbb{N}_0$, where sequence c_n is given by (3.11).

3.4 System (2.22)

Since the first equation in (2.22) is the same as in (2.19), formula (3.8) must hold, as well as the following one

$$\eta_n = \eta_0^{c_{n-2}}\eta_{-1}^{c_{n-1}}\eta_{-2}^{c_n}\eta_{-3}^{c_{n-3}}, \quad n \geq -3. \tag{3.41}$$

The following theorem follows from (2.18), (3.8) and (3.41).

Theorem 3.4. *If $a \neq 0$, then the general solution to (2.5) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},$$

for $n \geq -3$, where c_n is given by (3.11).

3.5 System (2.23)

The equations in (2.23) yield the relation

$$\zeta_n = \zeta_{n-4}\zeta_{n-6}\zeta_{n-7}, \quad n \geq 4. \tag{3.42}$$

We can write (3.42) as follows

$$\zeta_n = \zeta_{n-4}^{a_1}\zeta_{n-5}^{b_1}\zeta_{n-6}^{c_1}\zeta_{n-7}^{d_1}\zeta_{n-8}^{e_1}\zeta_{n-9}^{f_1}\zeta_{n-10}^{g_1}, \quad n \geq 4, \tag{3.43}$$

where, of course, the exponents are defined as follows

$$a_1 = 1, \quad b_1 = 0, \quad c_1 = d_1 = 1, \quad e_1 = f_1 = g_1 = 0. \tag{3.44}$$

From (3.42) and (3.43), we have

$$\begin{aligned}\zeta_n &= (\zeta_{n-8}\zeta_{n-10}\zeta_{n-11})^{a_1}\zeta_{n-5}^{b_1}\zeta_{n-6}^{c_1}\zeta_{n-7}^{d_1}\zeta_{n-8}^{e_1}\zeta_{n-9}^{f_1}\zeta_{n-10}^{g_1}\\ &= \zeta_{n-5}^{b_1}\zeta_{n-6}^{c_1}\zeta_{n-7}^{d_1}\zeta_{n-8}^{a_1+e_1}\zeta_{n-9}^{f_1}\zeta_{n-10}^{a_1+g_1}\zeta_{n-11}^{a_1}\\ &= \zeta_{n-5}^{a_2}\zeta_{n-6}^{b_2}\zeta_{n-7}^{c_2}\zeta_{n-8}^{d_2}\zeta_{n-9}^{e_2}\zeta_{n-10}^{f_2}\zeta_{n-11}^{g_2},\end{aligned}$$

for $n \geq 8$, where $a_2 := b_1$, $b_2 := c_1$, $c_2 := d_1$, $d_2 := a_1 + e_1$, $e_2 := f_1$, $f_2 := a_1 + g_1$ and $g_2 := a_1$.

It is natural to suppose that the following relations hold

$$\zeta_n = \zeta_{n-k-3}^{a_k}\zeta_{n-k-4}^{b_k}\zeta_{n-k-5}^{c_k}\zeta_{n-k-6}^{d_k}\zeta_{n-k-7}^{e_k}\zeta_{n-k-8}^{f_k}\zeta_{n-k-9}^{g_k}, \tag{3.45}$$

$$\begin{aligned} &a_k = b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = d_{k-1}, \quad d_k = a_{k-1} + e_{k-1}, \\ &e_k = f_{k-1}, \quad f_k = a_{k-1} + g_{k-1}, \quad g_k = a_{k-1}, \end{aligned} \tag{3.46}$$

for a $k \geq 2$ and $n \geq k+6$.

From (3.42) and (3.45), we have

$$\begin{aligned} \zeta_n &= \zeta_{n-k-3}^{a_k} \zeta_{n-k-4}^{b_k} \zeta_{n-k-5}^{c_k} \zeta_{n-k-6}^{d_k} \zeta_{n-k-7}^{e_k} \zeta_{n-k-8}^{f_k} \zeta_{n-k-9}^{g_k} \\ &= (\zeta_{n-k-7}\zeta_{n-k-9}\zeta_{n-k-10})^{a_k} \zeta_{n-k-4}^{b_k} \zeta_{n-k-5}^{c_k} \zeta_{n-k-6}^{d_k} \zeta_{n-k-7}^{e_k} \zeta_{n-k-8}^{f_k} \zeta_{n-k-9}^{g_k} \\ &= \zeta_{n-k-4}^{b_k} \zeta_{n-k-5}^{c_k} \zeta_{n-k-6}^{d_k} \zeta_{n-k-7}^{a_k+e_k} \zeta_{n-k-8}^{f_k} \zeta_{n-k-9}^{a_k+g_k} \zeta_{n-k-10}^{a_k} \\ &= \zeta_{n-k-4}^{a_{k+1}} \zeta_{n-k-5}^{b_{k+1}} \zeta_{n-k-6}^{c_{k+1}} \zeta_{n-k-7}^{d_{k+1}} \zeta_{n-k-8}^{e_{k+1}} \zeta_{n-k-9}^{f_{k+1}} \zeta_{n-k-10}^{g_{k+1}}, \end{aligned}$$

where

$$\begin{aligned} &a_{k+1} := b_k, \quad b_{k+1} := c_k, \quad c_{k+1} := d_k, \quad d_{k+1} := a_k + e_k, \\ &e_{k+1} := f_k, \quad f_{k+1} := a_k + g_k, \quad g_{k+1} := a_k, \end{aligned}$$

for a $k \geq 2$ and $n \geq k+7$. Thus, (3.45) and (3.46) are true for $2 \leq k \leq n-6$.

Relations (3.44) and (3.46) yield

$$a_n = a_{n-4} + a_{n-6} + a_{n-7}, \tag{3.47}$$

for $n \geq 8$ (in fact, for $n \in \mathbb{Z}$), and

$$a_0 = a_{-1} = a_{-2} = 0, \quad a_{-3} = 1, \quad a_{-j} = 0, \quad j = \overline{4,9}, \quad a_{-10} = 1, \quad a_{-11} = -1.$$

By choosing $k = n-6$ in (3.45), we get

$$\begin{aligned} \zeta_n &= \zeta_3^{a_{n-6}} \zeta_2^{b_{n-6}} \zeta_1^{c_{n-6}} \zeta_0^{d_{n-6}} \zeta_{-1}^{e_{n-6}} \zeta_{-2}^{f_{n-e}} \zeta_{-3}^{g_{n-6}} \\ &= (\eta_0\zeta_{-1})^{a_{n-6}} (\eta_{-1}\zeta_{-2})^{b_{n-6}} (\eta_{-2}\zeta_{-3})^{c_{n-6}} \zeta_0^{d_{n-6}} \zeta_{-1}^{e_{n-6}} \zeta_{-2}^{f_{n-e}} \zeta_{-3}^{g_{n-6}} \\ &= \zeta_0^{d_{n-6}} \zeta_{-1}^{a_{n-6}+e_{n-6}} \zeta_{-2}^{b_{n-6}+f_{n-6}} \zeta_{-3}^{c_{n-6}+g_{n-6}} \eta_0^{a_{n-6}} \eta_{-1}^{b_{n-6}} \eta_{-2}^{c_{n-6}} \\ &= \zeta_0^{a_{n-3}} \zeta_{-1}^{a_{n-2}} \zeta_{-2}^{a_{n-1}} \zeta_{-3}^{a_{n-4}+a_{n-7}} \eta_0^{a_{n-6}} \eta_{-1}^{a_{n-5}} \eta_{-2}^{a_{n-4}}, \end{aligned} \tag{3.48}$$

for $n \geq -3$.

Relations (2.23) and (3.48) yield

$$\begin{aligned} \eta_n &= \zeta_{n-3}\zeta_{n-4} \\ &= \zeta_0^{a_{n-6}+a_{n-7}} \zeta_{-1}^{a_{n-5}+a_{n-6}} \zeta_{-2}^{a_{n-4}+a_{n-5}} \zeta_{-3}^{a_{n-4}+a_{n-7}} \eta_0^{a_{n-9}+a_{n-10}} \eta_{-1}^{a_{n-8}+a_{n-9}} \eta_{-2}^{a_{n-7}+a_{n-8}}, \end{aligned} \tag{3.49}$$

for $n \in \mathbb{N}$. A direct check shows that (3.49) also holds for $n = 0$.

Let

$$P_7(\lambda) = \lambda^7 - \lambda^3 - \lambda - 1 = (\lambda^3 + 1)(\lambda^4 - \lambda - 1).$$

Clearly it is the characteristic polynomial of (3.47). Four roots of P_7 are those of (3.10), while $\lambda_{4+j} = e^{i\frac{\pi(2j+1)}{3}}$, $j = \overline{0,2}$. The roots are distinct. Lemma 2.1 shows that

$$a_n = \sum_{j=1}^{7} \frac{\lambda_j^{n+9}}{P_7'(\lambda_j)}, \quad n \in \mathbb{Z}, \tag{3.50}$$

is the solution to (3.47) satisfying the initial conditions $a_{-j} = 0$, $j = \overline{4,9} = 0$, $a_{-3} = 1$.

The following theorem follows from (2.18), (3.48) and (3.49).

Theorem 3.5. *If $a \neq 0$, then the general solution to (2.6) is*

$$x_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-6}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-6}}-1},$$

for $n \geq -3$, and

$$y_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-6}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-9}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-6}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-9}}-1},$$

for $n \in \mathbb{N}_0$, where a_n is given by (3.50) and $b_n = a_n + a_{n-1}$.

3.6 System (2.24)

From the equations in (2.24) we have $\zeta_n = \eta_n$, $n \in \mathbb{N}$. This together with (2.24), implies

$$\zeta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \quad n \geq 3.$$

If we use (3.8), we get

$$\begin{aligned}\zeta_n &= \zeta_4^{c_{n-6}}\zeta_3^{c_{n-5}}\zeta_2^{c_{n-4}}\zeta_1^{c_{n-7}}\\ &= (\eta_{-2}\zeta_0\zeta_{-3})^{c_{n-6}}(\eta_0\zeta_{-1})^{c_{n-5}}(\eta_{-1}\zeta_{-2})^{c_{n-4}}(\eta_{-2}\zeta_{-3})^{c_{n-7}}\\ &= \zeta_0^{c_{n-6}}\zeta_{-1}^{c_{n-5}}\zeta_{-2}^{c_{n-4}}\zeta_{-3}^{c_{n-3}}\eta_0^{c_{n-5}}\eta_{-1}^{c_{n-4}}\eta_{-2}^{c_{n-3}},\end{aligned} \tag{3.51}$$

for $n \in \mathbb{N}_0$, where c_n is given by (3.11).

Therefore

$$\eta_n = \zeta_0^{c_{n-6}}\zeta_{-1}^{c_{n-5}}\zeta_{-2}^{c_{n-4}}\zeta_{-3}^{c_{n-3}}\eta_0^{c_{n-5}}\eta_{-1}^{c_{n-4}}\eta_{-2}^{c_{n-3}}, \quad n \in \mathbb{N}. \tag{3.52}$$

The following theorem follows from (2.18), (3.51) and (3.52).

Theorem 3.6. *If $a \neq 0$, then the general solution to (2.7) is*

$$x_n = \sqrt{a}\,\frac{\prod_{j=0}^{3}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}+1}{\prod_{j=0}^{3}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}-1},$$

for $n \in \mathbb{N}_0$, and

$$y_n = \sqrt{a}\,\frac{\prod_{j=0}^{3}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}+1}{\prod_{j=0}^{3}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}-1},$$

for $n \in \mathbb{N}$, where c_n is given by (3.11).

3.7 System (2.25)

It is not difficult to see that in the case of the system the following relation holds

$$\zeta_n = \zeta_{n-4}^2\zeta_{n-6}\zeta_{n-8}^{-1}, \quad n \ge 5. \tag{3.53}$$

Let

$$\zeta_n^{(i)} = \zeta_{2n+i}, \quad n \ge -1,$$

for $i = -1, 0$, then we have

$$\zeta_n^{(i)} = (\zeta_{n-2}^{(i)})^2\zeta_{n-3}^{(i)}(\zeta_{n-4}^{(i)})^{-1}, \quad n \ge 3. \tag{3.54}$$

Let further

$$b_1 = 2, \quad c_1 = 1, \quad d_1 = -1, \quad e_1 = 0. \tag{3.55}$$

Then, we have

$$\begin{aligned}
\zeta_n^{(i)} &= (\zeta_{n-2}^{(i)})^{b_1}(\zeta_{n-3}^{(i)})^{c_1}(\zeta_{n-4}^{(i)})^{d_1}(\zeta_{n-5}^{(i)})^{e_1}\\
&= ((\zeta_{n-4}^{(i)})^2\zeta_{n-5}^{(i)}(\zeta_{n-6}^{(i)})^{-1})^{b_1}(\zeta_{n-3}^{(i)})^{c_1}(\zeta_{n-4}^{(i)})^{d_1}(\zeta_{n-5}^{(i)})^{e_1}\\
&= (\zeta_{n-3}^{(i)})^{c_1}(\zeta_{n-4}^{(i)})^{2b_1+d_1}(\zeta_{n-5}^{(i)})^{b_1+e_1}(\zeta_{n-6}^{(i)})^{-b_1}\\
&= (\zeta_{n-3}^{(i)})^{b_2}(\zeta_{n-4}^{(i)})^{c_2}(\zeta_{n-5}^{(i)})^{d_2}(\zeta_{n-6}^{(i)})^{e_2},
\end{aligned}$$

for $n \ge 5$, where $b_2 := c_1$, $c_2 := 2b_1 + d_1$, $d_2 := b_1 + e_1$ and $e_2 := -b_1$.

It is natural to assume that

$$\zeta_n^{(i)} = (\zeta_{n-k-1}^{(i)})^{b_k}(\zeta_{n-k-2}^{(i)})^{c_k}(\zeta_{n-k-3}^{(i)})^{d_k}(\zeta_{n-k-4}^{(i)})^{e_k}, \tag{3.56}$$

for a $k \ge 2$ and $n \ge k+3$, and

$$b_k = c_{k-1}, \quad c_k = 2b_{k-1} + d_{k-1}, \quad d_k = b_{k-1} + e_{k-1}, \quad e_k = -b_{k-1}. \tag{3.57}$$

From (3.54) and (3.56), we have

$$\begin{aligned}
\zeta_n^{(i)} &= (\zeta_{n-k-1}^{(i)})^{b_k}(\zeta_{n-k-2}^{(i)})^{c_k}(\zeta_{n-k-3}^{(i)})^{d_k}(\zeta_{n-k-4}^{(i)})^{e_k}\\
&= ((\zeta_{n-k-3}^{(i)})^2\zeta_{n-k-4}^{(i)}(\zeta_{n-k-5}^{(i)})^{-1})^{b_k}(\zeta_{n-k-2}^{(i)})^{c_k}(\zeta_{n-k-3}^{(i)})^{d_k}(\zeta_{n-k-4}^{(i)})^{e_k}\\
&= (\zeta_{n-k-2}^{(i)})^{c_k}(\zeta_{n-k-3}^{(i)})^{2b_k+d_k}(\zeta_{n-k-4}^{(i)})^{b_k+e_k}(\zeta_{n-k-5}^{(i)})^{-b_k}\\
&= (\zeta_{n-k-2}^{(i)})^{b_{k+1}}(\zeta_{n-k-3}^{(i)})^{c_{k+1}}(\zeta_{n-k-4}^{(i)})^{d_{k+1}}(\zeta_{n-k-5}^{(i)})^{e_{k+1}},
\end{aligned}$$

for $n \ge k+4$, where

$$b_{k+1} := c_k, \quad c_{k+1} := 2b_k + d_k, \quad d_{k+1} := b_k + e_k, \quad e_{k+1} := -b_k.$$

So, the method of induction shows that (3.56) and (3.57) hold for $2 \le k \le n-3$.

From (3.55) and (3.57) we get

$$b_n = 2b_{n-2} + b_{n-3} - b_{n-4}, \tag{3.58}$$

for $n \ge 5$ (in fact, for $n \in \mathbb{Z}$), and

$$b_0 = 0, \quad b_{-1} = 1, \quad b_{-j} = 0, \ j = \overline{2,4}, \quad b_{-5} = -1.$$

If we choose $k = n-3$ in (3.56), we get

$$\begin{aligned}\zeta_n^{(i)} &= (\zeta_2^{(i)})^{b_{n-3}}(\zeta_1^{(i)})^{c_{n-3}}(\zeta_0^{(i)})^{d_{n-3}}(\zeta_{-1}^{(i)})^{e_{n-3}}\\ &= (\zeta_2^{(i)})^{b_{n-3}}(\zeta_1^{(i)})^{b_{n-2}}(\zeta_0^{(i)})^{b_{n-1}-2b_{n-3}}(\zeta_{-1}^{(i)})^{-b_{n-4}},\end{aligned}$$

for $n \geq -1$, and $i = -1, 0$.

If $i = 0$, we obtain

$$\begin{aligned}\zeta_{2n} &= \zeta_4^{b_{n-3}}\zeta_2^{b_{n-2}}\zeta_0^{b_{n-1}-2b_{n-3}}\zeta_{-2}^{-b_{n-4}}\\ &= (\zeta_0\zeta_{-2}\eta_{-3})^{b_{n-3}}(\eta_{-1}\zeta_{-2})^{b_{n-2}}\zeta_0^{b_{n-1}-2b_{n-3}}\zeta_{-2}^{-b_{n-4}}\\ &= \zeta_0^{b_{n-1}-b_{n-3}}\zeta_{-2}^{b_n-b_{n-2}}\eta_{-1}^{b_{n-2}}\eta_{-3}^{b_{n-3}},\end{aligned} \tag{3.59}$$

whereas for $i = -1$, we get

$$\begin{aligned}\zeta_{2n-1} &= \zeta_3^{b_{n-3}}\zeta_1^{b_{n-2}}\zeta_{-1}^{b_{n-1}-2b_{n-3}}\zeta_{-3}^{-b_{n-4}}\\ &= (\eta_0\zeta_{-1})^{b_{n-3}}(\eta_{-2}\zeta_{-3})^{b_{n-2}}\zeta_{-1}^{b_{n-1}-2b_{n-3}}\zeta_{-3}^{-b_{n-4}}\\ &= \zeta_{-1}^{b_{n-1}-b_{n-3}}\zeta_{-3}^{b_{n-2}-b_{n-4}}\eta_0^{b_{n-3}}\eta_{-2}^{b_{n-2}},\end{aligned} \tag{3.60}$$

for $n \geq -1$.

Since (2.25) is symmetric, we get

$$\eta_{2n} = \eta_0^{b_{n-1}-b_{n-3}}\eta_{-2}^{b_n-b_{n-2}}\zeta_{-1}^{b_{n-2}}\zeta_{-3}^{b_{n-3}}, \tag{3.61}$$

$$\eta_{2n-1} = \eta_{-1}^{b_{n-1}-b_{n-3}}\eta_{-3}^{b_{n-2}-b_{n-4}}\zeta_0^{b_{n-3}}\zeta_{-2}^{b_{n-2}}, \tag{3.62}$$

for $n \geq -1$.

Let

$$\widehat{P}_4(\lambda) = \lambda^4 - 2\lambda^2 - \lambda + 1,$$

It is the characteristic polynomial of (3.58). Its zeros $\widehat{\lambda}_j$, $j = \overline{1,4}$, are distinct.

Therefore

$$b_n = \sum_{j=1}^{4} \frac{\widehat{\lambda}_j^{n+4}}{\widehat{P}_4'(\widehat{\lambda}_j)}, \quad n \in \mathbb{Z}, \tag{3.63}$$

is the solution to (3.58) satisfying the initial conditions $b_{-j} = 0$, $k = \overline{2,4}$, $b_{-1} = 1$.

The following theorem follows from (2.18), (3.59)–(3.62).

Theorem 3.7. *If $a \neq 0$, then the general solution to (2.8) is*

$$x_{2n} = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-3}}-1},$$

$$x_{2n-1} = \sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}-1}},$$

$$y_{2n} = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-3}}-1},$$

$$y_{2n-1}=\sqrt{a}\,\frac{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}}+1}{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}-1}},$$

for $n\ge -1$, where the sequence b_n is given by (3.63).

3.8 System (2.26)

By interchanging letters ζ and η, system (2.26) is obtained from (2.21). Hence

$$\zeta_{4n-3}=\zeta_{-3}\eta_0^{c_{4n-5}}\eta_{-1}^{c_{4n-4}}\eta_{-2}^{c_{4n-3}}\eta_{-3}^{c_{4n-6}-1},\tag{3.64}$$

$$\zeta_{4n-2}=\zeta_{-2}\eta_0^{c_{4n-4}}\eta_{-1}^{c_{4n-3}}\eta_{-2}^{c_{4n-2}-1}\eta_{-3}^{c_{4n-5}},\tag{3.65}$$

$$\zeta_{4n-1}=\zeta_{-1}\eta_0^{c_{4n-3}}\eta_{-1}^{c_{4n-2}-1}\eta_{-2}^{c_{4n-1}}\eta_{-3}^{c_{4n-4}},\tag{3.66}$$

$$\zeta_{4n}=\zeta_0\eta_0^{c_{4n-2}-1}\eta_{-1}^{c_{4n-1}}\eta_{-2}^{c_{4n}}\eta_{-3}^{c_{4n-3}},\tag{3.67}$$

for $n\in\mathbb{N}_0$,

$$\eta_n=\eta_0^{c_{n-2}}\eta_{-1}^{c_{n-1}}\eta_{-2}^{c_n}\eta_{-3}^{c_{n-3}},\quad n\ge -3.\tag{3.68}$$

The following theorem follows from (2.18), (3.64)–(3.68).

Theorem 3.8. *If $a\ne 0$, then the general solution to* (2.9) *is*

$$y_n=\sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},\quad n\ge -3,$$

$$x_{4n-3}=\sqrt{a}\,\frac{\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-5}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-6}-1}+1}{\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-5}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-6}-1}-1},$$

$$x_{4n-2}=\sqrt{a}\,\frac{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-5}}+1}{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-5}}-1},$$

$$x_{4n-1}=\sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-4}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-4}}-1},$$

$$x_{4n}=\sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-3}}-1},$$

for $n\in\mathbb{N}_0$, where sequence c_n is given by (3.11).

3.9 System (2.27)

It is easy to see that the following relation holds

$$\zeta_n=\zeta_{n-3}\zeta_{n-7}\zeta_{n-8},\quad n\ge 5.\tag{3.69}$$

We can write (3.69) as follows

$$\zeta_n = \zeta_{n-3}^{a_1}\zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{h_1}, \quad n \geq 5, \tag{3.70}$$

where, of course, the exponents are defined as follows

$$a_1 = 1, \quad b_1 = c_1 = d_1 = 0, \quad e_1 = f_1 = 1, \quad g_1 = h_1 = 0. \tag{3.71}$$

Employing (3.69) in (3.70), we have

$$\begin{aligned}\zeta_n &= (\zeta_{n-6}\zeta_{n-10}\zeta_{n-11})^{a_1}\zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{h_1}\\ &= \zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{a_1+d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{a_1+h_1}\zeta_{n-11}^{a_1}\\ &= \zeta_{n-4}^{a_2}\zeta_{n-5}^{b_2}\zeta_{n-6}^{c_2}\zeta_{n-7}^{d_2}\zeta_{n-8}^{e_2}\zeta_{n-9}^{f_2}\zeta_{n-10}^{g_2}\zeta_{n-11}^{h_2},\end{aligned}$$

for $n \geq 8$, where $a_2 := b_1$, $b_2 := c_1$, $c_2 := a_1 + d_1$, $d_2 := e_1$, $e_2 := f_1$, $f_2 := g_1$, $g_2 := a_1 + h_1$ and $h_2 := a_1$.

As in the case of equation (3.53) is obtained

$$\zeta_n = \zeta_{n-k-2}^{a_k}\zeta_{n-k-3}^{b_k}\zeta_{n-k-4}^{c_k}\zeta_{n-k-5}^{d_k}\zeta_{n-k-6}^{e_k}\zeta_{n-k-7}^{f_k}\zeta_{n-k-8}^{g_k}\zeta_{n-k-9}^{h_k}, \tag{3.72}$$

for a $k \geq 2$ and $n \geq k+6$, and

$$\begin{aligned}&a_k = b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = a_{k-1} + d_{k-1}, \quad d_k = e_{k-1}\\ &e_k = f_{k-1}, \quad f_k = g_{k-1}, \quad g_k = a_{k-1} + h_{k-1}, \quad h_k = a_{k-1}.\end{aligned} \tag{3.73}$$

Relations (3.71) and (3.73) imply

$$a_n = a_{n-3} + a_{n-7} + a_{n-8}, \tag{3.74}$$

for $n \geq 9$ (in fact, for $n \in \mathbb{Z}$), and

$$a_0 = a_{-1} = 0, \quad a_{-2} = 1, \quad a_{-j} = 0,\ j = \overline{3,9}, \quad a_{-10} = 1, \quad a_{-11} = -1.$$

By choosing $k = n-6$ in (3.72), it follows that

$$\begin{aligned}\zeta_n &= \zeta_4^{a_{n-6}}\zeta_3^{b_{n-6}}\zeta_2^{c_{n-6}}\zeta_1^{d_{n-6}}\zeta_0^{e_{n-6}}\zeta_{-1}^{f_{n-6}}\zeta_{-2}^{g_{n-6}}\zeta_{-3}^{h_{n-6}}\\ &= (\zeta_{-2}\eta_0\eta_{-3})^{a_{n-6}}(\zeta_0\eta_{-1})^{b_{n-6}}(\zeta_{-1}\eta_{-2})^{c_{n-6}}(\zeta_{-2}\eta_{-3})^{d_{n-6}}\zeta_0^{e_{n-6}}\zeta_{-1}^{f_{n-6}}\zeta_{-2}^{g_{n-6}}\zeta_{-3}^{h_{n-6}}\\ &= \zeta_0^{b_{n-6}+e_{n-6}}\zeta_{-1}^{c_{n-6}+f_{n-6}}\zeta_{-2}^{a_{n-6}+d_{n-6}+g_{n-6}}\zeta_{-3}^{h_{n-6}}\eta_0^{a_{n-6}}\eta_{-1}^{b_{n-6}}\eta_{-2}^{c_{n-6}}\eta_{-3}^{a_{n-6}+d_{n-6}}\\ &= \zeta_0^{a_{n-2}}\zeta_{-1}^{a_{n-1}}\zeta_{-2}^{a_n}\zeta_{-3}^{a_{n-7}}\eta_0^{a_{n-6}}\eta_{-1}^{a_{n-5}}\eta_{-2}^{a_{n-4}}\eta_{-3}^{a_{n-3}},\end{aligned} \tag{3.75}$$

for $n \geq -3$.

Relations (3.75) and (2.27) yield

$$\begin{aligned}\eta_n = \zeta_{n-3}\zeta_{n-4} = {}&\zeta_0^{a_{n-5}+a_{n-6}}\zeta_{-1}^{a_{n-4}+a_{n-5}}\zeta_{-2}^{a_{n-3}+a_{n-4}}\zeta_{-3}^{a_{n-10}+a_{n-11}}\\ &\times\eta_0^{a_{n-9}+a_{n-10}}\eta_{-1}^{a_{n-8}+a_{n-9}}\eta_{-2}^{a_{n-7}+a_{n-8}}\eta_{-3}^{a_{n-6}+a_{n-7}},\end{aligned} \tag{3.76}$$

for $n \geq -3$.

Let

$$\widetilde{P}_8(t) = t^8 - t^5 - t - 1 = (t^4 - t - 1)(t^4 + 1).$$

It is the characteristic polynomial associated to (3.74). Its roots are those of (3.10) and $t_{j+4} = e^{i\frac{\pi(2j+1)}{4}}$, $j = \overline{0,3}$. It is not difficult to see that they are distinct.

Lemma 2.1 shows that

$$a_n = \sum_{j=1}^{8} \frac{t_j^{n+9}}{\widetilde{P}_8'(t_j)}, \quad n \in \mathbb{Z}, \tag{3.77}$$

is the solution to (3.74) satisfying the initial conditions $a_{-j} = 0$, $j = \overline{3,9}$, $a_{-2} = 1$.

The following theorem follows from (2.18), (3.75) and (3.76).

Theorem 3.9. *If $a \neq 0$, then the general solution to (2.10) is*

$$x_n = \sqrt{a}\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-2}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-7}}\prod_{l=0}^{3}\left(\frac{y_{-l}+\sqrt{a}}{y_{-l}-\sqrt{a}}\right)^{a_{n+l-6}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-2}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-7}}\prod_{l=0}^{3}\left(\frac{y_{-l}+\sqrt{a}}{y_{-l}-\sqrt{a}}\right)^{a_{n+l-6}}-1},$$

$$y_n = \sqrt{a}\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-5}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-10}}\prod_{l=0}^{3}\left(\frac{y_{-l}+\sqrt{a}}{y_{-l}-\sqrt{a}}\right)^{b_{n+l-9}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-5}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-10}}\prod_{l=0}^{3}\left(\frac{y_{-l}+\sqrt{a}}{y_{-l}-\sqrt{a}}\right)^{b_{n+l-9}}-1},$$

for $n \geq -3$, where the sequence a_n is given by (3.77) and $b_n = a_n + a_{n-1}$.

3.10 System (2.28)

It is easy to see that the following relation holds

$$\zeta_n = \zeta_{n-3}^2\zeta_{n-6}^{-1}\zeta_{n-8}, \quad n \geq 5, \tag{3.78}$$

which can be written as

$$\zeta_n = \zeta_{n-3}^{a_1}\zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{h_1}, \tag{3.79}$$

for $n \geq 5$, where, of course, the exponents are given by

$$a_1 = 2, \quad b_1 = c_1 = 0, \quad d_1 = -1, \quad e_1 = 0, \quad f_1 = 1, \quad g_1 = h_1 = 0. \tag{3.80}$$

Relations (3.78) and (3.79) yield

$$\begin{aligned}\zeta_n &= \zeta_{n-3}^{a_1}\zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{h_1}\\ &= (\zeta_{n-6}^2\zeta_{n-9}^{-1}\zeta_{n-11})^{a_1}\zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{h_1}\\ &= \zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{2a_1+d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{-a_1+g_1}\zeta_{n-10}^{h_1}\zeta_{n-11}^{a_1}\\ &= \zeta_{n-4}^{a_2}\zeta_{n-5}^{b_2}\zeta_{n-6}^{c_2}\zeta_{n-7}^{d_2}\zeta_{n-8}^{e_2}\zeta_{n-9}^{f_2}\zeta_{n-10}^{g_2}\zeta_{n-11}^{h_2},\end{aligned}$$

for $n \geq 8$, where $a_2 := b_1$, $b_2 := c_1$, $c_2 := 2a_1 + d_1$, $d_2 := e_1$, $e_2 := f_1$, $f_2 := -a_1 + g_1$, $g_2 := h_1$ and $h_2 := a_1$.

As in (3.53) we obtain

$$\zeta_n = \zeta_{n-k-2}^{a_k}\zeta_{n-k-3}^{b_k}\zeta_{n-k-4}^{c_k}\zeta_{n-k-5}^{d_k}\zeta_{n-k-6}^{e_k}\zeta_{n-k-7}^{f_k}\zeta_{n-k-8}^{g_k}\zeta_{n-k-9}^{h_k}, \tag{3.81}$$

and

$$\begin{aligned} &a_k = b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = 2a_{k-1} + d_{k-1}, \quad d_k = e_{k-1}, \\ &e_k = f_{k-1}, \quad f_k = -a_{k-1} + g_{k-1}, \quad g_k = h_{k-1}, \quad h_k = a_{k-1}, \end{aligned} \tag{3.82}$$

for $2 \le k \le n-6$.

Relations (3.80) and (3.82) yield

$$a_n = 2a_{n-3} - a_{n-6} + a_{n-8}, \tag{3.83}$$

for $n \ge 9$ (in fact, for $n \in \mathbb{Z}$), and

$$a_0 = a_{-1} = 0, \quad a_{-2} = 1, \quad a_{-j} = 0,\ j = \overline{3,9}, \quad a_{-10} = 1, \quad a_{-11} = 0.$$

By choosing $k = n-6$ in (3.81), we obtain

$$\begin{aligned} \zeta_n &= \zeta_4^{a_{n-6}} \zeta_3^{b_{n-6}} \zeta_2^{c_{n-6}} \zeta_1^{d_{n-6}} \zeta_0^{e_{n-6}} \zeta_{-1}^{f_{n-6}} \zeta_{-2}^{g_{n-6}} \zeta_{-3}^{h_{n-6}} \\ &= (\zeta_{-2}\eta_0\eta_{-3})^{a_{n-6}} (\zeta_0\eta_{-1})^{b_{n-6}} (\zeta_{-1}\eta_{-2})^{c_{n-6}} (\zeta_{-2}\eta_{-3})^{d_{n-6}} \zeta_0^{e_{n-6}} \zeta_{-1}^{f_{n-6}} \zeta_{-2}^{g_{n-6}} \zeta_{-3}^{h_{n-6}} \\ &= \zeta_0^{b_{n-6}+e_{n-6}} \zeta_{-1}^{c_{n-6}+f_{n-6}} \zeta_{-2}^{a_{n-6}+d_{n-6}+g_{n-6}} \zeta_{-3}^{h_{n-6}} \eta_0^{a_{n-6}} \eta_{-1}^{b_{n-6}} \eta_{-2}^{c_{n-6}} \eta_{-3}^{a_{n-6}+d_{n-6}} \\ &= \zeta_0^{a_{n-2}-a_{n-5}} \zeta_{-1}^{a_{n-1}-a_{n-4}} \zeta_{-2}^{a_n-a_{n-3}} \zeta_{-3}^{a_{n-7}} \eta_0^{a_{n-6}} \eta_{-1}^{a_{n-5}} \eta_{-2}^{a_{n-4}} \eta_{-3}^{a_{n-3}-a_{n-6}}, \end{aligned} \tag{3.84}$$

for $n \ge -3$. System (2.28) is symmetric implying that

$$\eta_n = \eta_0^{a_{n-2}-a_{n-5}} \eta_{-1}^{a_{n-1}-a_{n-4}} \eta_{-2}^{a_n-a_{n-3}} \eta_{-3}^{a_{n-7}} \zeta_0^{a_{n-6}} \zeta_{-1}^{a_{n-5}} \zeta_{-2}^{a_{n-4}} \zeta_{-3}^{a_{n-3}-a_{n-6}}, \tag{3.85}$$

for $n \ge -3$.

Let

$$\widehat{P}_8(t) = t^8 - 2t^5 + t^2 - 1 = (t^4 - t - 1)(t^4 - t + 1).$$

It is the characteristic polynomial of (3.83). Let $\widetilde{t}_j$, $j = \overline{1,8}$, be the roots of $\widehat{P}_8$. They are simple. So, the solution to (3.83) such that $a_{-j} = 0$, $j = \overline{3,9}$, and $a_{-2} = 1$, is

$$a_n = \sum_{j=1}^{8} \frac{\widetilde{t}_j^{\,n+9}}{\widehat{P}_8'(\widetilde{t}_j)}, \quad n \in \mathbb{Z}. \tag{3.86}$$

The following theorem follows from (2.18), (3.84) and (3.85).

Theorem 3.10. *If $a \neq 0$, then the general solution to (2.11) is*

$$x_n = \sqrt{a}\, \frac{\prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-2}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-7}} \prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-6}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-3}} + 1}{\prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-2}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-7}} \prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-6}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-3}} - 1},$$

$$y_n = \sqrt{a}\, \frac{\prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-2}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-7}} \prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-6}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-3}} + 1}{\prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-2}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-7}} \prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-6}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-3}} - 1},$$

for $n \ge -3$, where the sequence a_n is given by (3.86) and $b_n = a_n - a_{n-3}$.

3.11 System (2.29)

We have $\zeta_n = \eta_n$, $n \in \mathbb{N}$, and consequently

$$\zeta_n = \zeta_{n-3}\zeta_{n-4},$$

for $n \geq 5$.

From (3.8), we obtain

$$\begin{aligned}\zeta_n &= \zeta_4^{c_{n-6}}\zeta_3^{c_{n-5}}\zeta_2^{c_{n-4}}\zeta_1^{c_{n-7}}\\ &= (\zeta_{-2}\eta_0\eta_{-3})^{c_{n-6}}(\zeta_0\eta_{-1})^{c_{n-5}}(\zeta_{-1}\eta_{-2})^{c_{n-4}}(\zeta_{-2}\eta_{-3})^{c_{n-7}}\\ &= \zeta_0^{c_{n-5}}\zeta_{-1}^{c_{n-4}}\zeta_{-2}^{c_{n-3}}\eta_0^{c_{n-6}}\eta_{-1}^{c_{n-5}}\eta_{-2}^{c_{n-4}}\eta_{-3}^{c_{n-3}},\end{aligned} \tag{3.87}$$

for $n \in \mathbb{N}$, where c_n is given by (3.11). Thus

$$\eta_n = \zeta_0^{c_{n-5}}\zeta_{-1}^{c_{n-4}}\zeta_{-2}^{c_{n-3}}\eta_0^{c_{n-6}}\eta_{-1}^{c_{n-5}}\eta_{-2}^{c_{n-4}}\eta_{-3}^{c_{n-3}}, \tag{3.88}$$

for $n \in \mathbb{N}_0$.

The following theorem follows from (2.18), (3.87) and (3.88).

Theorem 3.11. *If $a \neq 0$, then the general solution to (2.12) is*

$$x_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}\prod_{j=0}^{3}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}\prod_{j=0}^{3}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}-1},$$

for $n \in \mathbb{N}$, and

$$y_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}\prod_{j=0}^{3}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}\prod_{j=0}^{3}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}-1},$$

for $n \in \mathbb{N}_0$, where c_n is given by (3.11).

3.12 System (2.30)

By interchanging letters ζ and η, system (2.30) is got from (2.20). Hence

$$\zeta_{3n-2} = \zeta_{-2}\eta_0^{c_{3n-4}}\eta_{-1}^{c_{3n-3}}\eta_{-2}^{c_{3n-2}-1}\eta_{-3}^{c_{3n-5}}, \tag{3.89}$$

$$\zeta_{3n-1} = \zeta_{-1}\eta_0^{c_{3n-3}}\eta_{-1}^{c_{3n-2}-1}\eta_{-2}^{c_{3n-1}}\eta_{-3}^{c_{3n-4}}, \tag{3.90}$$

$$\zeta_{3n} = \zeta_0\eta_0^{c_{3n-2}-1}\eta_{-1}^{c_{3n-1}}\eta_{-2}^{c_{3n}}\eta_{-3}^{c_{3n-3}}, \tag{3.91}$$

for $n \in \mathbb{N}_0$, and

$$\eta_n = \eta_0^{c_{n-2}}\eta_{-1}^{c_{n-1}}\eta_{-2}^{c_n}\eta_{-3}^{c_{n-3}}, \tag{3.92}$$

for $n \geq -3$.

The following theorem follows (2.18), (3.89)–(3.92).

Theorem 3.12. *If $a \neq 0$, then the general solution to (2.13) is*

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1}, \quad n \geq -3,$$

$$x_{3n-2} = \sqrt{a}\,\frac{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-5}}+1}{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-5}}-1}$$

$$x_{3n-1} = \sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-4}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-4}}-1}$$

$$x_{3n} = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-3}}-1},$$

for $n \in \mathbb{N}$, where c_n is given by (3.11).

3.13 System (2.31)

It is easy to see that the following relation holds

$$\zeta_n = \zeta_{n-6}\zeta_{n-7}^2\zeta_{n-8}, \quad n \geq 5, \tag{3.93}$$

which can be written as follows

$$\zeta_n = \zeta_{n-6}^{a_1}\zeta_{n-7}^{b_1}\zeta_{n-8}^{c_1}\zeta_{n-9}^{d_1}\zeta_{n-10}^{e_1}\zeta_{n-11}^{f_1}\zeta_{n-12}^{g_1}\zeta_{n-13}^{h_1}, \tag{3.94}$$

for $n \geq 5$, where, of course, the exponents are given by

$$a_1 = 1, \quad b_1 = 2, \quad c_1 = 1, \quad d_1 = e_1 = f_1 = g_1 = h_1 = 0. \tag{3.95}$$

Relations (3.93) in (3.94) yield

$$\begin{aligned}
\zeta_n &= \zeta_{n-6}^{a_1}\zeta_{n-7}^{b_1}\zeta_{n-8}^{c_1}\zeta_{n-9}^{d_1}\zeta_{n-10}^{e_1}\zeta_{n-11}^{f_1}\zeta_{n-12}^{g_1}\zeta_{n-13}^{h_1}\\
&= (\zeta_{n-12}\zeta_{n-13}^2\zeta_{n-14})^{a_1}\zeta_{n-7}^{b_1}\zeta_{n-8}^{c_1}\zeta_{n-9}^{d_1}\zeta_{n-10}^{e_1}\zeta_{n-11}^{f_1}\zeta_{n-12}^{g_1}\zeta_{n-13}^{h_1}\\
&= \zeta_{n-7}^{b_1}\zeta_{n-8}^{c_1}\zeta_{n-9}^{d_1}\zeta_{n-10}^{e_1}\zeta_{n-11}^{f_1}\zeta_{n-12}^{a_1+g_1}\zeta_{n-13}^{2a_1+h_1}\zeta_{n-14}^{a_1}\\
&= \zeta_{n-7}^{a_2}\zeta_{n-8}^{b_2}\zeta_{n-9}^{c_2}\zeta_{n-10}^{d_2}\zeta_{n-11}^{e_2}\zeta_{n-12}^{f_2}\zeta_{n-13}^{g_2}\zeta_{n-14}^{h_2},
\end{aligned}$$

for $n \geq 11$, where $a_2 := b_1$, $b_2 := c_1$, $c_2 := d_1$, $d_2 := e_1$, $e_2 := f_1$, $f_2 := a_1 + g_1$, $g_2 := 2a_1 + h_1$ and $h_2 := a_1$.

As in (3.2) are obtained the folowing relations

$$\zeta_n = \zeta_{n-k-5}^{a_k}\zeta_{n-k-6}^{b_k}\zeta_{n-k-7}^{c_k}\zeta_{n-k-8}^{d_k}\zeta_{n-k-9}^{e_k}\zeta_{n-k-10}^{f_k}\zeta_{n-k-11}^{g_k}\zeta_{n-k-12}^{h_k}, \tag{3.96}$$

$$\begin{aligned}
&a_k = b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = d_{k-1}, \quad d_k = e_{k-1},\\
&e_k = f_{k-1}, \quad f_k = a_{k-1} + g_{k-1}, \quad g_k = 2a_{k-1} + h_{k-1}, \quad h_k = a_{k-1}.
\end{aligned} \tag{3.97}$$

for a $k \geq 2$ and $n \geq k+9$.

Relations (3.95) and (3.97) yield

$$a_k = a_{k-6} + 2a_{k-7} + a_{k-8}, \tag{3.98}$$

and

$$a_{-l} = 0, \quad l = \overline{0,4}, \quad a_{-5} = 1, \quad a_{-j} = 0, \quad j = \overline{6,12}.$$

By choosing $k = n-9$ in (3.96), we get

$$\begin{aligned}
\zeta_n &= \zeta_4^{a_{n-9}} \zeta_3^{b_{n-9}} \zeta_2^{c_{n-9}} \zeta_1^{d_{n-9}} \zeta_0^{e_{n-9}} \zeta_{-1}^{f_{n-9}} \zeta_{-2}^{g_{n-9}} \zeta_{-3}^{h_{n-9}} \\
&= (\zeta_{-2}\zeta_{-3}\eta_0)^{a_{n-9}} (\eta_0\eta_{-1})^{b_{n-9}} (\eta_{-1}\eta_{-2})^{c_{n-9}} (\eta_{-2}\eta_{-3})^{d_{n-9}} \zeta_0^{e_{n-9}} \zeta_{-1}^{f_{n-9}} \zeta_{-2}^{g_{n-9}} \zeta_{-3}^{h_{n-9}} \\
&= \zeta_0^{e_{n-9}} \zeta_{-1}^{f_{n-9}} \zeta_{-2}^{a_{n-9}+g_{n-9}} \zeta_{-3}^{a_{n-9}+h_{n-9}} \eta_0^{a_{n-9}+b_{n-9}} \eta_{-1}^{b_{n-9}+c_{n-9}} \eta_{-2}^{c_{n-9}+d_{n-9}} \eta_{-3}^{d_{n-9}} \\
&= \zeta_0^{a_{n-5}} \zeta_{-1}^{a_{n-4}} \zeta_{-2}^{a_{n-3}} \zeta_{-3}^{a_{n-9}+a_{n-10}} \eta_0^{a_{n-8}+a_{n-9}} \eta_{-1}^{a_{n-7}+a_{n-8}} \eta_{-2}^{a_{n-6}+a_{n-7}} \eta_{-3}^{a_{n-6}},
\end{aligned} \tag{3.99}$$

for $n \ge -3$.

System (2.31) is symmetric implying that

$$\eta_n = \eta_0^{a_{n-5}} \eta_{-1}^{a_{n-4}} \eta_{-2}^{a_{n-3}} \eta_{-3}^{a_{n-9}+a_{n-10}} \zeta_0^{a_{n-8}+a_{n-9}} \zeta_{-1}^{a_{n-7}+a_{n-8}} \zeta_{-2}^{a_{n-6}+a_{n-7}} \zeta_{-3}^{a_{n-6}} \tag{3.100}$$

for $n \ge -3$.

Let

$$\widehat{P}_8(t) = t^8 - t^2 - 2t - 1 = (t^4 - t - 1)(t^4 + t + 1).$$

It is the characteristic polynomial associated to (3.98). Let t_j, $j = \overline{1,8}$, be its roots. It is not difficult to see that they are simple. Then, the solution to (3.98) such that $a_{-j} = 0$, $j = \overline{6,12}$, and $a_{-5} = 1$, is given by

$$a_n = \sum_{j=1}^{8} \frac{t_j^{n+12}}{\widehat{P}_8'(t_j)}, \quad n \in \mathbb{Z}. \tag{3.101}$$

The following theorem follows from (2.18), (3.99) and (3.100).

Theorem 3.13. *If $a \neq 0$, then the general solution to (2.14) is*

$$x_n = \sqrt{a}\, \frac{\prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-5}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-9}} \prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-8}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-6}} + 1}{\prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-5}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-9}} \prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-8}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-6}} - 1},$$

$$y_n = \sqrt{a}\, \frac{\prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-5}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-9}} \prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-8}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-6}} + 1}{\prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-5}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-9}} \prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-8}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-6}} - 1},$$

for $n \ge -3$, where the sequence a_n is given by (3.101) and $b_n = a_n + a_{n-1}$.

3.14 System (2.32)

By interchanging letters ζ and η, (2.32) is got from (2.27). Hence

$$\zeta_n = \eta_0^{a_{n-5}+a_{n-6}} \eta_{-1}^{a_{n-4}+a_{n-5}} \eta_{-2}^{a_{n-3}+a_{n-4}} \eta_{-3}^{a_{n-10}+a_{n-11}} \zeta_0^{a_{n-9}+a_{n-10}} \zeta_{-1}^{a_{n-8}+a_{n-9}} \zeta_{-2}^{a_{n-7}+a_{n-8}} \zeta_{-3}^{a_{n-6}+a_{n-7}}, \tag{3.102}$$

$$\eta_n = \eta_0^{a_{n-2}} \eta_{-1}^{a_{n-1}} \eta_{-2}^{a_n} \eta_{-3}^{a_{n-7}} \zeta_0^{a_{n-6}} \zeta_{-1}^{a_{n-5}} \zeta_{-2}^{a_{n-4}} \zeta_{-3}^{a_{n-3}}, \tag{3.103}$$

for $n \ge -3$.

The following theorem follows from (2.18), (3.102) and (3.103).

Theorem 3.14. *If $a \neq 0$, then the general solution to (2.15) is*

$$x_n = \sqrt{a}\frac{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-5}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-10}}\prod_{l=0}^{3}\left(\frac{x_{-l}+\sqrt{a}}{x_{-l}-\sqrt{a}}\right)^{b_{n+l-9}}+1}{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-5}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-10}}\prod_{l=0}^{3}\left(\frac{x_{-l}+\sqrt{a}}{x_{-l}-\sqrt{a}}\right)^{b_{n+l-9}}-1},$$

$$y_n = \sqrt{a}\frac{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-2}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-7}}\prod_{l=0}^{3}\left(\frac{x_{-l}+\sqrt{a}}{x_{-l}-\sqrt{a}}\right)^{a_{n+l-6}}+1}{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-2}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-7}}\prod_{l=0}^{3}\left(\frac{x_{-l}+\sqrt{a}}{x_{-l}-\sqrt{a}}\right)^{a_{n+l-6}}-1},$$

for $n \geq -3$, where the sequence a_n is given by (3.77) and $b_n = a_n + a_{n-1}$.

3.15 System (2.33)

By interchanging letters ζ and η, (2.33) is got from (2.23). Hence

$$\zeta_n = \eta_0^{a_{n-6}+a_{n-7}}\eta_{-1}^{a_{n-5}+a_{n-6}}\eta_{-2}^{a_{n-4}+a_{n-5}}\eta_{-3}^{a_{n-4}+a_{n-7}}\zeta_0^{a_{n-9}+a_{n-10}}\zeta_{-1}^{a_{n-8}+a_{n-9}}\zeta_{-2}^{a_{n-7}+a_{n-8}}, \tag{3.104}$$

$$\eta_n = \eta_0^{a_{n-3}}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}}\eta_{-3}^{a_{n-4}+a_{n-7}}\zeta_0^{a_{n-6}}\zeta_{-1}^{a_{n-5}}\zeta_{-2}^{a_{n-4}}, \tag{3.105}$$

for $n \in \mathbb{N}$.

The following theorem follows from (2.18), (3.104) and (3.105).

Theorem 3.15. *If $a \neq 0$, then the general solution to (2.16) is*

$$x_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-6}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-9}}+1}{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-6}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-9}}-1},$$

for $n \in \mathbb{N}_0$,

$$y_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-6}}+1}{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-6}}-1},$$

for $n \geq -3$, and where a_n is given by (3.50) and $b_n = a_n + a_{n-1}$.

3.16 System (2.34)

By interchanging letters ζ and η, (2.34) is got from (2.19). Hence

$$\zeta_n = \eta_0^{c_{n-2}}\eta_{-1}^{c_{n-1}}\eta_{-2}^{c_n}\eta_{-3}^{c_{n-3}}, \quad n \in \mathbb{N}, \tag{3.106}$$

and

$$\eta_n = \eta_0^{c_{n-2}}\eta_{-1}^{c_{n-1}}\eta_{-2}^{c_n}\eta_{-3}^{c_{n-3}}, \quad n \geq -3. \tag{3.107}$$

The following theorem follows from (2.18), (3.106) and (3.107).

Theorem 3.16. *If $a \neq 0$, then the general solution to (2.17) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1}, \quad n \in \mathbb{N},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1}, \quad n \geq -3,$$

where c_n is given by (3.11).

Remark 3.17. From (2.18) we see that a solution to a system in (1.2) is well defined if and only if $\zeta_n \neq 1$ and $\eta_n \neq 1$ for every n belonging to the domain of the system. Using this fact, as well as above presented expressions for the sequences ζ_n and η_n, can be described the sets of not well defined solutions for each of the systems. We leave it to the reader.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **13**, 1–13; https://doi.org/10.14232/ejqtde.2021.1.13 www.math.u-szeged.hu/ejqtde/

Blow-up analysis in a quasilinear parabolic system coupled via nonlinear boundary flux

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Received 25 September 2020, appeared 26 February 2021
Communicated by Roberto Livrea

Abstract. This paper deals with the blow-up of the solution for a system of evolution p-Laplacian equations $u_{it} = \operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i)$ $(i = 1, 2, \dots, k)$ with nonlinear boundary flux. Under certain conditions on the nonlinearities and data, it is shown that blow-up will occur at some finite time. Moreover, when blow-up does occur, we obtain the upper and lower bounds for the blow-up time. This paper generalizes the previous results.

Keywords: blow-up, quasilinear parabolic system, nonlinear boundary flux.

2020 Mathematics Subject Classification: 35K55, 35K60.

1 Introduction

In this paper, we investigate the following parabolic equations

$$u_{it} = \operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i), \quad (i = 1, 2, \dots, k),\ (x, t) \in \Omega \times (0, t^*), \tag{1.1}$$

coupled via nonlinear boundary flux

$$|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu} = f_i(u_1, u_2, \dots, u_k), \quad (i = 1, 2, \dots, k),\ (x, t) \in \partial\Omega \times (0, t^*), \tag{1.2}$$

with initial data

$$u_i\,(x, 0) = u_{i0}\,(x) \geq 0, \quad (i = 1, 2, \dots, k),\ x \in \Omega, \tag{1.3}$$

where $p \geq 2$, $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u on the boundary $\partial\Omega$ assumed sufficiently smooth, Ω is a bounded star-shaped region in $\mathbb{R}^N$ $(N \geq 2)$ and t^* is the blow-up time if blow-up occurs, or else $t^* = \infty$. Moreover the non-negative initial functions $u_{i0}(x), i = 1, 2, \dots, k$ satisfy the compatibility conditions and $f_i(u_1, u_2, \dots, u_k) : \mathbb{R}^k \to \mathbb{R}, i = 1, 2, \dots, k$ are given functions to be specified later. It is well known that the functions $f_i(u_1, u_2, \dots, u_k), i = 1, 2, \dots, k$ may greatly affect the behavior of the solution $(u_1, u_2, \dots, u_k)$ with the development of time.

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The blow-up phenomena in nonlinear parabolic equations have been extensively investigated by many authors in the last decades (see [1–5, 8, 16, 24, 26] and the references therein). Nowadays, many methods are known and used in the study of various questions regarding the blow-up phenomena (such as blow-up criterion, blow-up rate and blow-up set, etc.) in nonlinear parabolic problems. In applications, due to the explosive nature of the solutions, it is more important to determine the lower bounds for the blow-up time. Therefore, there exist many interesting results about blow-up time in various problems, such as [11, 12, 14, 17] in parabolic problems, [13, 15, 22, 27] in chemotaxis systems, [23] even in fourth order wave equations, and so on.

In [20], Payne et al. considered the following semilinear heat equation with nonlinear boundary condition

$$\begin{cases} u_t = \Delta u - f(u), & (x,t) \in \Omega \times (0,t^*), \\ \frac{\partial u}{\partial \nu} = g(u), & (x,t) \in \partial\Omega \times (0,t^*), \\ u\left(x,0\right) = u_0\left(x\right), & x \in \Omega, \end{cases} \tag{1.4}$$

and established sufficient conditions on the nonlinearities to guarantee that the solution $u(x,t)$ exists for all time $t > 0$ or blows up in finite time t^*. Moreover, an upper bound for t^* was derived. Under more restrictive conditions, a lower bound for t^* was also obtained.

Moreover, Payne et al. [21] also studied the following initial-boundary problem

$$\begin{cases} u_t = \nabla(|\nabla u|^{2p}\nabla u), & (x,t) \in \Omega \times (0,t^*), \\ |\nabla u|^{2p}\frac{\partial u}{\partial \nu} = f(u), & (x,t) \in \partial\Omega \times (0,t^*), \\ u\left(x,0\right) = u_0\left(x\right), & x \in \Omega, \end{cases} \tag{1.5}$$

and obtained upper and lower bounds for the blow-up time under some conditions when blow-up does occur at some finite time.

Recently, for the special case $k = 2$ in (1.1), Liang [7] investigated the following system with nonlinear boundary flux

$$\begin{cases} u_t = \nabla(|\nabla u|^{p-2}\nabla u), v_t = \nabla(|\nabla v|^{p-2}\nabla v), & (x,t) \in \Omega \times (0,t^*), \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = f_1(u,v), |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = f_2(u,v), & (x,t) \in \partial\Omega \times (0,t^*), \\ u\left(x,0\right) = u_0\left(x\right), v\left(x,0\right) = v_0\left(x\right), & x \in \Omega, \end{cases} \tag{1.6}$$

and showed that under certain conditions on the nonlinearities and the data, blow-up will occur at some finite time and when blow-up does occur, upper and lower bounds for the blow-up time are obtained.

On the other hand, many authors have studied upper and lower bounds for the blow-up time to nonlinear parabolic equations with local or nonlocal sources (see [6, 9, 10, 18, 19, 25] and the references therein).

Motivated by the above works, we investigate the blow-up condition of the solution and derive upper and lower bounds for the blow-up time t^*. Throughout this paper, we take the functions $f_i(u_1, u_2, \ldots, u_k)$, $i = 1, 2, \ldots, k$ satisfying

$$f_i(u_1, u_2, \ldots, u_k) = a\left|\sum_{j=1}^{k} u_i\right|^{r-1}\left(\sum_{j=1}^{k} u_i\right) + b|u_i|^{\frac{r+1}{k}-2}u_i|u_1u_2\cdots u_{i-1}u_{i+1}\cdots u_k|^{\frac{r+1}{k}}, \tag{1.7}$$

where a, b are positive constants and r satisfies

$$\begin{cases} r > 1, & \text{if } N = 1,2, \\ 1 < r \leq \frac{N+2}{N-2}, & \text{if } N \geq 3. \end{cases} \tag{1.8}$$

Moreover it is easy to see that

$$\sum_{i=1}^{k} u_i f_i(u_1, u_2, \ldots, u_k) = (r+1)F(u_1, u_2, \ldots, u_k) \tag{1.9}$$

and

$$\frac{\partial F(u_1, u_2, \ldots, u_k)}{\partial u_i} = f_i(u_1, u_2, \ldots, u_k), \quad i = 1, 2, \ldots, k, \tag{1.10}$$

where

$$F(u_1, u_2, \ldots, u_k) = \frac{1}{r+1}\left[a\Big|\sum_{i=1}^{k} u_i\Big|^{r+1} + kb\Big|\prod_{i=1}^{k} u_i\Big|^{\frac{r+1}{k}}\right]. \tag{1.11}$$

Our main results of this paper are stated as follows.

Theorem 1.1. *Let $p \le r+1$. Assume that $(u_1, u_2, \ldots, u_k)$ is the nonnegative solution of problem (1.1)–(1.3). Moreover, suppose that $\Psi(0) > 0$ with*

$$\Psi(t) = p\int_{\partial\Omega} F(u_1, u_2, \ldots, u_k)ds - \sum_{i=1}^{k}\int_{\Omega} |\nabla u_i|^p dx, \tag{1.12}$$

where the function $F(u_1, u_2, \ldots, u_k)$ is defined by (1.11). Then for $p > 2$, the solution $(u_1, u_2, \ldots, u_k)$ of problem (1.1)–(1.3) blows up in finite time $t^* < T$ with

$$T = \frac{\Phi(0)}{(p-2)\Psi(0)}, \tag{1.13}$$

where

$$\Phi(t) = \sum_{i=1}^{k}\int_{\Omega} u_i^2 dx. \tag{1.14}$$

When $p = 2$, we have $T = \infty$.

Theorem 1.2. *Assume that $(u_1, u_2, \ldots, u_k)$ is the nonnegative solution of problem (1.1)–(1.3) in a bounded star-shaped domain $\Omega \subset \mathbb{R}^3$ assumed to be convex in two orthogonal directions. If the solution $(u_1, u_2, \ldots, u_k)$ does blow up in finite time t^*, then the blow-up time t^* is bounded from below by*

$$t^* \ge \int_{\Theta(0)}^{\infty} \frac{1}{\sum_{i=1}^{4} l_i \xi^{\alpha_i}} d\xi, \tag{1.15}$$

where

$$\Theta(t) = \sum_{i=1}^{k}\int_{\Omega} u_i^{m(r-1)} dx \quad \text{with } m \ge \max\left\{4, \frac{2}{r-1}\right\}, \tag{1.16}$$

and l_i, α_i $(i = 1, 2, 3, 4)$ are computable positive constants.

This paper is organized as follows. In Section 2, we obtain the blow-up condition of the solution and derive an upper bound estimate for the blow-up time t^*. Moreover, we also give the lower bound for the blow-up time t^* under appropriate assumptions on the data of problem (1.1)–(1.3), and prove Theorem 1.2 in Section 3.

2 Proof of Theorem 1.1

In this section, we obtain the blow-up condition of the solution and derive an upper bound estimate for the blow-up time t^*, and prove Theorem 1.1.

Proof of Theorem 1.1. Using the Green formula and the hypotheses stated in Theorem 1.1, we have

$$\begin{aligned}
\Phi'(t) &= 2\sum_{i=1}^{k}\int_{\Omega} u_i u_{it}dx \\
&= 2\sum_{i=1}^{k}\int_{\Omega} u_i \mathrm{div}(|\nabla u_i|^{p-2}\nabla u_i)dx \\
&= 2\sum_{i=1}^{k}\int_{\partial\Omega} u_i|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu}ds - 2\sum_{i=1}^{k}\int_{\Omega}|\nabla u_i|^p dx \\
&= 2\sum_{i=1}^{k}\int_{\partial\Omega} u_i f_i(u_1,u_2,\dots,u_k)ds - 2\sum_{i=1}^{k}\int_{\Omega}|\nabla u_i|^p dx \\
&= 2(r+1)\int_{\partial\Omega} F(u_1,u_2,\dots,u_k)ds - 2\sum_{i=1}^{k}\int_{\Omega}|\nabla u_i|^p dx \\
&\geq 2\left[p\int_{\partial\Omega} F(u_1,u_2,\dots,u_k)ds - \sum_{i=1}^{k}\int_{\Omega}|\nabla u_i|^p dx\right] \\
&= 2\Psi(t)
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
\Psi'(t) &= p\sum_{i=1}^{k}\int_{\partial\Omega} f_i(u_1,u_2,\dots,u_k)u_{it}ds - p\sum_{i=1}^{k}\int_{\Omega}|\nabla u_i|^{p-2}\nabla u_i\nabla u_{it}dx \\
&= p\sum_{i=1}^{k}\int_{\partial\Omega} f_i(u_1,u_2,\dots,u_k)u_{it}ds - p\sum_{i=1}^{k}\int_{\partial\Omega}|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu}u_{it}ds \\
&\quad + p\sum_{i=1}^{k}\int_{\Omega}\mathrm{div}(|\nabla u_i|^{p-2}\nabla u_i)u_{it}dx \\
&= p\sum_{i=1}^{k}\int_{\Omega}(u_{it})^2dx \geq 0.
\end{aligned} \tag{2.2}$$

It follows from $\Psi(0) > 0$ and (2.2) that $\Psi(t)$ is positive for all $t > 0$. By using Hölder's inequality and Cauchy's inequality, we deduce from (2.2) that

$$\begin{aligned}
\left(\sum_{i=1}^{k}\int_{\Omega} u_i u_{it}dx\right)^2 &\leq \left(\sum_{i=1}^{k}\left(\int_{\Omega} u_i^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega} u_{it}^2dx\right)^{\frac{1}{2}}\right)^2 \\
&\leq \left(\sum_{i=1}^{k}\int_{\Omega} u_i^2dx\right)\left(\sum_{i=1}^{k}\int_{\Omega} u_{it}^2dx\right) \\
&= \frac{1}{p}\Phi(t)\Psi'(t).
\end{aligned} \tag{2.3}$$

Therefore, it follows from (2.1)–(2.3) that

$$\Phi'(t)\Psi(t) \leq \frac{1}{2}(\Phi'(t))^2 = 2\left(\sum_{i=1}^{k}\int_{\Omega} u_i u_{it}dx\right)^2 \leq \frac{2}{p}\Phi(t)\Psi'(t), \tag{2.4}$$

that is,

$$\left(\Psi(t)\Phi^{-\frac{p}{2}}(t)\right)' \geq 0. \tag{2.5}$$

Integrating (2.5) over $(0, t)$, we obtain

$$\Psi(t)\Phi^{-\frac{p}{2}}(t) \geq \Psi(0)\Phi^{-\frac{p}{2}}(0) =: M. \tag{2.6}$$

Combining (2.1) with (2.6), we derive

$$\Phi'(t)\Phi^{-\frac{p}{2}}(t) \geq 2M. \tag{2.7}$$

If $p > 2$, then (2.7) can be written as

$$(\Phi^{1-\frac{p}{2}})'(t) \leq 2M\left(1 - \frac{p}{2}\right). \tag{2.8}$$

Integrating (2.8) over $(0, t)$ again, we have

$$\Phi(t) \geq \left[\Phi^{1-\frac{p}{2}}(0) - M(p-2)t\right]^{-\frac{2}{p-2}}, \tag{2.9}$$

which implies $\Phi(t) \to +\infty$ as $t \to T = \frac{\Phi^{1-\frac{p}{2}}(0)}{M(p-2)} = \frac{\Phi(0)}{(p-2)\Psi(0)}$. Therefore, for $p > 2$, we derive

$$t^* \leq T = \frac{\Phi(0)}{(p-2)\Psi(0)}. \tag{2.10}$$

If $p = 2$, then we infer from (2.7) that

$$\Phi(t) \geq \Phi(0)e^{2Mt}, \tag{2.11}$$

which implies $t^* = \infty$. The proof of Theorem 1.1 is complete. □

3 Proof of Theorem 1.2

In this section, under the assumption that $\Omega \subset \mathbb{R}^3$ is a convex bounded star-shaped domain in two orthogonal directions, we establish a lower bound for the blow-up time t^*. To do this, we need the following lemmas.

Lemma 3.1 (see [21, Lemma A.1]). *Let Ω be a bounded star-shaped domain in $\mathbb{R}^N$, $N \geq 2$. Then for any non-negative C^1-function u and $\gamma > 0$, we have*

$$\int_{\partial\Omega} u^\gamma ds \leq \frac{N}{\rho_0}\int_\Omega u^\gamma dx + \frac{\gamma d}{\rho_0}\int_\Omega u^{\gamma-1}|\nabla u|dx, \tag{3.1}$$

where

$$\rho_0 = \min_{x\in\partial\Omega}(x\cdot\nu) > 0 \quad \text{and} \quad d = \max_{x\in\overline{\Omega}}|x|. \tag{3.2}$$

Lemma 3.2 (see [21, Lemma A.2]). *Let Ω be a bounded domain in $\mathbb{R}^3$ assumed to be star-shaped and convex in two orthogonal directions. Then for any non-negative C^1-function u and $n \geq 1$, we have*

$$\int_\Omega u^{\frac{3n}{2}} dx \leq \left[\frac{3}{2\rho_0}\int_\Omega u^n dx + \frac{n}{2}\left(1 + \frac{d}{\rho_0}\right)\int_\Omega u^{n-1}|\nabla u|dx\right]^{\frac{3}{2}}, \tag{3.3}$$

where ρ_0 and d are defined in Lemma 3.1.

Proof of Theorem 1.2. Differentiating $\Theta(t)$ in (1.16), we obtain

$$\begin{aligned}\Theta'(t) &= m(r-1)\sum_{i=1}^{k}\int_{\Omega} u_i^{m(r-1)-1}u_{it}dx \\ &= m(r-1)\sum_{i=1}^{k}\int_{\Omega} u_i^{m(r-1)-1}\mathrm{div}(|\nabla u_i|^{p-2}\nabla u_i)dx \\ &= m(r-1)\sum_{i=1}^{k}\int_{\partial\Omega} u_i^{m(r-1)-1}f_i(u_1,u_2,\dots,u_k)ds \\ &\quad -m(r-1)[m(r-1)-1]\sum_{i=1}^{k}\int_{\Omega} u_i^{m(r-1)-2}|\nabla u_i|^p dx.\end{aligned} \tag{3.4}$$

By the definition of the functions $f_i, i=1,2,\dots,k$ and Lemma 3.1, we have

$$\begin{aligned}&\sum_{i=1}^{k}\int_{\partial\Omega} u_i^{m(r-1)-1}f_i(u_1,u_2,\dots,u_k)ds \\ &\quad\le C\sum_{i=1}^{k}\int_{\partial\Omega} u_i^{(m+1)(r-1)}ds \\ &\quad\le \frac{3C}{\rho_0}\sum_{i=1}^{k}\int_{\Omega} u_i^{(m+1)(r-1)}dx + \frac{C(m+1)(r-1)d}{\rho_0}\sum_{i=1}^{k}\int_{\Omega} u_i^{(m+1)(r-1)-1}|\nabla u_i|dx,\end{aligned} \tag{3.5}$$

where C is a positive constant. Combining (3.4) with (3.5), we derive

$$\begin{aligned}\Theta'(t) \le& \frac{3m(r-1)C}{\rho_0}I_1(t) + \frac{Cm(m+1)(r-1)^2 d}{\rho_0}I_2(t) \\ &- m(r-1)[m(r-1)-1]I_3(t),\end{aligned} \tag{3.6}$$

where

$$I_1(t) = \sum_{i=1}^{k}\int_{\Omega} u_i^{(m+1)(r-1)}dx = \sum_{i=1}^{k} I_{1i}(t), \tag{3.7}$$

$$I_2(t) = \sum_{i=1}^{k}\int_{\Omega} u_i^{(m+1)(r-1)-1}|\nabla u_i|dx = \sum_{i=1}^{k} I_{2i}(t), \tag{3.8}$$

and

$$I_3(t) = \sum_{i=1}^{k}\int_{\Omega} u_i^{m(r-1)-2}|\nabla u_i|^p dx = \sum_{i=1}^{k} I_{3i}(t). \tag{3.9}$$

By Lemma 3.2 and Hölder's inequality, we obtain

$$\begin{aligned}I_{1i}(t) &= \int_{\Omega} u_i^{(m+1)(r-1)}dx \\ &\le \left[\frac{3}{2\rho_0}\int_{\Omega} u_i^{\frac{2}{3}(m+1)(r-1)}dx + \frac{(m+1)(r-1)}{3}\left(1+\frac{d}{\rho_0}\right)\right. \\ &\quad\left.\times\int_{\Omega} u_i^{\frac{2}{3}(m+1)(r-1)-1}|\nabla u_i|dx\right]^{\frac{3}{2}} \\ &\le \left[\frac{3|\Omega|^{\frac{m-2}{3m}}}{2\rho_0}\left(\int_{\Omega} u_i^{m(r-1)}dx\right)^{\frac{2(m+1)}{3m}} + \frac{(m+1)(r-1)(\rho_0+d)}{3\rho_0}\right. \\ &\quad\left.\times\int_{\Omega} u_i^{\frac{2}{3}(m+1)(r-1)-1}|\nabla u_i|dx\right]^{\frac{3}{2}},\end{aligned} \tag{3.10}$$

where $i = 1, 2, \ldots, k$ and $|\Omega|$ is the measure of Ω. By using Hölder's inequality twice again, we have

$$\begin{aligned}
\int_\Omega u_i^{\frac{2}{3}(m+1)(r-1)-1}|\nabla u_i|dx &\le \left(\int_\Omega u_i^{\frac{2}{3}(m+1)(r-1)(1-\delta_1)}dx\right)^{\frac{p-1}{p}} \left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{1}{p}} \\
&\le \left(\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{2(m+1)(1-\delta_1)}{3m}} |\Omega|^{1-\frac{2(m+1)(1-\delta_1)}{3m}}\right)^{\frac{p-1}{p}} \\
&\quad \times \left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{1}{p}},
\end{aligned} \tag{3.11}$$

where $i = 1, 2, \ldots, k$ and $\delta_1 = \frac{(m-2)(r-1)+3p-6}{2(m+1)(r-1)(p-1)} \in (0, 1)$ due to (1.16). Therefore, it follows from (3.10) and (3.11) that

$$\begin{aligned}
I_{1i}(t) &\le \Bigg[\frac{3|\Omega|^{\frac{m-2}{3m}}}{2\rho_0}\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{2(m+1)}{3m}} + \frac{(m+1)(r-1)(\rho_0+d)}{3\rho_0} \\
&\quad \times \left(\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{2(m+1)(1-\delta_1)}{3m}} |\Omega|^{1-\frac{2(m+1)(1-\delta_1)}{3m}}\right)^{\frac{p-1}{p}} \left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{1}{p}}\Bigg]^{\frac{3}{2}} \\
&\le c_1\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{m+1}{m}} + c_2\left(\int_\Omega u_i^{m(r-1)}dx\right)^{\frac{(m+1)(p-1)(1-\delta_1)}{mp}} \left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{3}{2p}} \\
&\le c_1\Theta^{\frac{m+1}{m}}(t) + c_2\Theta^{\frac{(m+1)(p-1)(1-\delta_1)}{mp}}(t) I_3^{\frac{3}{2p}}(t), \quad i = 1, 2, \ldots, k,
\end{aligned} \tag{3.12}$$

where

$$c_1 = \frac{3\sqrt{3}}{2}\rho_0^{-\frac{3}{2}}|\Omega|^{\frac{m-2}{2m}} > 0 \tag{3.13}$$

and

$$c_2 = \frac{\sqrt{6}}{9}\left(\frac{(m+1)(r-1)(\rho_0+d)}{\rho_0}\right)^{\frac{3}{2}} |\Omega|^{\left(1-\frac{2(m+1)(1-\delta_1)}{3m}\right)\frac{3(p-1)}{2p}} > 0. \tag{3.14}$$

Hence, we infer from (3.12) that

$$I_1(t) = \sum_{i=1}^{k} I_{1i} \le kc_1\Theta^{\frac{m+1}{m}}(t) + kc_2\Theta^{\frac{(m+1)(p-1)(1-\delta_1)}{mp}}(t) I_3^{\frac{3}{2p}}(t). \tag{3.15}$$

Next, we estimate $I_2(t)$. By using Hölder's inequality, we have

$$\begin{aligned}
I_{2i}(t) &= \int_\Omega u_i^{(m+1)(r-1)-1}|\nabla u_i|dx \\
&\le \left(\int_\Omega u_i^{(m+2)(r-1)(1-\delta_2)}dx\right)^{\frac{p-1}{p}} \left(\int_\Omega u_i^{m(r-1)-2}|\nabla u_i|^p dx\right)^{\frac{1}{p}} \\
&\le \left(\left(\int_\Omega u_i^{(m+2)(r-1)}dx\right)^{1-\delta_2} |\Omega|^{\delta_2}\right)^{\frac{p-1}{p}} I_{3i}^{\frac{1}{p}}(t) \\
&= |\Omega|^{\frac{(p-1)\delta_2}{p}} \left(\int_\Omega u_i^{(m+2)(r-1)}dx\right)^{\frac{(p-1)(1-\delta_2)}{p}} I_{3i}^{\frac{1}{p}}(t), \quad i = 1, 2, \ldots, k,
\end{aligned} \tag{3.16}$$

where

$$\delta_2 = \frac{r(p-2)}{(m+2)(r-1)(p-1)} \in (0,1). \tag{3.17}$$

It follows from Lemma 3.2 and Hölder's inequality that

$$\begin{aligned}
\int_\Omega u_i^{(m+2)(r-1)} dx \leq & \left[\frac{3}{2\rho_0} \int_\Omega u_i^{\frac{2}{3}(m+2)(r-1)} dx + \frac{(m+2)(r-1)}{3} \left(1 + \frac{d}{\rho_0}\right) \right. \\
& \left. \times \int_\Omega u_i^{\frac{2}{3}(m+2)(r-1)-1} |\nabla u_i| dx \right]^{\frac{3}{2}} \\
\leq & \left[\frac{3|\Omega|^{\frac{m-4}{3m}}}{2\rho_0} \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{2(m+2)}{3m}} + \frac{(m+2)(r-1)(\rho_0+d)}{3\rho_0} \right. \\
& \left. \times \int_\Omega u_i^{\frac{2}{3}(m+2)(r-1)-1} |\nabla u_i| dx \right]^{\frac{3}{2}}, \quad i = 1, 2, \ldots, k.
\end{aligned} \tag{3.18}$$

By using Hölder's inequality twice again, we have

$$\begin{aligned}
\int_\Omega u_i^{\frac{2}{3}(m+2)(r-1)-1} |\nabla u_i| dx \leq & \left(\int_\Omega u_i^{\frac{2}{3}(m+2)(r-1)(1-\delta_3)} dx \right)^{\frac{p-1}{p}} \left(\int_\Omega u_i^{m(r-1)-2} |\nabla u_i|^p dx \right)^{\frac{1}{p}} \\
\leq & \left(\left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{2(m+2)(1-\delta_3)}{3m}} |\Omega|^{1-\frac{2(m+2)(1-\delta_3)}{3m}} \right)^{\frac{p-1}{p}} \\
& \times \left(\int_\Omega u_i^{m(r-1)-2} |\nabla u_i|^p dx \right)^{\frac{1}{p}} \\
= & |\Omega|^{(1-\frac{2(m+2)(1-\delta_3)}{3m})\frac{p-1}{p}} \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{2(m+2)(p-1)(1-\delta_3)}{3mp}} (t) I_{3i}^{\frac{1}{p}}(t),
\end{aligned} \tag{3.19}$$

where $i = 1, 2, \ldots, k$ and

$$\delta_3 = \frac{(m-4)(r-1)+3p-6}{2(m+2)(r-1)(p-1)} < \delta_1 < 1. \tag{3.20}$$

Combining (3.18) with (3.19), we obtain

$$\begin{aligned}
& \int_\Omega u_i^{(m+2)(r-1)} dx \\
& \quad \leq \frac{3\sqrt{3}}{2} |\Omega|^{\frac{m-4}{2m}} \rho_0^{-\frac{3}{2}} \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{m+2}{m}} (t) + \frac{\sqrt{6}}{9} \left(\frac{(m+2)(r-1)(\rho_0+d)}{\rho_0} \right)^{\frac{3}{2}} \\
& \quad \times |\Omega|^{\left(1-\frac{2(m+2)(1-\delta_3)}{3m}\right)\frac{3(p-1)}{2p}} \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{(m+2)(p-1)(1-\delta_3)}{mp}} (t) I_{3i}^{\frac{3}{2p}}(t), \quad i = 1, 2, \ldots, k.
\end{aligned} \tag{3.21}$$

Substituting (3.21) into (3.16) and applying the following inequality

$$(a_1 + a_2)^s \leq 2^s (a_1^s + a_2^s), \quad a_1, a_2 > 0 \quad \text{and} \quad s > 0,$$

we derive

$$\begin{aligned}
I_{2i}(t) &\le |\Omega|^{\frac{(p-1)\delta_2}{p}} \left(\frac{3\sqrt{3}}{2} |\Omega|^{\frac{m-4}{2m}} \rho_0^{-\frac{3}{2}} \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{m+2}{m}} (t) + \frac{\sqrt{6}}{9} \left(\frac{(m+2)(r-1)(\rho_0+d)}{\rho_0} \right)^{\frac{3}{2}} \right. \\
&\quad \left. \times |\Omega|^{\left(1-\frac{2(m+2)(1-\delta_3)}{3m}\right)\frac{3(p-1)}{2p}} \cdot \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{(m+2)(p-1)(1-\delta_3)}{mp}} (t) I_{3i}^{\frac{3}{2p}}(t) \right)^{\frac{(p-1)(1-\delta_2)}{p}} I_{3i}^{\frac{1}{p}}(t) \\
&\le c_3 \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{\alpha(m+2)(1-\delta_2)}{m}} (t) I_{3i}^{\frac{1}{p}}(t) + c_4 \left(\int_\Omega u_i^{m(r-1)} dx \right)^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m}} (t) I_{3i}^{\beta}(t) \\
&\le c_3 \Theta^{\frac{\alpha(m+2)(1-\delta_2)}{m}} (t) I_3^{\frac{1}{p}}(t) + c_4 \Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m}} (t) I_3^{\beta}(t), \quad i = 1, 2, \ldots, k,
\end{aligned} \tag{3.22}$$

where

$$\alpha = 1 - \frac{1}{p}, \quad \beta = \frac{1}{p} + \frac{3\alpha(1-\delta_2)}{2p} < 1, \tag{3.23}$$

$$c_3 = \left(3\sqrt{3}\rho_0^{-\frac{3}{2}} \right)^{\alpha(1-\delta_2)} |\Omega|^{\frac{\alpha}{2m}(m-4+(m+4)\delta_2)}, \tag{3.24}$$

and

$$c_4 = \left(\frac{2\sqrt{6}}{9} \right)^{\alpha(1-\delta_2)} \left(\frac{(m+2)(r-1)(\rho_0+d)}{\rho_0} \right)^{\frac{3\alpha(1-\delta_2)}{2}} |\Omega|^{\left(1-\frac{2(m+2)(1-\delta_3)}{3m}\right)\frac{3\alpha^2(1-\delta_2)}{2}+\alpha\delta_2}. \tag{3.25}$$

Hence, we deduce from (3.22) that

$$I_2(t) = \sum_{i=1}^{k} I_{2i}(t) \le kc_3 \Theta^{\frac{\alpha(m+2)(1-\delta_2)}{m}} (t) I_3^{\frac{1}{p}}(t) + kc_4 \Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m}} (t) I_3^{\beta}(t). \tag{3.26}$$

Therefore, it follows from (3.6), (3.15) and (3.26) that

$$\begin{aligned}
\Theta'(t) &\le l_1 \Theta^{\frac{m+1}{m}}(t) + \widetilde{l}_2 \Theta^{\frac{(m+1)(p-1)(1-\delta_1)}{mp}} (t) I_3^{\frac{3}{2p}}(t) + \widetilde{l}_3 \Theta^{\frac{\alpha(m+2)(1-\delta_2)}{m}} (t) I_3^{\frac{1}{p}}(t) \\
&\quad + \widetilde{l}_4 \Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m}} (t) I_3^{\beta}(t) - m(r-1)[m(r-1)-1] I_3(t),
\end{aligned} \tag{3.27}$$

where

$$l_1 = \frac{9\sqrt{3}mk(r-1)C}{2\rho_0} \rho_0^{-\frac{3}{2}} |\Omega|^{\frac{m-2}{2m}} > 0, \tag{3.28}$$

$$\widetilde{l}_2 = \frac{\sqrt{6}mk(r-1)C}{3\rho_0} \left(\frac{(m+1)(r-1)(\rho_0+d)}{\rho_0} \right)^{\frac{3}{2}} |\Omega|^{\left(1-\frac{2(m+1)(1-\delta_1)}{3m}\right)\frac{3(p-1)}{2p}} > 0, \tag{3.29}$$

$$\widetilde{l}_3 = \frac{mk(m+1)(r-1)^2 Cd}{\rho_0} \left(3\sqrt{3}\rho_0^{-\frac{3}{2}} \right)^{\alpha(1-\delta_2)} |\Omega|^{\frac{\alpha}{2m}(m-2+(m+2)\delta_2)} > 0, \tag{3.30}$$

and

$$\begin{aligned}
\widetilde{l}_4 &= \left(\frac{2\sqrt{6}}{9} \right)^{\alpha(1-\delta_2)} \frac{mk(m+1)(r-1)^2 Cd}{\rho_0} \left(\frac{(m+2)(r-1)(\rho_0+d)}{\rho_0} \right)^{\frac{3\alpha(1-\delta_2)}{2}} \\
&\quad \times |\Omega|^{\left(1-\frac{2(m+2)(1-\delta_3)}{3m}\right)\frac{3\alpha^2(1-\delta_2)}{2}+\alpha\delta_2} > 0.
\end{aligned} \tag{3.31}$$

Next, by using the fundamental inequality

$$a_1^{r_1} a_2^{r_2} \le r_1 a_1 + r_2 a_2, \quad a_1, a_2 > 0, r_1, r_2 > 0 \quad \text{and} \quad r_1 + r_2 = 1, \tag{3.32}$$

we have

$$\begin{aligned}\Theta^{\frac{(m+1)(p-1)(1-\delta_1)}{mp}}(t) I_3^{\frac{3}{2p}}(t) &= (\varepsilon_1 I_3(t))^{\frac{3}{2p}} \left[\frac{\Theta^{\frac{2(m+1)(1-\delta_1)(p-1)}{m(2p-3)}}(t)}{\varepsilon_1^{\frac{3}{2p-3}}} \right]^{1-\frac{3}{2p}} \\ &\le \frac{3}{2p}\varepsilon_1 I_3(t) + \left(1 - \frac{3}{2p}\right) \varepsilon_1^{\frac{3}{3-2p}} \Theta^{\frac{2(m+1)(1-\delta_1)(p-1)}{m(2p-3)}}(t),\end{aligned} \tag{3.33}$$

where ε_1 is an arbitrary positive constant.

Similarly, we obtain

$$\Theta^{\frac{\alpha(m+2)(1-\delta_2)}{m}}(t) I_3^{\frac{1}{p}}(t) \le \frac{1}{p}\varepsilon_2 I_3(t) + \left(1 - \frac{1}{p}\right) \varepsilon_2^{\frac{1}{1-p}} \Theta^{\frac{\alpha p(m+2)(1-\delta_2)}{m(p-1)}}(t) \tag{3.34}$$

and

$$\Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m}}(t) I_3^{\beta}(t) \le \beta \varepsilon_3 I_3(t) + (1-\beta)\, \varepsilon_3^{\frac{\beta}{\beta-1}} \Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m(1-\beta)}}(t), \tag{3.35}$$

where $\varepsilon_i, i = 2, 3$ are arbitrary positive constants.

Choosing the arbitrary positive constants ε_i $(i = 1, 2, 3)$ such that

$$\frac{3}{2p}\varepsilon_1 \widetilde{l_2} + \frac{1}{p}\varepsilon_2 \widetilde{l_3} + \beta \varepsilon_3 \widetilde{l_4} - m(r-1)[m(r-1) - 1] = 0, \tag{3.36}$$

it follows from (3.27),(3.33)–(3.35) that

$$\begin{aligned}\Theta'(t) &\le l_1 \Theta^{\frac{m+1}{m}}(t) + l_2 \Theta^{\frac{2(m+1)(1-\delta_1)(p-1)}{m(2p-3)}}(t) + l_3 \Theta^{\frac{\alpha p(m+2)(1-\delta_2)}{m(p-1)}}(t) + l_4 \Theta^{\frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m(1-\beta)}}(t) \\ &= l_1 \Theta^{\alpha_1}(t) + l_2 \Theta^{\alpha_2}(t) + l_3 \Theta^{\alpha_3}(t) + l_4 \Theta^{\alpha_4}(t),\end{aligned} \tag{3.37}$$

where

$$l_2 = \left(1 - \frac{3}{2p}\right) \varepsilon_1^{\frac{3}{3-2p}} \widetilde{l_2}, \tag{3.38}$$

$$l_3 = \left(1 - \frac{1}{p}\right) \varepsilon_2^{\frac{1}{1-p}} \widetilde{l_3}, \tag{3.39}$$

$$l_4 = (1-\beta)\, \varepsilon_3^{\frac{\beta}{\beta-1}} \widetilde{l_4}, \tag{3.40}$$

$$\alpha_1 = \frac{m+1}{m}, \tag{3.41}$$

$$\alpha_2 = \frac{2(m+1)(1-\delta_1)(p-1)}{m(2p-3)}, \tag{3.42}$$

$$\alpha_3 = \frac{\alpha p(m+2)(1-\delta_2)}{m(p-1)}, \tag{3.43}$$

and

$$\alpha_4 = \frac{\alpha^2(m+2)(1-\delta_2)(1-\delta_3)}{m(1-\beta)}. \tag{3.44}$$

Integrating (3.37) over $(0,t)$, we derive

$$t \geq \int_{\Theta(0)}^{\Theta(t)} \frac{1}{\sum\limits_{i=1}^{4} l_i \xi^{\alpha_i}} d\xi. \tag{3.45}$$

As $(u_1, u_2, \ldots, u_k)$ blows up, we obtain the lower bound for the blow-up time t^* as follows

$$t \geq \int_{\Theta(0)}^{\infty} \frac{1}{\sum\limits_{i=1}^{4} l_i \xi^{\alpha_i}} d\xi. \tag{3.46}$$

Clearly, it is unlikely that the quantity $\int_{\Theta(0)}^{\infty} \frac{1}{\sum\limits_{i=1}^{4} l_i \xi^{\alpha_i}} d\xi$ can be evaluated exactly. However a lower bound for the integral may be obtained as follows. Let

$$g(\Theta) = \begin{cases} L\Theta^{\alpha_m}, & \text{if } \Theta(t) < 1, \\ L\Theta^{\alpha_M}, & \text{if } \Theta(t) > 1, \end{cases} \tag{3.47}$$

where $\alpha_m = \min_i\{\alpha_i\}$, $\alpha_M = \max_i\{\alpha_i\}$, $(i = 1,2,3,4)$ and $L = \sum_{i=1}^{4} l_i$. Then we have

$$t \geq \int_{\Theta(0)}^{\infty} \frac{1}{\sum\limits_{i=1}^{4} l_i \xi^{\alpha_i}} d\xi \geq \int_{\Theta(0)}^{\infty} \frac{1}{g(\xi)} d\xi. \tag{3.48}$$

The proof of Theorem 1.2 is complete. □

Acknowledgements

The authors would like to deeply thank the editors and reviewers for their insightful and constructive comments. This work is partially supported by National Natural Science Foundation of China (Grant Nos: 11601053, 11526042), Natural Science Foundation of Chongqing (Grant No. cstc2019jcyj-msxmX0082), Research Training Program of CQUPT (Grant Nos. A2019-83, A2020-166).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 14, 1–24; https://doi.org/10.14232/ejqtde.2021.1.14 www.math.u-szeged.hu/ejqtde/

Null controllability for a singular heat equation with a memory term

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Received 17 August 2020, appeared 2 March 2021
Communicated by László Simon

Abstract. In this paper we focus on the null controllability problem for the heat equation with the so-called inverse square potential and a memory term. To this aim, we first establish the null controllability for a nonhomogeneous singular heat equation by a new Carleman inequality with weights which do not blow up at $t = 0$. Then the null controllability property is proved for the singular heat equation with memory under a condition on the kernel, by means of Kakutani's fixed-point theorem.

Keywords: controllability, heat equation with memory, singular potential, Carleman estimates.

2020 Mathematics Subject Classification: 93B05, 35K05, 35K67, 35R09.

1 Introduction

In this paper we address the null controllability for the following singular heat equation with memory:

$$\begin{cases} y_t - y_{xx} - \dfrac{\mu}{x^2} y = \displaystyle\int_0^t a(t,r,x) y(r,x)\,dr + 1_\omega u, & (t,x) \in Q, \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{1.1}$$

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*The author thanks the MAECI (Ministry of Foreign Affairs and International Cooperation, Italy) for funding that greatly facilitated scientific collaboration between Université Hassan 1^{er} (Morocco) and Università di Bari Aldo Moro (Italy)

**The author is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and she is supported by the FFABR *Fondo per il finanziamento delle attività base di ricerca* 2017, by the INdAM - GNAMPA Project 2019 *Controllabilità di PDE in modelli fisici e in scienze della vita*, by Fondi di Ateneo 2017/18 of the University of Bari *Problemi differenziali non linearii* and by PRIN 2017-2019 *Qualitative and quantitative aspects of nonlinear PDEs*.

where $y_0 \in L^2(0,1)$, $T > 0$ is fixed, μ is a real parameter, $Q := (0,T) \times (0,1)$ and 1_ω stands for a characteristic function of a nonempty open subset ω of $(0,1)$. Here y and u are the state variable and the control variable, respectively; a is a given L^∞ function defined on $(0,T) \times Q$.

The analysis of evolution equations involving memory terms is a topic in continuous development. In the last decades, many researchers have started devoting their attention to this branch of mathematics, motivated by many applications in modelling phenomena in which the processes are affected not only by its current state but also by its history. Indeed, there is a large spectrum of situations in which the presence of the memory may render the description of the phenomena more accurate. This is particularly the case for models such as heat conduction in materials with memory, viscoelasticity, theory of population dynamics and nuclear reactors, where one often needs to reflect the effects of the memory of the system (see for instance [4,8,32,38]).

Controllability problems for evolution equations with memory terms have been extensively studied in the past. Among other contributions, we mention [5,21,24,27,28,30,33,39,42] which, as in our case, deal with parabolic type equations. We also refer to [37] for an overview of the bibliography on control problems for systems with persistent memory. The first results for a degenerate parabolic equation with memory can be found in [1].

In this work, for the first time to our knowledge, we study the null controllability for (1.1). We underline that here we consider not only a memory term but also a singular potential one. In other words, given any $y_0 \in L^2(0,1)$, we want to show that there exists a control function $u \in L^2(Q)$ such that the corresponding solution y to (1.1) satisfies $y(T,x) = 0$ for every $x \in [0,1]$. First results in this direction are obtained in [46] in the absence of a memory term when $\mu \le \frac{1}{4}$ (see also [45] for the wave and Schrödinger equations and [11] for boundary singularity). Indeed, for the equation

$$u_t - \Delta u - \mu \frac{1}{|x|^2} u = 0, \qquad (t,x) \in (0,T) \times \Omega, \tag{1.2}$$

with associated Dirichlet boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^N$ containing the singularity $x = 0$ in the interior, the value of the parameter μ determines the behavior of the equation: if $\mu \le 1/4$ (which is the optimal constant of the Hardy inequality, see [9]) global positive solutions exist, while, if $\mu > 1/4$, instantaneous and complete blow-up occurs (for other comments on this argument we refer to [44]). In the case of global positive solutions, hence if $\mu \le \frac{1}{4}$, using Carleman estimates, it has been proved that such equations can be controlled (in any time $T > 0$) by a locally distributed control (see [46]). On the contrary, if $\mu > \frac{1}{4}$, the null controllability fails as shown in [14]. After these first results, several other works followed extending them in various situations (see for instance [6,7,11,15–20,36,44]).

However, when $\mu = 0$ and $a = 1$, (1.1) reduces to the following control system associated to the classical heat equation with memory:

$$\begin{cases} y_t - y_{xx} = \displaystyle\int_0^t y(s)\,dr + 1_\omega u, & (t,x) \in Q, \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1). \end{cases} \tag{1.3}$$

In this case, as shown in [24, 49], there exists a set of initial conditions such that the null controllability property for (1.3) fails whenever the control region ω is fixed, independent of time. For some related works in this respect we also refer to [12,28,48].

Nevertheless, since the positive controllability results are important in real world applications, it is natural to analyze whether it is possible that control properties for (1.1) could be obtained. For this reason, under suitable conditions on the singularity parameter μ and on the kernel a (see (3.1)), we establish that (1.1) is null controllable.

Our approach is inspired from the techniques presented in the work [42] for the heat equation perturbed with a memory-type kernel, suitably adapted in order to deal with the additional inverse-square potential.

We recall that a natural technique for showing controllability results for parabolic equations is to prove an observability estimate for their adjoint systems by Carleman inequalities. However, this classical strategy does not seem to be appropriate for studying the controllability problem for integro-differential parabolic equations like (1.1). In fact, as in [10, 42], in this case we shall argue by a fixed point procedure. For this reason, we shall introduce a nonhomogeneous singular heat equation for which we prove a null controllability result by a modified Carleman inequality with weighted functions that do not blow up at $t = 0$. This is crucial in order to get the null controllability of the memory system (1.1) by weakening the assumptions on the kernel a. Finally, we mention that Carleman inequalities for singular equations without memory have been obtained in [44, 46], but the employment of a weight blowing up at $t = 0$ and $t = T$ in the Carleman inequality does not permit to consider a general kernel a.

The paper is organized as follows: Section 2 is devoted to the study of null controllability for a nonhomogeneous singular heat equation without memory via new Carleman estimates. In Section 3, the null controllability for the singular heat equation with memory (1.1) is proved.

A final comment on the notation: by C we shall denote universal positive constants, which are allowed to vary from line to line.

2 Nonhomogeneous singular heat equation

In this section, we prove the null controllability for a nonhomogeneous singular heat equation using a new modified Carleman inequality. This null controllability result is the key tool for the controllability of the heat equation with memory. Thus, as a first step, we consider the following problem:

$$\begin{cases} y_t - y_{xx} - \dfrac{\mu}{x^2} y = f + 1_\omega u(t), & (t,x) \in Q := (0,T) \times (0,1), \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{2.1}$$

where $f \in L^2(Q)$ is a given source term.

Prior to null controllability is the well-posedness of (2.1), a question we address in the next subsection.

2.1 Functional framework and well-posedness

We analyze here existence and uniqueness of solutions for the heat problem (2.1). To simplify the presentation, we first focus on the well-posedness of the following inhomogeneous singular problem

$$\begin{cases} y_t - y_{xx} - \dfrac{\mu}{x^2}y = f, & (t,x) \in Q, \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1). \end{cases} \tag{2.2}$$

In this framework, in order to deal with the singularity of the potential, a fundamental tool is the very famous Hardy inequality. To fix the ideas, we recall here the basic form of the Hardy inequality in dimension one (see, for example, [29, Theorem 327] or [13, Lemma 5.3.1]):

$$\frac{1}{4}\int_0^1 \frac{y^2}{x^2}\,dx \le \int_0^1 y_x^2\,dx, \tag{2.3}$$

which is valid for every $y \in H^1(0,1)$ with $y(0) = 0$.

Now, for any $\mu \le \frac{1}{4}$, we define

$$H_0^{1,\mu}(0,1) := \left\{ y \in L^2(0,1) \cap H^1_{loc}((0,1]) \mid y(0) = y(1) = 0, \text{ and } \int_0^1 \left(y_x^2 - \mu \frac{y^2}{x^2} \right) dx < +\infty \right\}.$$

Note that $H_0^{1,\mu}(0,1)$ is a Hilbert space obtained as the completion of $C_c^\infty(0,1)$, or $H_0^1(0,1)$, with respect to the norm

$$\|y\|_\mu := \left(\int_0^1 (y_x^2 - \mu \frac{y^2}{x^2})\,dx \right)^{\frac{1}{2}}, \qquad \forall\, y \in H_0^1(0,1).$$

In the case of a sub-critical parameter $\mu < \frac{1}{4}$, thanks to the Hardy inequality (2.3), one can see that $\|\cdot\|_\mu$ is equivalent to the standard norm of $H_0^1(0,1)$, and thus $H_0^{1,\mu}(0,1) = H_0^1(0,1)$. In the critical case $\mu = \frac{1}{4}$, it is proved (see [47]) that this identification does not hold anymore and the space $H_0^{1,\mu}(0,1)$ is slightly (but strictly) larger than $H_0^1(0,1)$.

Now, define the operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ corresponding to the heat equation with an inverse square potential in the following way:

$$Ay := y_{xx} + \frac{\mu}{x^2}y$$

$$\forall\, y \in D(A) := \left\{ y \in H^2_{loc}((0,1]) \cap H_0^{1,\mu}(0,1) : y_{xx} + \frac{\mu}{x^2}y \in L^2(0,1) \right\}.$$

In this context, A is self-adjoint, nonpositive on $L^2(0,1)$ and it generates an analytic semigroup of contractions in $L^2(0,1)$ for the equation (2.2) (see [47]). Consequently, the singular heat equation (2.2) is well-posed. To be precise, the next result holds.

Theorem 2.1. *For all $f \in L^2(Q)$ and $y_0 \in L^2(0,1)$, there exists a unique solution*

$$y \in \mathcal{W} := C\big([0,T]; L^2(0,1)\big) \cap L^2\big(0,T; H_0^{1,\mu}(0,1)\big)$$

of (2.2) such that

$$\sup_{t\in[0,T]} \|y(t)\|^2_{L^2(0,1)} + \int_0^T \|y(t)\|^2_\mu dt \le C_T \left(\|y_0\|^2_{L^2(0,1)} + \|f\|^2_{L^2(Q)} \right), \tag{2.4}$$

for some positive constant C_T. Moreover, if $y_0 \in H_0^{1,\mu}(0,1)$, then

$$y \in \mathcal{Z} := H^1\big(0,T; L^2(0,1)\big) \cap L^2\big(0,T; D(A)\big) \cap C\big([0,T]; H_0^{1,\mu}(0,1)\big), \tag{2.5}$$

and there exists a positive constant C such that

$$\sup_{t\in[0,T]}\left(\|y(t)\|_\mu^2\right)+\int_0^T\left(\|y_t\|_{L^2(0,1)}^2+\left\|y_{xx}+\frac{\mu}{x^2}y\right\|_{L^2(0,1)}^2\right)dt\le C\left(\|y_0\|_\mu^2+\|f\|_{L^2(Q)}^2\right). \tag{2.6}$$

Proof. In [47], the authors use semigroup theory to obtain the well-posedness result for the problem (2.2) (see also [36]). Thus, in the rest of the proof, we will prove only (2.4)–(2.6). First, being A the generator of a strongly continuous semigroup on $L^2(0,1)$, if $y_0\in L^2(0,1)$, then the solution y of (2.2) belongs to $C([0,T];L^2(0,1))\cap L^2(0,T;H_0^{1,\mu}(0,1))$, while, if $y_0\in D(A)$, then $y\in H^1(0,T;L^2(0,1))\cap L^2(0,T;D(A))$.

Now, by a usual energy method we shall prove (2.5) and (2.6), from which the last required regularity property for y will follow by standard linear arguments. First, take $y_0\in D(A)$ and multiply the equation of (2.2) by y. By the Cauchy–Schwarz inequality we obtain for every $t\in(0,T]$,

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_{L^2(0,1)}^2+\|y(t)\|_\mu^2\le\frac{1}{2}\|f(t)\|_{L^2(0,1)}^2+\frac{1}{2}\|y(t)\|_{L^2(0,1)}^2. \tag{2.7}$$

From (2.7) and using Gronwall's inequality, we get

$$\|y(t)\|_{L^2(0,1)}^2\le e^T\left(\|y(0)\|_{L^2(0,1)}^2+\|f\|_{L^2(Q)}^2\right) \tag{2.8}$$

for every $t\le T$. From (2.7) and (2.8) we immediately obtain

$$\int_0^T\|y(t)\|_\mu^2dt\le C_T\left(\|y(0)\|_{L^2(0,1)}^2+\|f\|_{L^2(Q)}^2\right) \tag{2.9}$$

for some universal constant $C_T>0$. Thus, by (2.8) and (2.9), (2.4) follows if $y_0\in D(A)$. Since $D(A)$ is dense in $L^2(0,1)$ (see [43,47]), the same inequality holds if $y_0\in L^2(0,1)$.

Now, multipling the equation by $-y_{xx}-\frac{\mu}{x^2}y$, integrating on $(0,1)$ and using the Cauchy–Schwarz inequality, we easily get

$$\frac{d}{dt}\|y(t)\|_\mu^2+\|y_{xx}(t)+\frac{\mu}{x^2}y(t)\|_{L^2(0,1)}^2\le\|f(t)\|_{L^2(0,1)}^2$$

for every $t\in[0,T]$, so that, as before, we find $C_T'>0$ such that

$$\|y(t)\|_\mu^2+\int_0^T\|y_{xx}(t)+\frac{\mu}{x^2}y(t)\|_{L^2(0,1)}^2dt\le C_T'\left(\|y(0)\|_\mu+\|f\|_{L^2(Q)}^2\right) \tag{2.10}$$

for every $t\le T$. Finally, from $y_t=y_{xx}+\dfrac{\mu}{x^2}y+f$, squaring and integrating on Q, we find

$$\int_0^T\|y_t(t)\|_{L^2(0,1)}^2\le C\left(\int_0^T\|y_{xx}+\frac{\mu}{x^2}y\|_{L^2(0,1)}^2+\|f\|_{L^2(Q)}^2\right),$$

and together with (2.10) we have

$$\int_0^T\|y_t(t)\|_{L^2(0,1)}^2\le C\left(\|y(0)\|_\mu^2+\|f\|_{L^2(Q)}^2\right). \tag{2.11}$$

In conclusion, (2.7), (2.8), (2.10) and (2.11) give (2.4) and (2.6). Notice that, (2.5) and (2.6) hold also if $y_0\in H_0^{1,\mu}(0,1)$. □

2.2 Carleman estimates for a singular problem

In this subsection we prove a new Carleman estimate for the adjoint parabolic equation associated to (2.1), which will provide that the nonhomogeneous singular heat equation (2.1) is null controllable. Hence, in the following, we concentrate on the next adjoint problem

$$\begin{cases} -z_t - z_{xx} - \dfrac{\mu}{x^2} z = g, & (t,x) \in Q, \\ z(t,0) = z(t,1) = 0, & t \in (0,T), \\ z(T,x) = z_T(x), & x \in (0,1). \end{cases} \tag{2.12}$$

Following [46], for every $0 < \gamma < 2$, let us introduce the weight function

$$\varphi(t,x) := \theta(t)\psi(x), \tag{2.13}$$

where

$$\psi(x) := \mathfrak{c}(x^2 - d), \qquad \theta(t) := \left(\frac{1}{t(T-t)}\right)^k, \qquad k := 1 + \frac{2}{\gamma}, \tag{2.14}$$

$\mathfrak{c} > 0$ and $d > 1$. A more precise restriction on the parameters $k, \mathfrak{c}$ and d will be needed later. Observe that $\lim_{t\to 0^+} \theta(t) = \lim_{t\to T^-} \theta(t) = +\infty$, and

$$\psi(x) < 0 \quad \text{for every } x \in [0,1].$$

Using the previous weight functions and the following improved Hardy–Poincaré inequality given in [44]:

For all $\eta > 0$, there exists some positive constant $C = C(\eta) > 0$ such that, for all $z \in C_c^\infty(0,1)$:

$$\int_0^1 x^\eta z_x^2 \, dx \le C \int_0^1 \left(z_x^2 - \frac{1}{4}\frac{z^2}{x^2} \right) dx, \tag{2.15}$$

one can prove the following Carleman estimate for the case of a purely singular parabolic equation:

Lemma 2.2 ([44, Theorem 5.1])**.** *Assume that $\mu \le \frac{1}{4}$. Then, there exists $C > 0$ and $s_0 > 0$ such that, for all $s \ge s_0$, every solution z of* (2.12) *satisfies*

$$\begin{aligned} &\iint_Q s^3\theta^3 x^2 z^2 e^{2s\varphi} \, dx\,dt + \iint_Q s\theta \left(z_x^2 - \mu\frac{z^2}{x^2} \right) e^{2s\varphi} \, dx\,dt + \iint_Q s\theta \frac{z^2}{x^\gamma} e^{2s\varphi} \, dx\,dt \\ &\quad \le C \left(\iint_Q g^2 e^{2s\varphi} \, dx\,dt + \int_0^T s\theta z_x^2(t,1) e^{2s\varphi(t,1)} \, dx\,dt \right). \end{aligned} \tag{2.16}$$

Observe that, if the term

$$\iint_Q s\theta \left(z_x^2 - \mu\frac{z^2}{x^2} \right) e^{2s\varphi} \, dx\,dt$$

is not positive, then the estimate (2.16) is not of great importance. In fact, the Hardy inequality (2.3) only ensures the positivity of the quantity

$$\iint_Q s\theta \left(z_x^2 - \mu\frac{z^2}{x^2} \right) dx\,dt.$$

However, from [44, Remark 3] and similarly as in [25], we will rewrite the result given in Lemma 2.2 in a more practical way.

Lemma 2.3. *Assume that $\mu \leq \frac{1}{4}$. Then, there exist $C > 0$ and $s_0 > 0$ such that, for all $s \geq s_0$, every solution z of (2.12) satisfies*

$$\mathfrak{I}_{\varphi,\eta,\gamma}(z) \leq C\left(\iint_Q g^2 e^{2s\varphi}\,dx\,dt + \int_0^T s\theta z_x^2(t,1)e^{2s\varphi(t,1)}\,dx\,dt\right), \tag{2.17}$$

where

$$\mathfrak{I}_{\varphi,\eta,\gamma}(z) = \iint_Q s^3\theta^3 x^2 z^2 e^{2s\varphi}\,dx\,dt + \iint_Q s\theta z_x^2 e^{2s\varphi}\,dx\,dt + \iint_Q s\theta \frac{z^2}{x^2} e^{2s\varphi}\,dx\,dt, \tag{2.18}$$

if $\mu < \frac{1}{4}$, and

$$\mathfrak{I}_{\varphi,\eta,\gamma}(z) = \iint_Q s^3\theta^3 x^2 z^2 e^{2s\varphi}\,dx\,dt + \iint_Q s\theta x^\eta z_x^2 e^{2s\varphi}\,dx\,dt + \iint_Q s\theta \frac{z^2}{x^\gamma} e^{2s\varphi}\,dx\,dt, \tag{2.19}$$

if $\mu = \frac{1}{4}$. We recall that $0 < \gamma < 2$.

Proof. **Case 1:** If $\mu < \frac{1}{4}$.

Let $Z = ze^{s\varphi}$. In order to prove [44, Theorem 5.1], the author has derived the following estimate

$$\begin{aligned}\iint_Q s^3\theta^3 x^2 Z^2\,dx\,dt + \iint_Q s\theta\left(Z_x^2 - \mu\frac{Z^2}{x^2}\right)dx\,dt + \iint_Q s\theta\frac{Z^2}{x^\gamma}\,dx\,dt \\ \leq C\left(\iint_Q g^2 e^{2s\varphi}\,dx\,dt + \int_0^T s\theta Z_x^2(t,1)\,dx\,dt\right).\end{aligned} \tag{2.20}$$

Let $\delta < \inf(1,(1-4\mu))$ be a fixed positive constant. We have

$$\begin{aligned}\iint_Q s\theta\left(Z_x^2 - \mu\frac{Z^2}{x^2}\right)dx\,dt = (1-\delta)\iint_Q s\theta\left(Z_x^2 - \frac{1}{4}\frac{Z^2}{x^2}\right)dx\,dt \\ + \delta\iint_Q s\theta Z_x^2\,dx\,dt + \left(\frac{1}{4}(1-\delta)-\mu\right)\iint_Q s\theta\frac{Z^2}{x^2}\,dx\,dt.\end{aligned} \tag{2.21}$$

By (2.20) and (2.21), we obtain

$$\begin{aligned}\iint_Q s^3\theta^3 x^2 Z^2\,dx\,dt + (1-\delta)\iint_Q s\theta\left(Z_x^2 - \frac{1}{4}\frac{Z^2}{x^2}\right)dx\,dt + \delta\iint_Q s\theta Z_x^2\,dx\,dt \\ + \left(\frac{1}{4}(1-\delta)-\mu\right)\iint_Q s\theta\frac{Z^2}{x^2}\,dx\,dt + \iint_Q s\theta\frac{Z^2}{x^\gamma}\,dx\,dt \\ \leq C\left(\iint_Q g^2 e^{2s\varphi}\,dx\,dt + \int_0^T s\theta Z_x^2(t,1)\,dx\,dt\right).\end{aligned}$$

On the other hand, from (2.15), for all $\eta > 0$ there exists a constant $c_0 = c_0(\eta) > 0$ such that

$$\iint_Q s\theta\left(Z_x^2 - \frac{1}{4}\frac{Z^2}{x^2}\right)dx\,dt \geq c_0\iint_Q s\theta x^\eta Z_x^2\,dx\,dt. \tag{2.22}$$

Hence,

$$\begin{aligned}\iint_Q s^3\theta^3 x^2 Z^2\,dx\,dt + (1-\delta)c_0\iint_Q s\theta x^\eta Z_x^2\,dx\,dt + \delta\iint_Q s\theta Z_x^2\,dx\,dt \\ + \left(\frac{1}{4}(1-\delta)-\mu\right)\iint_Q s\theta\frac{Z^2}{x^2}\,dx\,dt + \iint_Q s\theta\frac{Z^2}{x^\gamma}\,dx\,dt \\ \leq C\left(\iint_Q g^2 e^{2s\varphi}\,dx\,dt + \int_0^T s\theta Z_x^2(t,1)\,dx\,dt\right).\end{aligned} \tag{2.23}$$

Using the definition of Z, we have

$$Z^2 = z^2 e^{2s\varphi}, \tag{2.24}$$

$$Z_x = z_x e^{s\varphi} + s\theta\psi_x Z \quad \text{and} \quad z_x^2 e^{2s\varphi} \le 2Z_x^2 + cs^2\theta^2 x^2 Z^2, \tag{2.25}$$

for a positive constant c. Then,

$$\iint_Q s\theta z_x^2 e^{2s\varphi}\,dx\,dt \le 2\iint_Q s\theta Z_x^2\,dx\,dt + c\iint_Q s^3\theta^3 x^2 Z^2\,dx\,dt. \tag{2.26}$$

Combining (2.23)–(2.26), we obtain the desired estimate (2.17). Indeed, defining

$$a_0 = \min\left\{\frac{1}{1+c}, \frac{\delta}{2}, \left(\frac{1}{4}(1-\delta) - \mu\right)\right\} > 0,$$

we have

$$\begin{aligned}
& a_0\left(\iint_Q s^3\theta^3 x^2 z^2 e^{2s\varphi}\,dx\,dt + \iint_Q s\theta z_x^2 e^{2s\varphi}\,dx\,dt + \iint_Q s\theta\frac{z^2}{x^2}e^{2s\varphi}\,dx\,dt + \iint_Q s\theta\frac{z^2}{x^\gamma}e^{2s\varphi}\,dx\,dt\right)\\
&\le a_0\left((1+c)\iint_Q s^3\theta^3x^2Z^2\,dx\,dt + 2\iint_Q s\theta Z_x^2 dxdt + \iint_Q s\theta\frac{Z^2}{x^2}\,dx\,dt + \iint_Q s\theta\frac{Z^2}{x^\gamma}\,dx\,dt\right)\\
&\le \iint_Q s^3\theta^3x^2Z^2\,dx\,dt + \delta\iint_Q s\theta Z_x^2 dxdt + \left(\frac{1}{4}(1-\delta)-\mu\right)\iint_Q s\theta\frac{Z^2}{x^2}\,dx\,dt + \iint_Q s\theta\frac{Z^2}{x^\gamma}\,dx\,dt\\
&\le \iint_Q s^3\theta^3x^2Z^2\,dx\,dt + (1-\delta)c_0\iint_Q s\theta x^\eta Z_x^2\,dx\,dt + \delta\iint_Q s\theta Z_x^2\,dx\,dt\\
&\quad + \left(\frac{1}{4}(1-\delta)-\mu\right)\iint_Q s\theta\frac{Z^2}{x^2}\,dx\,dt + \iint_Q s\theta\frac{Z^2}{x^\gamma}\,dx\,dt\\
&\le C\left(\iint_Q g^2e^{2s\varphi}\,dx\,dt + \int_0^T s\theta Z_x^2(t,1)\,dx\,dt\right).
\end{aligned}$$

Thus, the conclusion follows.

Case 2: If $\mu = \frac{1}{4}$.

As before, let $Z = ze^{s\varphi}$ and define

$$a_0 = \min\left\{\frac{1}{1+c}, \frac{c_0}{2}\right\} > 0,$$

where c_0 and c are the constants of (2.22) and (2.25), respectively. Then, by (2.20), (2.22), (2.24) and (2.25), that still hold if $\mu = \frac{1}{4}$, we have

$$\begin{aligned}
& a_0\left(\iint_Q s^3\theta^3x^2z^2e^{2s\varphi}\,dx\,dt + \iint_Q s\theta x^\eta z_x^2 e^{2s\varphi}\,dx\,dt + \iint_Q s\theta\frac{z^2}{x^\gamma}e^{2s\varphi}\,dx\,dt\right)\\
&\le a_0\left(\iint_Q s^3\theta^3x^2Z^2\,dx\,dt + 2\iint_Q s\theta x^\eta Z_x^2\,dx\,dt + c\iint_Q s^3\theta^3x^2Z^2\,dx\,dt + \iint_Q s\theta\frac{Z^2}{x^\gamma}\,dx\,dt\right)\\
&\le a_0(1+c)\iint_Q s^3\theta^3x^2Z^2\,dx\,dt + a_0\frac{2}{c_0}\iint_Q s\theta\left(Z_x^2 - \frac{1}{4}\frac{Z^2}{x^2}\right)\,dx\,dt + a_0\iint_Q s\theta\frac{Z^2}{x^\gamma}\,dx\,dt\\
&\quad \text{(by (2.20))}\\
&\le C\left(\iint_Q g^2e^{2s\varphi}\,dx\,dt + \int_0^T s\theta z_x^2(t,1)e^{2s\varphi(t,1)}\,dx\,dt\right).
\end{aligned} \tag{2.27}$$

Hence, also in this case the conclusion follows.

□

We point out that the Carleman estimates stated above are not appropriate to achieve our goal. In fact, all these estimates does not have the observation term in the interior of the domain. However, we use them to obtain the main Carleman estimate stated in Proposition 2.5. More precisely, from the boundary Carleman estimates (2.17), we will deduce a global Carleman estimate for the adjoint problem (2.12) with a distributed observation on a subregion

$$\omega' := (\alpha', \beta') \subset\subset \omega. \tag{2.28}$$

To do so, we recall the following weight functions associated to nonsingular Carleman estimates which are suited to our purpose:

$$\Phi(t,x) := \theta(t)\Psi(x)$$

where θ is defined in (2.14) and $\Psi(x) = e^{\rho\sigma} - e^{2\rho\|\sigma\|_\infty}$. Here $\rho > 0$, $\sigma \in C^2([0,1])$ is such that $\sigma(x) > 0$ in $(0,1)$, $\sigma(0) = \sigma(1) = 0$ and $\sigma_x(x) \neq 0$ in $[0,1] \setminus \tilde{\omega}$, being $\tilde{\omega}$ an arbitrary open subset of ω.

In the following, we choose the constant $\mathfrak{c}$ in (2.14) so that

$$\mathfrak{c} \geq \frac{e^{2\rho\|\sigma\|_\infty} - 1}{d-1}.$$

By this choice one can prove that the function φ defined in (2.13) satisfies the next estimate

$$\varphi(t,x) \leq \Phi(t,x) \qquad \text{for every } (t,x) \in [0,T] \times [0,1]. \tag{2.29}$$

Thanks to this property, we can prove the main Carleman estimate of this paper whose proof is based also on the following Caccioppoli's inequality:

Proposition 2.4 (Caccioppoli's inequality). *Let ω' and ω'' be two nonempty open subsets of $(0,1)$ such that $\overline{\omega''} \subset \omega'$ and $\phi(t,x) = \theta(t)\varrho(x)$, where $\varrho \in C^2(\overline{\omega'}, \mathbb{R})$. Then, there exists a constant $C > 0$ such that any solution z of (2.12) satisfies*

$$\iint_{Q_{\omega''}} z_x^2 e^{2s\phi}\, dx\, dt \leq C \iint_{Q_{\omega'}} (g^2 + s^2\theta^2 z^2) e^{2s\phi}\, dx\, dt, \tag{2.30}$$

where $Q_\omega := (0,T) \times \omega$.

The proof of the previous result is similar to the one given, for instance, in [3, Lemma 6.1], so we omit it.

Now, we are ready to prove the following result:

Proposition 2.5. *Assume that $\mu \leq \frac{1}{4}$. Then, there exist two positive constants C and s_0 such that, the solution z of equation (2.12) satisfies, for all $s \geq s_0$*

$$\mathfrak{I}_{\varphi,\eta,\gamma}(z) \leq C \left(\iint_Q g^2 e^{2s\Phi}\, dx\, dt + \iint_{Q_{\omega'}} s^3\theta^3 z^2 e^{2s\Phi}\, dx\, dt \right). \tag{2.31}$$

Here $\mathfrak{I}_{\varphi,\eta,\gamma}(\cdot)$ is defined in (2.18) or (2.19).

Proof. Let us set $\omega'' = (\alpha'', \beta'') \subset\subset \omega'$ and consider a smooth cut-off function $\xi \in C^\infty([0,1])$ such that $0 \leq \xi(x) \leq 1$ for $x \in (0,1)$, $\xi(x) = 1$ for $x \in [0, \alpha'']$ and $\xi(x) = 0$ for $x \in [\beta'', 1]$. Define $w := \xi z$ where z is the solution of (2.12). Then, w satisfies the following problem:

$$\begin{cases} -w_t - w_{xx} - \dfrac{\mu}{x^2} w = \xi g - \xi_{xx} z - 2\xi_x z_x, & (t,x) \in Q, \\ w(t,1) = w(t,0) = 0, & t \in (0,T), \\ w(T,x) = \xi(x) z_T(x), & x \in (0,1). \end{cases} \tag{2.32}$$

First of all, we prove the first intermediate Carleman estimate for z in $(0,T)\times(0,\alpha')$ (recall that $z\equiv w$ in $[0,\alpha']$):

$$\begin{aligned}\mathfrak{I}_{\varphi,\eta,\gamma}(w) &\le C\left(\iint_Q \xi^2 g^2 e^{2s\varphi}\,dx\,dt + \iint_{Q_{\omega'}} (g^2+s^2\theta^2 z^2)e^{2s\varphi}\,dx\,dt\right)\\ &\le C\left(\iint_Q \xi^2 g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_{\omega'}} (g^2+s^2\theta^2 z^2)e^{2s\Phi}\,dx\,dt\right).\end{aligned} \tag{2.33}$$

The second inequality in (2.33) follows by (2.29), thus it is sufficient to prove the first inequality of (2.33). Applying the Carleman estimate (2.17) to (2.32), we obtain

$$\mathfrak{I}_{\varphi,\eta,\gamma}(w) \le C\iint_Q \left(\xi^2 g^2 + \left(\xi_{xx} z + 2\xi_x z_x\right)^2\right)e^{2s\varphi}\,dx\,dt. \tag{2.34}$$

From the definition of ξ and the Caccioppoli inequality (2.30), we obtain

$$\begin{aligned}\iint_Q \left(\xi_{xx} z + 2\xi_x z_x\right)^2 e^{2s\varphi}\,dx\,dt &\le C\iint_{Q_{\omega''}} (z^2+z_x^2)e^{2s\varphi}\,dx\,dt\\ &\le C\iint_{Q_{\omega'}} (g^2+s^2\theta^2 z^2)e^{2s\varphi}\,dx\,dt.\end{aligned} \tag{2.35}$$

Combining (2.34) and (2.35) we obtain (2.33).

Now, using the non singular Carleman estimate of Corollary 5.2, we are going to show a second estimate of z in $(0,T)\times(\beta',1)$. For this purpose, let $v=\zeta z$ where $\zeta := 1-\xi$ (hence $z\equiv v$ in $[\beta',1]$). Clearly, the function v is a solution of the uniformly parabolic equation

$$\begin{cases} -v_t - v_{xx} - \dfrac{\mu}{x^2} v = \zeta g - \zeta_{xx} z - 2\zeta_x z_x, & (t,x)\in(0,T)\times(\alpha',1),\\ v(t,1)=v(t,\alpha')=0, & t\in(0,T),\\ v(T,x)=\zeta(x) z_T(x), & x\in(\alpha',1).\end{cases} \tag{2.36}$$

Since ζ has its support in $[\alpha'',\beta'']$, by Corollary 5.2 we have

$$\begin{aligned}\iint_Q \left(s\theta v_x^2 + s^3\theta^3 v^2\right)e^{2s\Phi}\,dx\,dt &= \int_0^T\int_{\alpha'}^1 \left(s\theta v_x^2 + s^3\theta^3 v^2\right)e^{2s\Phi}\,dx\,dt\\ &\le C\left(\int_0^T\int_{\alpha'}^1 \left(\zeta^2 g^2 + \left(\zeta_{xx} z + 2\zeta_x z_x\right)^2\right)e^{2s\Phi}\,dx\,dt + \iint_{Q_{\omega''}} s^3\theta^3 v^2 e^{2s\Phi}\,dx\,dt\right)\\ &\le C\left(\iint_Q \zeta^2 g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_{\omega''}} (z^2+z_x^2)e^{2s\Phi}\,dx\,dt + \iint_{Q_{\omega''}} s^3\theta^3 v^2 e^{2s\Phi}\,dx\,dt\right).\end{aligned}$$

Therefore, by the previous estimate, by (2.29) and using the Caccioppoli inequality (2.30), we deduce

$$\begin{aligned}\iint_Q \left(s\theta v_x^2 + s^3\theta^3 v^2\right)e^{2s\varphi}\,dx\,dt &\le \iint_Q \left(s\theta v_x^2 + s^3\theta^3 v^2\right)e^{2s\Phi}\,dx\,dt\\ &\le C\left(\iint_Q \zeta^2 g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_{\omega'}} (g^2+s^3\theta^3 z^2)e^{2s\Phi}\,dx\,dt\right).\end{aligned} \tag{2.37}$$

Thus, since $v = \zeta z$ has its support in $[0, T] \times [\alpha'', 1]$, that is far away from the singularity point $x = 0$, one can prove that there exists a constant $C > 0$ such that:

$$\begin{aligned}\mathfrak{I}_{\varphi,\eta,\gamma}(v) &\leq C \iint_Q \left(s\theta v_x^2 + s^3\theta^3 v^2\right) e^{2s\varphi}\,dx\,dt \\ &\text{(by (2.37))} \\ &\leq C\left(\iint_Q \zeta^2 g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_{\omega'}} (g^2 + s^3\theta^3 z^2) e^{2s\Phi}\,dx\,dt\right).\end{aligned} \tag{2.38}$$

Note that

$$z^2 = (w+v)^2 \leq 2(w^2+v^2) \qquad \text{and} \qquad z_x^2 = (w_x+v_x)^2 \leq 2(w_x^2+v_x^2).$$

Therefore, adding (2.33) and (2.38), (2.31) follows immediately. □

For our purposes in the next section, we concentrate now on a Carleman inequality for solutions of (2.12) obtained via weight functions not exploding at $t = 0$. To this end, we will apply a classical argument that can be found, for instance, in [22] and recently in [1] for a degenerate parabolic equation with memory. More precisely, let us consider the function:

$$\nu(t) = \begin{cases} \theta\left(\dfrac{T}{2}\right), & t \in \left[0, \dfrac{T}{2}\right], \\ \theta(t), & t \in \left[\dfrac{T}{2}, T\right], \end{cases} \tag{2.39}$$

and the following associated weight functions:

$$\begin{aligned} &\tilde{\varphi}(t,x) := \nu(t)\psi(x), \qquad \tilde{\Phi}(t,x) := \nu(t)\Psi(x), \\ &\hat{\Phi}(t) := \max_{x\in[0,1]} \tilde{\Phi}(t,x), \qquad \hat{\varphi}(t) := \max_{x\in[0,1]} \tilde{\varphi}(t,x) \quad \text{and} \quad \check{\varphi}(t) := \min_{x\in[0,1]} \tilde{\varphi}(t,x).\end{aligned} \tag{2.40}$$

Now we are ready to state and prove this new modified Carleman estimate for the adjoint problem (2.12).

Lemma 2.6. *Assume that $\mu \leq \frac{1}{4}$. Then, there exist two positive constants C and s_0 such that every solution z of (2.12) satisfies, for all $s \geq s_0$*

$$\begin{aligned}&\|e^{s\hat{\varphi}(0)} z(0)\|^2_{L^2(0,1)} + \iint_Q \nu z^2 e^{2s\tilde{\varphi}}\,dx\,dt \\ &\quad \leq C e^{2s[\hat{\varphi}(0) - \check{\varphi}(\frac{5T}{8})]} \left(\iint_Q g^2 e^{2s\tilde{\Phi}}\,dx\,dt + \iint_{Q_\omega} s^3\nu^3 z^2 e^{2s\tilde{\Phi}}\,dx\,dt\right).\end{aligned} \tag{2.41}$$

Proof. By the definitions of ν and $\tilde{\varphi}$ and using Proposition 2.5, it results that there exists a positive constant C such that all the solutions to equation (2.12) satisfy

$$\begin{aligned}\int_{\frac{T}{2}}^{T}\!\!\int_0^1 \nu z^2 e^{2s\tilde{\varphi}}\,dx\,dt &= \int_{\frac{T}{2}}^{T}\!\!\int_0^1 \theta z^2 e^{2s\varphi}\,dx\,dt \leq C \int_{\frac{T}{2}}^{T}\!\!\int_0^1 s\theta \frac{z^2}{x^\gamma} e^{2s\varphi}\,dx\,dt \\ &\leq C\left(\iint_Q g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_{\omega'}} s^3\theta^3 z^2 e^{2s\Phi}\,dx\,dt\right).\end{aligned} \tag{2.42}$$

Let us introduce a function $\tau \in C^1([0,T])$ such that $\tau = 1$ in $\left[0, \frac{T}{2}\right]$ and $\tau \equiv 0$ in $\left[\frac{5T}{8}, T\right]$. Denote $\tilde{\tau} = e^{s\hat{\varphi}(0)}\sqrt{\nu}\tau$, where $e^{s\hat{\varphi}(0)} = \max_{0\leq t\leq T} e^{s\hat{\varphi}(t)}$.

Let $\tilde{z} = \tilde{\tau} z$, then $\tilde{z}$ satisfies

$$\begin{cases} -\tilde{z}_t - \tilde{z}_{xx} - \dfrac{\mu}{x^2}\tilde{z} = -\tilde{\tau}_t z + \tilde{\tau} g, & (t,x) \in Q, \\ \tilde{z}(t,0) = \tilde{z}(t,1) = 0, & t \in (0,T), \\ \tilde{z}(T,x) = 0, & x \in (0,1). \end{cases} \tag{2.43}$$

Thanks to the estimate of $\sup_{t\in[0,T]} \|\tilde{z}(t)\|^2_{L^2(0,1)}$ (see the energy estimate (2.4)), we have

$$\|\tilde{z}(0)\|^2_{L^2(0,1)} + \|\tilde{z}\|^2_{L^2(Q)} \le C \iint_Q (\tilde{\tau}_t z + \tilde{\tau} g)^2 \,dx\,dt,$$

which implies

$$\nu(0)\|e^{s\hat{\varphi}(0)} z(0)\|^2_{L^2(0,1)} + \|e^{s\hat{\varphi}(0)}\sqrt{\nu}\tau z\|^2_{L^2(Q)} \le C \iint_Q (\tilde{\tau}_t z + \tilde{\tau} g)^2 \,dx\,dt.$$

By using the boundedness of θ in $\left[\frac{T}{2}, \frac{5T}{8}\right]$, the definitions of τ and of ν in $\left[0, \frac{5T}{8}\right]$ and the fact that $\nu_t(t) = 0$ in $\left[0, \frac{T}{2}\right]$ and $\tau(t) = 0$ in $\left[\frac{5T}{8}, T\right]$, it holds that

$$\begin{aligned} &\bar{c}\left(\|e^{s\hat{\varphi}(0)} z(0)\|^2_{L^2(0,1)} + \int_0^{\frac{5T}{8}}\int_0^1 \nu\tau^2 z^2 e^{2s\hat{\varphi}} \,dx\,dt \right) \\ &\quad\le \nu(0)\|e^{s\hat{\varphi}(0)} z(0)\|^2_{L^2(0,1)} + \int_0^{\frac{5T}{8}}\int_0^1 \nu\tau^2 z^2 e^{2s\hat{\varphi}} \,dx\,dt \\ &\quad\le C\left(\int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 (\theta^2(t)+\theta(t)) z^2 e^{2s\hat{\varphi}(0)} \,dx\,dt + \int_0^{\frac{5T}{8}}\int_0^1 \nu g^2 e^{2s\hat{\varphi}(0)} \,dx\,dt \right) \\ &\quad\le C\left(\int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 z^2 e^{2s\hat{\varphi}(0)} \,dx\,dt + \int_0^{\frac{5T}{8}}\int_0^1 g^2 e^{2s\hat{\varphi}(0)} \,dx\,dt \right), \end{aligned}$$

where $\bar{c} := \min\{\nu(0), 1\}$. That is,

$$\begin{aligned} &\|e^{s\hat{\varphi}(0)} z(0)\|^2_{L^2(0,1)} + \int_0^{\frac{T}{2}}\int_0^1 \nu z^2 e^{2s\tilde{\varphi}} \,dx\,dt \\ &\quad\le C\left(\int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 z^2 e^{2s(\hat{\varphi}(0)-\tilde{\varphi})} e^{2s\tilde{\varphi}} \,dx\,dt + \int_0^{\frac{5T}{8}}\int_0^1 g^2 e^{2s(\hat{\varphi}(0)-\tilde{\varphi})} e^{2s\tilde{\varphi}} \,dx\,dt \right). \end{aligned}$$

Observe that

$$\check{\varphi}\left(\frac{5T}{8}\right) \le \tilde{\varphi} \quad \text{in } \left(0, \frac{5T}{8}\right) \times (0,1)$$

so that,

$$\begin{aligned} &\|e^{s\hat{\varphi}(0)} z(0)\|^2_{L^2(0,1)} + \int_0^{\frac{T}{2}}\int_0^1 \nu z^2 e^{2s\tilde{\varphi}} \,dx\,dt \\ &\quad\le C e^{2s(\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8}))}\left(\int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 z^2 e^{2s\tilde{\varphi}} \,dx\,dt + \int_0^{\frac{5T}{8}}\int_0^1 g^2 e^{2s\tilde{\varphi}} \,dx\,dt \right). \end{aligned} \tag{2.44}$$

As in (2.42), one can prove that there exists a positive constant C such that

$$\int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 z^2 e^{2s\tilde{\varphi}} \,dx\,dt \le C\left(\iint_Q g^2 e^{2s\Phi} \,dx\,dt + \iint_{Q_\omega} s^3\theta^3 z^2 e^{2s\Phi} \,dx\,dt \right).$$

Using this last inequality in (2.44), we have

$$\begin{aligned}
&\|e^{s\hat{\varphi}(0)}z(0)\|^2_{L^2(0,1)} + \int_0^{\frac{T}{2}}\int_0^1 \nu z^2 e^{2s\tilde{\varphi}}\,dx\,dt \\
&\quad \le Ce^{2s(\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8}))}\left(\iint_Q g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_\omega} s^3\theta^3 z^2 e^{2s\Phi}\,dx\,dt + \int_0^{\frac{5T}{8}}\int_0^1 g^2 e^{2s\tilde{\varphi}}\,dx\,dt\right). \quad (2.45)
\end{aligned}$$

From (2.29) and by the definition of the modified weights, notice that, in particular $\tilde{\varphi} \le \tilde{\Phi}$ and $\Phi \le \tilde{\Phi}$ in Q. This, together with (2.42) and (2.45), implies that

$$\begin{aligned}
&\|e^{s\hat{\varphi}(0)}z(0)\|^2_{L^2(0,1)} + \int_0^T\int_0^1 \nu z^2 e^{2s\tilde{\varphi}}\,dx\,dt \\
&\quad \le Ce^{2s(\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8}))}\left(\iint_Q g^2 e^{2s\tilde{\Phi}}\,dx\,dt + \iint_{Q_\omega} s^3\theta^3 z^2 e^{2s\Phi}\,dx\,dt\right). \quad (2.46)
\end{aligned}$$

To conclude, it suffices to remark that for $c > 0$, the function $s \mapsto s^3e^{-cs}$ is nonincreasing for s sufficiently large. So, since $\nu(t) \le \theta(t)$ by taking s large enough, one has

$$s^3\theta^3 e^{2s\Phi} \le s^3\nu^3 e^{2s\tilde{\Phi}},$$

which, together with (2.46), provides the desired inequality. □

2.3 Null controllability result

Following the classical method as in [22], with the modified Carleman inequality proved in the previous subsection, we can get a null controllability result for (2.1). However, as explained in [42], this null controllability result cannot help to solve the controllability for integro-differential equations. Indeed, we will need to prove the null controllability of the singular heat equation (2.1), for more regular solutions. For this reason, to formulate our results we introduce the following function space where the controllability will be solved:

$$X_s := \left\{y \in \mathcal{Z} : e^{-s\tilde{\Phi}}y \in L^2(Q)\right\}$$

equipped with the norm

$$\|y\|_{X_s} := \|e^{-s\tilde{\Phi}}y\|_{L^2(Q)}.$$

Observe that, since $\tilde{\Phi} < 0$, we have that the function $e^{-s\tilde{\Phi}}$ tends to $+\infty$ for $t \to T^-$. Therefore, $y \in X_s$ requires that the solution y has more regularity than the one in Lemma 2.1. Moreover,

$$\text{if } y \in X_s \text{ then } y(T,x) = 0 \text{ in } (0,1). \quad (2.47)$$

From now on, we denote by s_0 the parameter defined in Lemma 2.6. Our first result, stated as follows, ensures the null controllability for (2.1).

Theorem 2.7. *Assume that $\mu \le \frac{1}{4}$ and $y_0 \in H_0^{1,\mu}(0,1)$. If $e^{-s\tilde{\varphi}}f \in L^2(Q)$ with $s \ge s_0$, then there exists a control function $u \in L^2(Q)$, such that the associated solution y of (2.1) belongs to X_s.*

Moreover, there exists a positive constant C such that y satisfies the following estimate:

$$\begin{aligned}
&\iint_Q y^2 e^{-2s\tilde{\Phi}}\,dx\,dt + \iint_{Q_\omega} s^{-3}\nu^{-3}u^2 e^{-2s\tilde{\Phi}}\,dx\,dt \\
&\quad \le Ce^{2s[\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8})]}\left(\iint_Q f^2 e^{-2s\tilde{\varphi}}\,dx\,dt + \|y_0 e^{-s\hat{\varphi}(0)}\|^2_{L^2(0,1)}\right). \quad (2.48)
\end{aligned}$$

Proof. Following the ideas in [10,42], fixed $s \geq s_0$, let us consider the functional

$$J(y,u) = \left(\iint_Q y^2 e^{-2s\tilde{\Phi}}\,dx\,dt + \iint_{Q_\omega} s^{-3}\nu^{-3}u^2 e^{-2s\tilde{\Phi}}\,dx\,dt \right), \tag{2.49}$$

where (y,u) satisfies

$$\begin{cases} y_t - y_{xx} - \dfrac{\mu}{x^2}y = f + 1_\omega u(t), & (t,x) \in Q, \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), \quad y(T,x) = 0, & x \in (0,1), \end{cases} \tag{2.50}$$

with $u \in L^2(Q)$.

By means of standard arguments, it is easy to prove (see [34,35]) that J attains its minimizer at a unique point denoted as $(\bar{y},\bar{u})$.

We set

$$L_\mu y := y_t - y_{xx} - \frac{\mu}{x^2}y \quad \text{in } Q.$$

We will first prove that there exists a dual variable $\bar{z}$ such that

$$\begin{cases} \bar{y} = e^{2s\tilde{\Phi}}L_\mu^\star \bar{z}, & \text{in } Q, \\ \bar{u} = -s^3\nu^3 e^{2s\tilde{\Phi}}\bar{z}, & \text{in } (0,T)\times\omega, \\ \bar{z} = 0, & \text{on } (0,T)\times\{0,1\}, \end{cases} \tag{2.51}$$

where $L_\mu^\star$ is the (formally) adjoint operator of L_μ.

Let us start by introducing the following linear space

$$\mathcal{P}_0 = \{z \in C^\infty(\overline{Q}) : z = 0 \ \text{ on } \ (0,T)\times\{0,1\}\},$$

and introduce the bilinear form a:

$$a(z_1,z_2) = \iint_Q e^{2s\tilde{\Phi}}L_\mu^\star z_1 L_\mu^\star z_2\,dx\,dt + \iint_{Q_\omega} s^3\nu^3 e^{2s\tilde{\Phi}}z_1z_2\,dx\,dt, \qquad \forall\, z_1,z_2 \in \mathcal{P}_0.$$

Then, if the functions $\bar{y}$ and $\bar{u}$ given by (2.51) satisfy the parabolic problem (2.50), we must have

$$a(\bar{z},z) = \iint_Q fz\,dx\,dt + \int_0^1 y_0 z(0)\,dx, \qquad \forall\, z \in \mathcal{P}_0. \tag{2.52}$$

The key idea in this proof is to show that there exists exactly one $\bar{z}$ satisfying (2.52) in an appropriate class. We will then define $\bar{y}$ and $\bar{u}$ using (2.51) and we will check that the couple $(\bar{y},\bar{u})$ fulfills the desired properties.

Observe that the modified Carleman inequality (2.41) holds for all $z \in \mathcal{P}_0$. Consequently,

$$\|e^{s\hat{\varphi}(0)}z(0)\|^2_{L^2(0,1)} + \iint_Q \nu z^2 e^{2s\tilde{\varphi}}\,dx\,dt \leq Ce^{2s[\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8})]}a(z,z). \tag{2.53}$$

In particular, $a(\cdot,\cdot)$ is a strictly positive and symmetric bilinear form, that is, $a(\cdot,\cdot)$ is a scalar product in $\mathcal{P}_0$.

Denote by $\mathcal{P}$ the Hilbert space which is the completion of $\mathcal{P}_0$ with respect to the norm associated to $a(\cdot,\cdot)$ (which we denote by $\|\cdot\|_{\mathcal{P}}$). Let us now consider the linear form l, given by

$$l(z) = \iint_Q fz\,dx\,dt + \int_0^1 y_0 z(0)\,dx, \qquad \forall\, z \in \mathcal{P}.$$

By the Cauchy–Schwarz inequality and in view of (2.53), we have that

$$
\begin{aligned}
|l(z)| &\leq \left\| f\frac{e^{-s\tilde{\varphi}}}{\sqrt{\nu}} \right\|_{L^2(Q)} \|z\sqrt{\nu}e^{s\tilde{\varphi}}\|_{L^2(Q)} + \|y_0 e^{-s\hat{\varphi}(0)}\|_{L^2(0,1)} \|z(0)e^{s\hat{\varphi}(0)}\|_{L^2(0,1)} \\
&\leq Ce^{s[\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8})]} \left(\|fe^{-s\tilde{\varphi}}\|_{L^2(Q)} + \|y_0 e^{-s\hat{\varphi}(0)}\|_{L^2(0,1)} \right) \|z\|_{\mathcal{P}},
\end{aligned}
$$

and then l is a linear continuous form on $\mathcal{P}$. Hence, in view of Lax–Milgram's Lemma, there exists one and only one $\bar{z} \in \mathcal{P}$ satisfying

$$
a(\bar{z}, z) = l(z), \qquad \forall\, z \in \mathcal{P}. \tag{2.54}
$$

Moreover, we have

$$
\|\bar{z}\|_{\mathcal{P}} \leq Ce^{s[\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8})]} \left(\|fe^{-s\tilde{\varphi}}\|_{L^2(Q)} + \|y_0 e^{-s\hat{\varphi}(0)}\|_{L^2(0,1)} \right). \tag{2.55}
$$

Let us set

$$
\bar{y} = e^{2s\tilde{\Phi}} L_\mu^\star \bar{z} \quad \text{and} \quad \bar{u} = -1_\omega s^3 \nu^3 e^{2s\tilde{\Phi}} \bar{z}. \tag{2.56}
$$

With these definitions and by (2.55), it is easy to check that $\bar{y}$ and $\bar{u}$ satisfy

$$
\begin{aligned}
&\iint_Q \bar{y}^2 e^{-2s\tilde{\Phi}}\, dx\, dt + \iint_{Q_\omega} s^{-3}\nu^{-3}\bar{u}^2 e^{-2s\tilde{\Phi}}\, dx\, dt \\
&\qquad \leq Ce^{2s[\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8})]} \left(\|fe^{-s\tilde{\varphi}}\|^2_{L^2(Q)} + \|y_0 e^{-s\hat{\varphi}(0)}\|^2_{L^2(0,1)} \right),
\end{aligned} \tag{2.57}
$$

which implies (2.48).

It remains to check that $\bar{y}$ is the solution of (2.50) corresponding to $\bar{u}$. First of all, it is immediate that $\bar{y} \in X_s$ and $\bar{u} \in L^2(Q)$. Denote by $\tilde{y}$ the (weak) solution of (2.1) associated to the control function $u = \bar{u}$, then $\tilde{y}$ is also the unique solution of (2.1) defined by transposition. In other words, $\tilde{y}$ is the unique function in $L^2(Q)$ satisfying

$$
\iint_Q \tilde{y}h\, dx\, dt = \iint_Q 1_\omega \bar{u} z\, dx\, dt + \iint_Q fz\, dx\, dt + \int_0^1 y_0 z(0)\, dx, \qquad \forall\, h \in L^2(Q), \tag{2.58}
$$

where z is the solution to

$$
\begin{cases}
-z_t - z_{xx} - \frac{\mu}{x^2} z = h, & (t,x) \in Q, \\
z(t,0) = z(t,1) = 0, & t \in (0,T), \\
z(T,x) = 0 & x \in (0,1).
\end{cases}
$$

According to (2.54) and (2.56), we see that $\bar{y}$ also satisfies (2.58). Therefore, $\bar{y} = \tilde{y}$. Consequently, the control $\bar{u} \in L^2(\omega \times (0,T))$ drives the state $\bar{y} \in X_s$ exactly to zero at time T. $\square$

3 Singular heat equation with memory

Prior to null controllability is the well-posedness of problem (1.1). From the results in [23], we recall that in the nonsingular case ($\mu = 0$), it is well known that the heat operator with memory gives rise to well-posed Cauchy–Dirichlet problems. Likewise in [23], by an application of the Contraction Mapping Principle and invoking Theorem 2.1, we have that (1.1) is well-posed in the following sense:

Proposition 3.1. *Assume that $\mu \le \frac{1}{4}$. If $y_0 \in L^2(0,1)$ and $u \in L^2(Q)$, then there exists a unique solution y of* (1.1) *such that*

$$y \in C\big([0,T];L^2(0,1)\big) \cap L^2\big(0,T;H_0^{1,\mu}(0,1)\big).$$

Now, we pass to derive our main result, which concerns the null controllability of the singular heat equation with memory (1.1). Hence, in what follows, we assume that the function a satisfies

$$e^{\frac{4^k s\mathfrak{c}d}{T^k(T-t)^k}} a \in L^\infty((0,T)\times Q), \tag{3.1}$$

where $\mathfrak{c}, d, k$ are the constants defined in (2.14) and s is the same of Theorem 2.7.

Remark 3.2. It is worth mentioning that, from the results in Guerrero and Imanuvilov [24], it seems that the null controllability property of parabolic equations with memory may fail without any additional conditions on the kernel. On the other hand, observe that the condition (3.1) may appear as a quite strong restriction on the admissible function a, but it is a natural one. Indeed, the only thing that we are asking is its integrability with respect to the Carleman weight: it just restricts the function a very near T, which is due to the fact that the function ν blows up only at $t = T$ (see also [6]).

For our proof, we are going to employ a fixed point strategy. For $R > 0$, we define

$$X_{s,R} = \{w \in X_s : \|e^{-s\tilde{\Phi}} w\|_{L^2(Q)} \le R\},$$

which is a bounded, closed, and convex subset of $L^2(Q)$.

For any $w \in X_{s,R}$, let us consider the control problem

$$\begin{cases} y_t - y_{xx} - \dfrac{\mu}{x^2} y = \displaystyle\int_0^t a(t,r,x)w(r,x)\,dr + 1_\omega u, & (t,x) \in Q, \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1). \end{cases} \tag{3.2}$$

By Theorem 2.7 we first derive a null controllability result for (3.2); then, as a second step, we will obtain the same controllability result for (1.1) applying Kakutani's fixed point Theorem.

Our main result is thus the following.

Theorem 3.3. *Assume that $\mu \le \frac{1}{4}$. If the function a satisfies* (3.1)*, then for any $y_0 \in H_0^{1,\mu}(0,1)$, there exists a control function $u \in L^2(Q)$ such that the associated solution $y \in \mathcal{Z}$ of* (1.1) *satisfies*

$$y(T,\cdot) = 0 \quad \textit{in } (0,1). \tag{3.3}$$

Proof. Setting $C_0 := \frac{4^k \mathfrak{c}d}{T^k}$, by (3.1) and the estimate $e^{-s\tilde{\varphi}} \le e^{\frac{sC_0}{(T-t)^k}}$, we get that

$$\begin{aligned} \iint_Q \left(e^{-s\tilde{\varphi}} \int_0^t a(t,r,x)w(r,x)\,dr \right)^2 dx\,dt &\le C \iint_Q \int_0^t e^{\frac{2C_0 s}{(T-t)^k}} a^2(t,r,x)w^2(r,x)\,dr\,dx\,dt \\ &\le C \iint_Q w^2\,dx\,dt \le C \left(\sup_{(t,x)\in\overline{Q}} e^{2s\tilde{\Phi}}\right) \iint_Q e^{-2s\tilde{\Phi}} w^2\,dx\,dt \le CR^2 < +\infty. \end{aligned}$$

(recall that $w \in X_{s,R}$). Thus, the result in Theorem 2.7 holds for the equation (3.2), i.e. for any $y_0 \in H_0^{1,\mu}(0,1)$, there exists a control function $u \in L^2(Q)$ such that the associated solution y of (3.2) is in X_s and

$$y(T,\cdot) = 0 \quad \text{in } (0,1).$$

Let us now introduce, for every $w \in X_{s,R}$, the multivalued map

$$\Lambda : X_{s,R} \subset X_s \to 2^{X_s}$$

with

$$\Lambda(w) = \Big\{ y \in X_s : \text{ for some } u \in L^2(Q) \text{ satisfying} \\ \iint_{Q_\omega} s^{-3}\nu^{-3}u^2 e^{-2s\tilde{\Phi}}\,dx\,dt \le Ce^{2s[\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8})]}\left(R^2 + \int_0^1 y_0^2 e^{-2s\hat{\varphi}(0)}\,dx\right) \\ y \text{ solves } (3.2)\Big\}.$$

Observe that if $y \in \Lambda(w)$, then $y(T,\cdot) = 0$ in $(0,1)$ via (2.47).

To achieve our goal, it will suffice to show that Λ possesses at least one fixed point. To this purpose, we shall apply Kakutani's fixed point Theorem (see [10, Theorem 2.3]).

It is readily seen that $\Lambda(w)$ is a nonempty, closed and convex subset of $L^2(Q)$ for every $w \in X_{s,R}$. Then, we prove that $\Lambda(X_{s,R}) \subset X_{s,R}$ with sufficiently large $R > 0$. By (2.48) and condition (3.1), and arguing as before we have

$$\begin{aligned}
&\iint_Q y^2 e^{-2s\tilde{\Phi}}\,dx\,dt + \iint_{Q_\omega} s^{-3}\nu^{-3}u^2 e^{-2s\tilde{\Phi}}\,dx\,dt \\
&\quad \le Ce^{2s[\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8})]}\left(\iint_Q e^{-2s\tilde{\varphi}}\left(\int_0^t a(t,r,x)w(r,x)\,dr\right)^2 dx\,dt + e^{-2s\hat{\varphi}(0)}\int_0^1 y_0^2\,dx\right) \\
&\quad \le Ce^{2s[\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8})]}\left(\iint_Q w^2(t,x)\,dx\,dt + e^{-2s\hat{\varphi}(0)}\int_0^1 y_0^2\,dx\right) \\
&\quad \le Ce^{2s[\hat{\varphi}(0)-\check{\varphi}(\frac{5T}{8})]}\left(\sup_{(t,x)\in\overline{Q}} e^{2s\tilde{\Phi}}\right)\left(\iint_Q e^{-2s\tilde{\Phi}(t,x)}w^2(t,x)\,dx\,dt\right) + Ce^{-2s\check{\varphi}(\frac{5T}{8})}\int_0^1 y_0^2\,dx.
\end{aligned}$$

By virtue of $\hat{\varphi}(0) \le \hat{\Phi}(0)$ and $\tilde{\Phi} \le \hat{\Phi}(0)$ in Q, we get

$$\begin{aligned}
&\iint_Q y^2 e^{-2s\tilde{\Phi}}\,dx\,dt + \iint_{Q_\omega} s^{-3}\nu^{-3}u^2 e^{-2s\tilde{\Phi}}\,dx\,dt \\
&\quad \le Ce^{s[2\hat{\varphi}(0)-2\check{\varphi}(\frac{5T}{8})+2\hat{\Phi}(0)]}\iint_Q e^{-2s\tilde{\Phi}(t,x)}w^2(t,x)\,dx\,dt + Ce^{-2s\check{\varphi}(\frac{5T}{8})}\int_0^1 y_0^2\,dx \\
&\quad \le Ce^{s[4\hat{\Phi}(0)-2\check{\varphi}(\frac{5T}{8})]}R^2 + Ce^{-2s\check{\varphi}(\frac{5T}{8})}\int_0^1 y_0^2\,dx.
\end{aligned} \tag{3.4}$$

Now, choosing the constant $\mathfrak{c}$ (see (2.14)) in the interval

$$\left(\frac{e^{2\rho\|\sigma\|_\infty}-1}{d-1}, \frac{16}{15}\frac{e^{2\rho\|\sigma\|_\infty}-e^{\rho\|\sigma\|_\infty}}{d-1}\right),$$

which is not empty for ρ sufficiently large, we have

$$\begin{aligned}
2\hat{\Phi}(0) - \check{\varphi}\left(\frac{5T}{8}\right) &= \left(\frac{4}{T^2}\right)^k\left[2(e^{\rho\|\sigma\|_\infty} - e^{2\rho\|\sigma\|_\infty}) + \mathfrak{c}d\left(\frac{16}{15}\right)^k\right] \\
&< \left(\frac{4}{T^2}\right)^k\left(-2 + \frac{d}{d-1}\left(\frac{16}{15}\right)^{k+1}\right)(e^{2\rho\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty}).
\end{aligned}$$

Therefore, taking the parameters d and k defined in (2.14) in such a way that $d > 3$ and $2 < k < \frac{\ln(4/3)}{\ln(16/15)} - 1$, we infer that

$$2\hat{\Phi}(0) - \check{\varphi}(\frac{5T}{8}) < 0.$$

Hence for s sufficiently large, increasing the parameter s_0 if necessary, we obtain

$$\iint_Q y^2 e^{-2s\tilde{\Phi}}\,dx\,dt + \iint_{Q_\omega} s^{-3}\nu^{-3}u^2 e^{-2s\tilde{\Phi}}\,dx\,dt \le \frac{1}{2}R^2 + Ce^{-2s\check{\varphi}(\frac{5T}{8})}\int_0^1 y_0^2\,dx.$$

Then, for s and R large enough, we obtain

$$\iint_Q y^2 e^{-2s\tilde{\Phi}}\,dx\,dt \le R^2.$$

It follows that $\Lambda(X_{s,R}) \subset X_{s,R}$. Furthermore, let $\{w_n\}$ be a sequence of $X_{s,R}$. The regularity assumption on y_0 and Theorem 2.1, imply that the associated solutions $\{y_n\}$ are bounded in $H^1(0,T;L^2(0,1)) \cap L^2(0,T;D(A))$. Therefore, $\Lambda(X_{s,R})$ is a relatively compact subset of $L^2(Q)$ by the Aubin–Lions Theorem [41].

In order to conclude, we have to prove that Λ is upper-semicontinuous under the L^2 topology. First, observe that for any $w \in X_{s,R}$, we have at least $u \in L^2(Q)$ such that the corresponding solution $y \in X_{s,R}$. Hence, taking $\{w_n\}$ a sequence in $X_{s,R}$, we can find a sequence of controls $\{u_n\}$ such that the corresponding solutions $\{y_n\}$ is in $L^2(Q)$. Thus, let $\{w_n\}$ be a sequence satisfying $w_n \to w$ in $X_{s,R}$ and $y_n \in \Lambda(w_n)$ such that $y_n \to y$ in $L^2(Q)$. We must prove that $y \in \Lambda(w)$. For every n, we have a control $u_n \in L^2(Q)$ such that the system

$$\begin{cases} y_{n,t} - y_{n,xx} - \dfrac{\mu}{x^2}y_n = \displaystyle\int_0^t a(t,r,x)w_n(r,x)\,dr + 1_\omega u_n, & (t,x) \in Q, \\ y_n(t,0) = y_n(t,1) = 0, & t \in (0,T), \\ y_n(0,x) = y_0(x), & x \in (0,1) \end{cases} \tag{3.5}$$

has a least one solution $y_n \in L^2(Q)$ that satisfies

$$y_n(T,\cdot) = 0 \qquad \text{in} \quad (0,1).$$

From Theorem 2.1 and (3.4), it follows (at least for a subsequence) that

$$\begin{aligned} u_n \to u \quad & \text{weakly in } L^2(Q), \\ y_n \to y \quad & \text{weakly in } H^1(0,T;L^2(0,1)) \cap L^2(0,T;D(A)), \\ & \text{strongly in } C(0,T;L^2(0,1)). \end{aligned}$$

Passing to the limit in (3.5), we obtain a control $u \in L^2(Q)$ such that the corresponding solution y to (3.2) satisfies (3.3). This shows that $y \in \Lambda(w)$ and, therefore, the map Λ is upper-semicontinuous.

Hence, the multivalued map Λ possesses at least one fixed point, i.e., there exists $y \in X_{s,R}$ such that $y \in \Lambda(y)$. By the definition of Λ, this implies that there exists at least one pair (y,u) satisfying the conditions of Theorem 3.3. The uniqueness of y follows by Proposition 3.1. This ends the proof of Theorem 3.3. □

As a consequence of the previous theorem one has the next result.

Theorem 3.4. *Assume that $\mu \leq \frac{1}{4}$. If the function a satisfies (3.1), then for any $y_0 \in L^2(0,1)$, there exists a control function $u \in L^2(Q)$ such that the associated solution $y \in \mathcal{W}$ of (1.1) satisfies*

$$y(T,\cdot) = 0 \quad \text{in } (0,1).$$

Proof. Consider the following singular parabolic problem:

$$\begin{cases} w_t - w_{xx} - \dfrac{\mu}{x^2} w = \displaystyle\int_0^t a(t,r,x) w(r,x)\, dr, & (t,x) \in \left(0, \dfrac{T}{2}\right) \times (0,1), \\ w(t,0) = w(t,1) = 0, & t \in \left(0, \dfrac{T}{2}\right), \\ w(0,x) = y_0(x), & x \in (0,1), \end{cases}$$

where $y_0 \in L^2(0,1)$ is the initial condition in (1.1).

By Theorem 2.1, the solution of this system belongs to

$$\mathcal{W}\left(0, \frac{T}{2}\right) := L^2\left(0, \frac{T}{2}; H_0^{1,\mu}(0,1)\right) \cap C\left(\left[0, \frac{T}{2}\right]; L^2(0,1)\right).$$

Then, there exists $t_0 \in (0, \frac{T}{2})$ such that $w(t_0, \cdot) := \tilde{w}(\cdot) \in H_0^{1,\mu}(0,1)$.

Now, we consider the following controlled parabolic system:

$$\begin{cases} z_t - z_{xx} - \dfrac{\mu}{x^2} z = \displaystyle\int_0^t a(t,r,x) z(r,x)\, dr + 1_\omega h & (t,x) \in (t_0, T) \times (0,1), \\ z(t,0) = z(t,1) = 0, & t \in (t_0, T), \\ z(t_0, x) = \tilde{w}(x), & x \in (0,1). \end{cases}$$

We start by observing that, since Theorem 3.3 holds also in a general domain $(t_0, T) \times (0,1)$ with suitable changes, we can see that there exists a control function $h \in L^2((t_0,T) \times (0,1))$ such that the associated solution

$$z \in \mathcal{Z}(t_0, T) := L^2(t_0, T; D(A)) \cap H^1(t_0, T; L^2(0,1)) \cap C\left([t_0, T]; H_0^{1,\mu}(0,1)\right)$$

satisfies

$$z(T, \cdot) = 0 \quad \text{in } (0,1).$$

Finally, setting

$$y := \begin{cases} w, & \text{in } [0, t_0], \\ z, & \text{in } [t_0, T] \end{cases} \quad \text{and} \quad u := \begin{cases} 0, & \text{in } [0, t_0], \\ h, & \text{in } [t_0, T], \end{cases}$$

one can prove that $y \in \mathcal{W}$ is the solution to the system (1.1) corresponding to u and satisfies

$$y(T, \cdot) = 0 \quad \text{in } (0,1).$$

Hence, our assertion is proved. □

4 Conclusions and perspectives

In this work, we have addressed the problem of null controllability for a class of one dimensional heat equations with an inverse square potential and a memory type-kernel. Using Carleman-based techniques and a fixed point argument, we have proved that under suitable decaying conditions on the memory kernel, the null controllability of the system is ensured by means of a distributed control.

In what follows, we highlight a few possible directions related with the topics addressed in this work.

Memory-type null controllability of singular parabolic equation: This work addresses only the null controllability property for system (1.1). It would be of interest to consider the problem of memory-type controllability (see [12] for the corresponding definition). The goal is then not only to drive the solution to rest at some time-instant, but also to require the memory term to vanish at the same time, ensuring that the whole process reaches the equilibrium. In the spirit of previous results in [12, 31], it would be interesting to analyse this memory-type null controllability problem for system (1.1), provided the support of the control moves, covering the whole domain where the equation evolves.

Coupled singular parabolic systems with memory: Inspired by the results in [2, 26, 40], it would be quite interesting to consider the null controllability of coupled system of parabolic equations with singular potentials and memory effects, with less controls than equations (and ideally only one control if possible).

Degenerate and singular parabolic equation with memory: In [1], the null controllability for a one-dimensional degenerate heat equation is investigated. So, in this regard, following the method of proof used in this paper (see also [1]), we think that it is possible to combine the techniques in both papers and obtain a result for the degenerate/singular equation with memory-type kernel.

5 Appendix

In this section, we recall a classical Carleman estimate for the following nonsingular heat equation

$$\begin{cases} y_t - y_{xx} - by = f, & (t,x) \in Q, \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{5.1}$$

where $b \in L^\infty(Q)$ and $f \in L^2(Q)$.

Following [22], we introduce the weight functions

$$\tilde{\Phi}(t,x) := \theta(t)(e^{-\rho\sigma(x)} - e^{2\rho\|\sigma\|_\infty}) \quad \text{and} \quad \phi(t,x) := \theta(t)e^{\rho\sigma(x)},$$

where θ, ρ and σ are defined in Subsection 2.2. Then, [22, Lemma 1.2] gives the following.

Lemma 5.1. *There exists a positive constant ρ_0 such that for an arbitrary $\rho \geq \rho_0$ there exists $s_0(\rho_0) > 0$ such that for each $s \geq s_0(\rho_0)$ the solutions of (5.1) satisfy the inequality*

$$\iint_Q \left(s\phi v_x^2 + s^3\phi^3 v^2\right)(e^{2s\Phi} + e^{2s\tilde{\Phi}})\,dx\,dt$$
$$\leq C\left(\iint_Q f^2(e^{2s\Phi} + e^{2s\tilde{\Phi}})\,dx\,dt + \iint_{Q_\omega} s^3\phi^3 v^2(e^{2s\Phi} + e^{2s\tilde{\Phi}})\,dx\,dt\right).$$

Since $0 \le \sigma(x) \le \|\sigma\|_\infty$, one has $\tilde{\Phi} \le \Phi$ and $\theta(t) \le \phi(t,x) \le \theta(t)e^{\rho\|\sigma\|_\infty}$ for all $(t,x) \in Q$. Hence, from Lemma 5.1 one can easily deduce the following result.

Corollary 5.2. *There exists a positive constant ρ_0 such that for an arbitrary $\rho \ge \rho_0$ there exists $s_0(\rho_0) > 0$ such that for each $s \ge s_0(\rho_0)$ the solutions of (5.1) satisfy*

$$\iint_Q \left(s\theta v_x^2 + s^3\theta^3 v^2\right) e^{2s\Phi}\,dx\,dt \le C\left(\iint_Q f^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_\omega} s^3\theta^3 v^2 e^{2s\Phi}\,dx\,dt\right).$$

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **15**, 1–27; https://doi.org/10.14232/ejqtde.2021.1.15 www.math.u-szeged.hu/ejqtde/

Stability index of linear random dynamical systems

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Received 28 February 2020, appeared 19 March 2021
Communicated by Mihály Pituk

Abstract. Given a homogeneous linear discrete or continuous dynamical system, its stability index is given by the dimension of the stable manifold of the zero solution. In particular, for the n dimensional case, the zero solution is globally asymptotically stable if and only if this stability index is n. Fixed n, let X be the random variable that assigns to each linear random dynamical system its stability index, and let p_k with $k = 0, 1, \ldots, n$, denote the probabilities that $P(X = k)$. In this paper we obtain either the exact values p_k, or their estimations by combining the Monte Carlo method with a least square approach that uses some affine relations among the values $p_k, k = 0, 1, \ldots, n$. The particular case of n-order homogeneous linear random differential or difference equations is also studied in detail.

Keywords: stability index, random differential equations, random difference equations, random dynamical systems.

2020 Mathematics Subject Classification: 37H10, 34F05, 39A25, 37C75.

1 Introduction

Nowadays it is unnecessary to emphasize the importance of ordinary differential equations and discrete dynamical systems to model real world phenomena, from physics to biology, from economics to sociology. These dynamical systems, a concept that includes both continuous and discrete models (and even dynamic equations in time-scales), can have undetermined coefficients that in the case of real applications must be adjusted to fixed values that serve to make good predictions: this is known as the identification process. Once these coefficients are fixed we obtain a deterministic model.

In recent years some authors have highlighted the utility of considering random rather than deterministic coefficients to incorporate effects due to errors in the identification process,

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natural variability in some of the physical parameters, or as a method to treat and to incorporate uncertainties in the model, see [5,6,21] for examples coming from biological modeling and [11] for examples coming from mechanical systems.

In the same aim that inspires some works like [1,7,14], in this paper we focus on giving a statistical measure of the stability for both discrete and continuous linear dynamical systems,

$$\dot{\mathbf{x}} = A\,\mathbf{x} \quad \text{or} \quad \mathbf{x}_{k+1} = A\,\mathbf{x}_k, \tag{1.1}$$

where both $\mathbf{x}, \mathbf{x}_k \in \mathbb{R}^n$ and A is an $n \times n$ real matrix.

More concretely, in the continuous (resp. discrete) case we define the *stability index* of the origin, $s(A)$, as the number of eigenvalues, taking into account their multiplicities, of A with negative real part (resp. modulus smaller than 1). This index coincides with the dimension of the invariant stable manifold of the origin. Notice also that if $s(A) = n$ (resp. $s(A) = 0$) the origin is a global stable attractor (resp. a global unstable repeller).

In this work we study the probabilities p_k for a linear dynamic system (1.1) to have a given stability index k when the parameters of the matrix A are random variables with a given natural distribution. As we will see in Section 2, this distribution must be that all the elements of A are *independent and identically distributed* (i.i.d.) normal random variables with zero mean. We also will study the same question for linear n-th order differential equations and for linear difference equations.

We also remark that our results can be extrapolated to know a measure of the stability behaviour of critical or fixed points for general non-linear dynamical systems, because near them they can be written as

$$\dot{\mathbf{x}} = A\,\mathbf{x} + f(\mathbf{x}), \quad \text{or} \quad \mathbf{x}_{k+1} = A\,\mathbf{x}_k + f(\mathbf{x}_k),$$

with f being a non-linear term vanishing at zero. Moreover, while the situation where the origin is non-hyperbolic is negligible, in the complementary situation, the stability index of the linear part coincides with the dimension of the local stable manifold at the point.

In the continuous case, the key tool to know the stability index of a matrix is the Routh–Hurwitz criterion, see for instance [10, §15.715, p. 1076]. This approach allows to know the number of roots of a polynomial with negative real part in terms of algebraic inequalities among its coefficients. Similarly, its counterpart for the discrete case is called the Jury criterion. It is worth observing that in fact both are equivalent and it is possible to get one from the other by using a Möbius transformation that sends the left hand part of the complex plane into the complex ball of radius 1.

In all the cases, when we do not know how to compute analytically the true probabilities, we introduce a two step approach to obtain estimations of them:

- **Step 1:** We start using the celebrated Monte Carlo method. Recall that this computational algorithm relies on repeated random sampling and gives estimations of the true probabilities based on the law of large numbers and the law of iterated logarithm, see [2,3,13,18]. It is the case that using M samples this approach gives the true value with an absolute error of order $O\big(((\log\log M)/M)^{1/2}\big)$, which practically behaves as $O(M^{-1/2})$, where O stands for the usual Landau notation. In all our simulations we will work with $M = 10^8$, so our first approaches to the desired probabilities will have an asymptotic absolute error of order 10^{-4}. More detailed explanations of the sharpness of our estimations for this value of M are given in Section 3.2 by using the Chebyshev inequality and the Central limit theorem.

We have used the default in-built pseudo-random number generator in the `Statistics` package of Maple in our simulations*. This procedure use the Mersenne Twister method with period $2^{19937}-1$ to generate uniformly-distributed pseudo-random numbers, and then the Ziggurat method, which is a kind of rejection sampling algorithm, to obtain the normally-distributed pseudo-random numbers, see [16] and [17]. Observe that our sample size $M = 10^8$ is much smaller than the period of the pseudo-random number generator, which is greater that 10^{6001}.

- **Step 2:** Since the results of the plain Monte Carlo simulations do not satisfy certain linear constrains concerning the true probabilities, we propose to correct them by using a least squares approach. We take as final estimates of the true probabilities the least squares solution ([20, Def. 6.1]) of the inconsistent overdetermined system obtained when relative frequencies of the Monte Carlo simulation are forced to satisfy these linear constrains. See Section 3.2 for more details. We would like to remark that there are other options to improve plain Monte Carlo simulations like variance reduction and quasi-Monte Carlo methods [2, 13].

To have a flavour of the type of results that we will obtain we describe several consequences of some of our results for linear homogeneous differential or difference equations of order n with constant coefficients (see Sections 5 and 7). A first result is that in both cases the expected stability index is $n/2$. Moreover, let r_n denote the probability of the 0 solution to be a global stable attractor (stability index equals n) for them. Then, for differential equations, $r_n \le 1/2^n$. Furthermore, $r_1 = 1/2$, $r_2 = 1/4$, $r_3 = 1/16$ and our two step approach gives that $r_4 \simeq 0.00925$, $r_5 \simeq 0.00071$, and that r_k is smaller that 10^{-4} for bigger k. In the case of difference equations we prove that $r_1 = 1/2$ and $r_2 = \frac{1}{\pi}\arctan(\sqrt{2}) \simeq 0.304$.

2 A suitable probability space

In our approach, the starting point is to determine which is the natural choice of the probability space and the distribution law of the coefficients of the linear dynamical system. Only after this step is fixed we can ask for the probabilities of some dynamical features or some phase portraits.

For completeness, we start with some previous considerations and with an example, already considered in the literature, see [1,14,23]. Consider the planar linear differential system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.1}$$

where A, B, C, D are random variables, so we can set the sample space to be $\Omega = \mathbb{R}^4$. It is plausible to require that these real random variables are independent and identically distributed (i.i.d.) and continuous. Also, according to the *principle of indifference* (or principle of insufficient reason) [8], it would seem reasonable to impose that these variables were such that the random vector (A, B, C, D) had some kind of uniform distribution in $\mathbb{R}^4$. But there is no uniform distribution for unbounded probability spaces. Nevertheless, there is a natural choice for the distribution of the variables A, B, C and D.

Indeed, it is well-known that the phase portrait of the above system does not vary if we multiply the right-hand side of both equations by a positive constant (which corresponds to a

*Concretely, we use the commands `RandomVariable(Normal(0,1))` and `Sample`.

proportional change in the time scale). This means that in the space of parameters, $\mathbb{R}^4$, all the systems with parameters belonging to the same half-straight line passing through the origin are topologically equivalent and in particular have the same stability index. Hence, we can ask for a probability distribution density f of the coefficients such that the random vector

$$\left(\frac{A}{S},\frac{B}{S},\frac{C}{S},\frac{D}{S}\right), \quad \text{with} \quad S=\sqrt{A^2+B^2+C^2+D^2}, \tag{2.2}$$

has a uniform distribution on the sphere $\mathbb{S}^3\subset\mathbb{R}^4$. This achieves our objective, since $\mathbb{S}^3$ is a compact set.

The question is: which are the probability densities f that give rise to a uniform distribution of the vector (2.2) on the sphere? The answer is that, just assuming that f is continuous and positive, f must be the density of a normal random variable with zero mean. Moreover, this result is true for arbitrary dimension: see the next theorem. We remark that the converse result is well-known [15, 19].

Theorem 2.1. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. one-dimensional random variables with a continuous positive density function f. The random vector*

$$\left(\frac{X_1}{S},\frac{X_2}{S},\ldots,\frac{X_n}{S}\right), \quad \textit{with} \quad S=\left(\sum_{i=1}^{n}X_i^2\right)^{1/2},$$

has a uniform distribution in $\mathbb{S}^{n-1}\subset\mathbb{R}^n$ if and only if each X_i is a normal random variable with zero mean.

Curiously, in the case that we cannot assign uniform distributions, there is an extension of the indifference principle which suggests to use those distributions that maximize the entropy, i.e. the quantity $h(f)=-\int_\Omega f(x)\ln(f(x))\mathrm{d}x$ for any given density f. The one-dimensional random variables with continuous probability density function f on $\Omega=\mathbb{R}$ that maximize the entropy are again the Gaussian ones, [8, Thm 3.2].

Of course, if instead of properties concerning general dynamical systems one focuses on particular models in which the parameters have specific restrictions—due to physical or biological reasons—one must consider other type of distributions, see for instance [21].

Using Theorem 2.1, and going back to the initial motivating example, in order to study (2.1) we have to consider the probability space $(\Omega,\mathcal{F},P)$ where $\Omega=\mathbb{R}^4$, $\mathcal{F}$ is the σ-algebra generated by the open sets of $\mathbb{R}^4$ and $P:\mathcal{F}\to[0,1]$ is the probability function with density $\frac{1}{4\pi^2}\mathrm{e}^{-(a^2+b^2+c^2+d^2)/2}$, where for simplicity we take variance 1 in each marginal density function.

For instance, assume that we want to compute the probability α of system (2.1) to have exactly one eigenvalue with negative real part. Next, we observe that the probability of having one null eigenvalue is zero. This is because the event which characterizes this possibility is a subset of an event which is itself described by an algebraic equality between the random variables A,B,C,D. This subset has Lebesgue measure zero and therefore, by virtue of the fact that the joint distribution is continuous, the probability of this event, and therefore the event characterizing the null eigenvalue, must also be zero. Thus we have that α coincides with the probability of having a saddle (stability index 1) at the origin, i.e. $AD-BC<0$. Then, the open set $\mathcal{U}:=\{(a,b,c,d)\in\mathbb{R}^4 : ad-bc<0\}$ belongs to $\mathcal{F}$ and

$$\alpha=P(AD-BC<0)=\frac{1}{4\pi^2}\int_{\mathcal{U}}\mathrm{e}^{-\frac{a^2+b^2+c^2+d^2}{2}}\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d,$$

which is $1/2$ by symmetry, as we will see.

Proof of Theorem 2.1. Let $(X_1, \dots, X_n)$ be the random vector associated with the random variables of the statement, with joint continuous density function $g(x_1, \dots, x_n)$. We claim that

$$g(x_1, \dots, x_n) = h(x_1^2 + \cdots + x_n^2), \tag{2.3}$$

for some continuous function h.

Taking spherical coordinates, we consider the new random vector $(R, \Theta) \in \mathbb{R}^n$ where $R = (X_1^2 + \cdots + X_n^2)^{1/2}$ and $\Theta = (\Theta_1, \dots, \Theta_{n-1})$. We have $X_1 = R\cos\Theta_1$, $X_2 = R\sin\Theta_1\cos\Theta_2, \dots$ $X_{n-1} = R\sin\Theta_1\sin\Theta_2\cdots\sin\Theta_{n-2}\cos\Theta_{n-1}$ and $X_n = R\sin\Theta_1\sin\Theta_2\cdots\sin\Theta_{n-2}\sin\Theta_{n-1}$. By the change of variables theorem, the joint density function of (R, Θ) is

$$g_{R,\Theta}(r, \theta) = g(r\cos(\theta_1), \dots, r\sin(\theta_1)\cdots\sin(\theta_{n-1}))\, r^{n-1}\sin^{n-2}(\theta_1)\sin^{n-1}(\theta_2)\cdots\sin(\theta_{n-2})\cdot\chi$$

where $\theta = (\theta_1, \dots, \theta_{n-1})$, and

$$\chi := \chi_{[0,\infty)}(r)\cdot\chi_{[0,2\pi)}(\theta_{n-1})\cdot\prod_{i=1}^{n-2}\chi_{[0,\pi)}(\theta_i),$$

where χ_A stands for the characteristic function of the set A.

The density function of (R, Θ) conditioned to R, $g_{\Theta|R}$, is

$$g_{\Theta|R}(r, \theta) := \frac{g_{R,\Theta}(r, \theta)}{g_R(r)},$$

where $g_R(r)$ is the marginal density of R:

$$g_R(r) := \int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi} g(r\cos(\theta_1), \dots, r\sin(\theta_1)\cdots\sin(\theta_{n-1}))\,\mathrm{d}\mathcal{S},$$

where $\mathrm{d}\mathcal{S} = r^{n-1}\sin^{n-2}(\theta_1)\sin^{n-1}(\theta_2)\cdots\sin(\theta_{n-2})\mathrm{d}\theta_{n-1}\cdots\mathrm{d}\theta_1$ is the n-dimensional surface element in spherical coordinates.

To prove the statement, we need to characterize which are the joint density functions $g(x_1, \dots, x_n)$ such that when we fix $R = r$, the probability on the $(n-1)$-dimensional sphere of radius r, denoted by $\mathbb{S}^{n-1}(r)$, is uniformly distributed. In such a case the partial spherical segment $\Sigma_r = \{R = r, \theta_i \in [\alpha_i, \beta_i] \text{ for } i = 1, \dots, n-1\}$ must have probability $P(\Sigma_r) = \mathcal{S}(\Sigma_r)/\mathcal{S}(\mathbb{S}^{n-1}(r))$ where $\mathcal{S}$ denotes the surface area. Set $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ and $\beta = (\beta_1, \dots, \beta_{n-1})$. Notice that

$$\mathcal{S}(\Sigma_r) = \int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_{n-1}}^{\beta_{n-1}}\mathrm{d}\mathcal{S} = r^{n-1}A(\alpha, \beta)$$

where

$$A(\alpha, \beta) = \int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_{n-1}}^{\beta_{n-1}}\sin^{n-2}(\theta_1)\sin^{n-1}(\theta_2)\cdots\sin(\theta_{n-2})\mathrm{d}\theta_{n-1}\cdots\mathrm{d}\theta_1$$

and $\mathcal{S}(\mathbb{S}^{n-1}(r)) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}r^{n-1}$. Hence, on the one hand,

$$P(\Sigma_r) = \Gamma\left(\frac{n}{2}\right)\frac{A(\alpha, \beta)}{2\pi^{\frac{n}{2}}},$$

which does not depend on r. On the other hand,

$$P(\Sigma_r) = \int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_{n-1}}^{\beta_{n-1}} g_{\Theta|R}\,\mathrm{d}\theta$$

where $\mathrm{d}\theta = \mathrm{d}\theta_{n-1}\cdots \mathrm{d}\theta_2\mathrm{d}\theta_1$. This implies that

$$\int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_{n-1}}^{\beta_{n-1}} \frac{g(r\cos(\theta_1),\ldots,r\sin(\theta_1)\cdots\sin(\theta_{n-1}))\,r^{n-1}\sin^{n-2}(\theta_1)\sin^{n-1}(\theta_2)\cdots\sin(\theta_{n-2})\cdot\chi}{g_R(r)}\,\mathrm{d}\theta$$
$$= \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}\int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_{n-1}}^{\beta_{n-1}} \sin^{n-2}(\theta_1)\sin^{n-1}(\theta_2)\cdots\sin(\theta_{n-2})\mathrm{d}\theta,$$

for all $\alpha_i,\beta_i \in [0,\pi)$ for $i = 1,\ldots,n-2$ with $\alpha_i < \beta_i$ and $\alpha_{n-2},\beta_{n-2} \in [0,2\pi)$ with $\alpha_{n-2} < \beta_{n-2}$. This last equality implies that almost everywhere

$$\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} = \frac{g(r\cos(\theta_1),\ldots,r\sin(\theta_1)\cdots\sin(\theta_{n-1}))\,r^{n-1}}{g_R(r)},$$

and therefore $g(r\cos(\theta_1),\ldots,r\sin(\theta_1)\cdots\sin(\theta_{n-1}))$ is a function that only depends on r. In consequence, writing this fact in Cartesian coordinates, we get that almost everywhere $g(x_1,\ldots,x_n) = h(x_1^2+\cdots+x_n^2)$, for some continuous function h and the claim (2.3) follows.

Now we complete the proof. Since $X_1,\ldots,X_n$ are i.i.d. with positive density f, we know that $g(x_1,\ldots,x_n) = f(x_1)\cdots f(x_n)$. So equation (2.3) can be expressed as

$$f(x_1)\cdots f(x_n) \;=\; h(x_1^2+\cdots+x_n^2) \quad \text{for all} \quad (x_1,\ldots,x_n)\in\mathbb{R}^n$$

where h is a positive function. Taking $x_2 = \cdots = x_n = 0$ we have that $f(x_1)\,f(0)^{n-1} = h(x_1^2)$ and $h(0) = (f(0))^n > 0$. Thus,

$$f(x_1)\cdots f(x_n) = \frac{h(x_1^2)}{(f(0))^{n-1}}\cdots\frac{h(x_n^2)}{(f(0))^{n-1}} = \frac{h(x_1^2)}{(f(0))^{n}}\cdots\frac{h(x_n^2)}{(f(0))^{n}}\,(f(0))^n = h(x_1^2+\cdots+x_n^2).$$

Hence, using that $h(0) = (f(0))^n > 0$,

$$\frac{h(x_1^2)}{h(0)}\cdots\frac{h(x_n^2)}{h(0)} = \frac{h(x_1^2+\cdots+x_n^2)}{h(0)}.$$

Taking $H(\xi) := h(\xi)/h(0)$, and $u_i = x_i^2$, it holds that

$$H(u_1)\cdots H(u_n) \;=\; H(u_1+\cdots+u_n) \quad \text{with} \quad H(0) = 1. \tag{2.4}$$

Hence, $\varphi(u) = \log(H(u))$ is a continuous function that satisfies Cauchy's functional equation

$$\varphi(u_1)+\cdots+\varphi(u_n) \;=\; \varphi(u_1+\cdots+u_n) \quad \text{with} \quad \varphi(0) = 0.$$

It is well-known that all its continuous solutions are $\varphi(x) = ax$, for some $a\in\mathbb{R}$. Hence all continuous solutions of (2.4) are $H(x) = \mathrm{e}^{ax}$.

As a consequence, $f(x) = b\,\mathrm{e}^{ax^2}$ for some $(a,b)\in\mathbb{R}^2$. Since f is a density function, $a<0$. Moreover, using $\int_{-\infty}^{\infty} b\mathrm{e}^{ax^2}\mathrm{d}x = b\sqrt{-\pi/a} = 1$, and setting $a = -1/(2\sigma^2)$, we get that

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\mathrm{e}^{-\frac{x^2}{2\sigma^2}},$$

so each variable X_i is a normal random variable $\mathrm{N}(0,\sigma^2)$.

The converse part is straightforward and well-known [15,19]. □

Remark 2.2. The continuity condition for f in Theorem 2.1 is relevant since Equation (2.4) also admits non-continuous solutions that can be constructed, for instance, from non-continuous solutions of Cauchy's functional equation known for $n = 2$, see [12].

3 A preliminary result and methodology

We will investigate the probabilities of having a certain stability index for several linear dynamical systems with random coefficients. In particular we consider:

(a) Differential systems $\dot{\mathbf{x}} = A\,\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^n$ and A is a real constant $n \times n$ matrix,

(b) Homogeneous linear differential equation of order n with constant coefficients: $a_n x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0$,

(c) Linear discrete systems $b\,\mathbf{x}_{k+1} = A\,\mathbf{x}_k$ where $\mathbf{x}_k \in \mathbb{R}^n$, $b \in \mathbb{R}$; and A is a real constant $n \times n$ matrix,

(d) Linear homogeneous difference equation of order n with constant coefficients $a_n x_{k+n} + a_{n-1}x_{k+n-1} + \cdots + a_1 x_{k+1} + a_0 x_k = 0$.

Notice that in the four situations the behaviour of the dynamical systems does not change if we multiply all the involved constants by the same positive real number. This fact situates the four problems in the same context that the motivating example (2.1). Hence, following the results of Section 2, in all the cases, we may take the coefficients to be i.i.d. random normal variables with zero mean and variance 1.

Hence in every case we have a well-defined probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \mathbb{R}^m$, with $m = n^2, n+1, n^2+1$ or $n+1$ according we are in case (a), (b), (c) or (d), respectively, $\mathcal{F}$ is the σ-algebra generated by the open sets and for each $\mathcal{A} \in \mathcal{F}$,

$$P(\mathcal{A}) = \frac{1}{(\sqrt{2\pi})^m} \int_{\mathcal{A}} \mathrm{e}^{-||\mathbf{a}||^2/2} \mathrm{d}\mathbf{a}, \tag{3.1}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, $||\mathbf{a}||^2 = \sum_{j=1}^m a_j^2$ and $\mathrm{d}\mathbf{a} = \mathrm{d}a_1\,\mathrm{d}a_2 \ldots \mathrm{d}a_m$. For instance the matrices A appearing in case (a) and (c) are the so called *random matrices*.

The use of Routh–Hurwitz algorithm is a very useful tool to count the number of roots of a polynomial with negative real parts and it is implemented in many computer algebra systems. These conditions are given in terms of algebraic inequalities among the coefficients of the polynomials. Let us recall how to use it to count the number of roots with modulus less than one of a polynomial and, hence, to obtain the so called Jury conditions.

Given any polynomial $Q(\lambda) = q_n\lambda^n + q_{n-1}\lambda^{n-1} + \cdots + q_1\lambda + q_0$ with $q_j \in \mathbb{C}$, by using the conformal transformation $\lambda = \frac{z+1}{z-1}$, we get the associated polynomial

$$Q^\star(z) = q_n(z+1)^n + q_{n-1}(z+1)^{n-1}(z-1) + \cdots + q_0(z-1)^n. \tag{3.2}$$

It is straightforward to observe that $\lambda_0 \in \mathbb{C}$ is a root of of $Q(\lambda)$ such that $|\lambda_0| < 1$ if and only if $z_0 = (\lambda_0+1)/(\lambda_0-1)$ is a root of $Q^\star(z)$ such that $\mathrm{Re}(z_0) < 0$.

Hence, because Routh–Hurwitz and Jury conditions are semi-algebraic, in every case the random variable X that assigns to each dynamical system its stability index $k, 0 \leq k \leq n$, is measurable. Hence $\mathcal{A}_k := \{\mathbf{a} \in \mathbb{R}^m : X(\mathbf{a}) = k\} \in \mathcal{F}$ and its probability $p_k := P(\mathcal{A}_k)$ is well-defined. Observe also that the non-hyperbolic cases are totally negligible because in their characterization some algebraic equalities appear. In this paper we will either calculate or estimate in the four situations the values p_k for $k \leq 10$.

3.1 A preliminary result

In three of the above considered cases we will apply the following auxiliary result:

Lemma 3.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $Y : \Omega \to \mathbb{R}$ be a discrete random variable with image* $\mathrm{Im}(Y) = \{0, 1, \ldots, n\}$*, and probability mass function $p_k = P(Y = k)$ such that $p_k = p_{n-k}$ for all $k = 0, \ldots, n$. Then $E(Y) = \sum_{k=0}^{n} kp_k = n/2$. Moreover*

(a) If n is odd then $2\sum_{k=0}^{\frac{n-1}{2}} p_k = 1$. In particular, when $n = 1$, $p_0 = p_1 = \frac{1}{2}$.

(b) If n is even and $n \geq 2$ then $2\sum_{k=0}^{\frac{n}{2}-1} p_k + p_{\frac{n}{2}} = 1$.

If, additionally, n is even[**] *and $\sum_{i\,\mathrm{odd}} p_i = \sum_{i\,\mathrm{even}} p_i = \frac{1}{2}$, then*

(c) If $\frac{n}{2}$ is even, then $2\sum_{k=0,\,k\,\mathrm{even}}^{\frac{n}{2}-2} p_k + p_{\frac{n}{2}} = \frac{1}{2}$, and $2\sum_{k=1,\,k\,\mathrm{odd}}^{\frac{n}{2}-1} p_k = \frac{1}{2}$. In particular, when $n = 4$, $p_1 = p_3 = \frac{1}{4}$, $p_2 = \frac{1}{2} - 2p_0$ and $p_4 = p_0$.

(d) If $\frac{n}{2}$ is odd, then $2\sum_{k=0,\,k\,\mathrm{even}}^{\frac{n}{2}-1} p_k = \frac{1}{2}$, and $\sum_{k=1,\,k\,\mathrm{odd}}^{\frac{n}{2}-2} p_k + p_{\frac{n}{2}} = \frac{1}{2}$. In particular, when $n = 2$, $p_0 = p_2 = \frac{1}{4}$ and $p_1 = \frac{1}{2}$.

Proof. We start proving that $E(Y) = n/2$. Assume for instance that n is odd. Since $p_k = p_{n-k}$, its holds that $kp_k + (n-k)p_{n-k} = np_k$, for each $k \leq (n-1)/2$. Hence,

$$\begin{aligned} E(Y) &= np_0 + np_1 + \cdots + np_{\frac{n-1}{2}} = \frac{n}{2}\left(2p_0 + 2p_1 + \cdots + 2p_{\frac{n-1}{2}}\right) \\ &= \frac{n}{2}\big((p_0 + p_n) + (p_1 + p_{n-1}) + \cdots + (p_{\frac{n-1}{2}} + p_{\frac{n+1}{2}})\big) = \frac{n}{2}. \end{aligned}$$

When n is even the proof is similar.

The proof of all the four items is straightforward and we omit it. □

3.2 Experimental methodology

In every case considered in the paper, when we can not give an exact value of the probabilities p_k we start estimating them by using the *Monte Carlo* method, see [18]. The estimates obtained (namely, the observed relative frequencies) are then improved via the *least squares* method, by using the linear constraints given in Corollaries 4.2, 5.2 and 7.4.

In every case we will use Monte Carlo method with $M = 10^8$ to obtain an estimation, say $\widetilde{p}$, for a probability $p := P(\mathcal{A})$ like the one given in equality (3.1) for different measurable sets $\mathcal{A}$. Further details for each concrete situation are given in each of the following subsections.

In brief, recall that $\widetilde{p}$ is given by the proportion of samples that are in $\mathcal{A}$. For studying, for a given M, how close are p and $\widetilde{p}$, let $B_j, j = 1, \ldots, M$ be i.i.d. Bernoulli random variables, where each one of them that takes the value 1 with probability p and the value 0 with probability $1 - p$.

Define $P_M = \frac{1}{M}\sum_{j=1}^{M} B_j$. Then, the value obtained for the random variable P_M, $\widetilde{p}$ is the approximation of p given by the Monte Carlo method. Let us see, by using Chebyshev inequality or the Central limit theorem, that with very high probability, $\widetilde{p}$ is a good approximation of p.

Notice first that $\mathrm{E}(P_M) = p$ and due to the independence of the B_j,

$$\mathrm{Var}(P_M) = \mathrm{Var}\left(\frac{1}{M}\sum_{j=1}^{M} B_j\right) = \frac{1}{M^2} M\,\mathrm{Var}(B_1) = \frac{p(1-p)}{M} \leq \frac{1}{4M}$$

[**] When n is odd the imposed equalities automatically hold.

because $p(1-p) \leq 1/4$. Recall also that for each $\varepsilon > 0$ and any random variable X, with $\mathrm{E}(X^2) < \infty$, the Chebyshev inequality reads as

$$P\left(|X - \mathrm{E}(X)| < \varepsilon\right) \geq 1 - \frac{\mathrm{Var}(X)}{\varepsilon^2}.$$

Hence, applying the Chebyshev inequality to $X = P_M$ we get that

$$P\left(|P_M - p| < \varepsilon\right) \geq 1 - \frac{p(1-p)}{M\varepsilon^2} \geq 1 - \frac{1}{4M\varepsilon^2}.$$

Taking $M = 10^8$, as in our computations, denoting $\widetilde{p} = P_{10^8}$, and considering $\varepsilon = 10^{-3}$ we get that the above probability gives the following conservative estimate of the reliability of the method

$$P\left(|\widetilde{p} - p| < 10^{-3}\right) \geq 1 - \frac{1}{400} = \frac{399}{400} = 0.9975.$$

Let us see, by using the Central limit theorem, that the above probability seems to be much bigger. By this theorem we know that for M big enough, and $p(1-p)M$ also big enough, the distribution of the random variable

$$\frac{P_M - \mathrm{E}(P_M)}{\sqrt{\mathrm{Var}(P_M)}} = \frac{P_M - p}{\sqrt{\frac{p(1-p)}{M}}}$$

can be practically considered to be a random variable Z with distribution $\mathrm{N}(0,1)$. In fact, in Statistics it is usually imposed that $p(1-p)M > 18$. Hence, unless p is very close to 0 or 1, the value $M = 10^8$ is big enough. Hence

$$\begin{aligned} P\left(|P_M - p| < \varepsilon\right) &= P\left(\frac{\sqrt{M}|P_M - p|}{\sqrt{p(1-p)}} < \frac{\varepsilon\sqrt{M}}{\sqrt{p(1-p)}}\right) \simeq P\left(|Z| < \frac{\varepsilon\sqrt{M}}{\sqrt{p(1-p)}}\right) \\ &= 2\Phi\left(\frac{\varepsilon\sqrt{M}}{\sqrt{p(1-p)}}\right) - 1 > 2\Phi\left(2\varepsilon\sqrt{M}\right) - 1, \end{aligned}$$

where Φ is the distribution function of a $\mathrm{N}(0,1)$ random variable. Taking again $M = 10^8$ and either $\varepsilon = 10^{-3}$ or $\varepsilon = 2 \times 10^{-4}$ we get

$$P\left(|\widetilde{p} - p| < 10^{-3}\right) \gtrsim 2\Phi\left(20\right) - 1 > 1 - 10^{-88}$$

or

$$P\left(|\widetilde{p} - p| < 2 \times 10^{-4}\right) \gtrsim 2\Phi\left(4\right) - 1 > 0.99993.$$

In fact, for instance looking at the values p_k of Table 4.1 for $n = 2$ in Section 4, that can also be obtained analytically, we get that $|\widetilde{p}_k - p_k| \leq 6 \times 10^{-5}$, for $k = 0, 1, 2$. So, the actual bound is smaller that the bounds obtained above.

Finally, to illustrate how the error decays when the sample size increases, we show the evolution of the errors in one case where the true probabilities are known. We consider the second order difference equation $A_2 x_{k+2} + A_1 x_{k+1} + A_0 x_k = 0$ where A_i are i.i.d. random variables with $\mathrm{N}(0,1)$ distribution. The stability index is given by the number of zeroes with modulus smaller than 1 of the characteristic polynomial $Q(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$. Let X be the random variable that counts the number of roots with modulus smaller than 1 of $Q(\lambda)$, and $p_k = P(X = k)$ for $k = 0, 1, 2$. The true value of the probabilities p_k is obtained in Corollary 7.4.

Performing Monte Carlo simulations with $M = 10^m$ with $m = 2, \ldots, 10$ we obtain the observed frequencies $\widetilde{p}_2(m)$ shown in Table 3.1. These frequencies are the estimated probabilities for the origin to be asymptotically stable. Notice that in Proposition 7.3 and in Corollary 7.4 we prove that $p_0 = p_2 = \arctan(\sqrt{2})/\pi$ and, of course, $p_1 = 1 - p_0 - p_2 = 2\arctan(1/\sqrt{2})/\pi$. For $M = 10^m$ we denote the absolute error $e_m = |\widetilde{p}_2(m) - p_2|$:

$M = 10^2$	$M = 10^3$	$M = 10^4$
$\widetilde{p}_2(2) = 0.37$	$\widetilde{p}_2(3) = 0.319$	$\widetilde{p}_2(4) = 0.3102$
$e_2 \approx 0.065913276015$	$e_3 \approx 0.014913276015$	$e_4 \approx 0.006113276015$
$M = 10^5$	$M = 10^6$	$M = 10^7$
$\widetilde{p}_2(5) = 0.30416$	$\widetilde{p}_2(6) = 0.303892$	$\widetilde{p}_2(7) = 0.3041241$
$e_5 \approx 0.000073276015$	$e_6 \approx 0.000194723985$	$e_7 \approx 0.000037376015$
$M = 10^8$	$M = 10^9$	$M = 10^{10}$
$\widetilde{p}_2(8) = 0.30406079$	$\widetilde{p}_2(9) = 0.304076699$	$\widetilde{p}_2(10) = 0.304079098$
$e_8 \approx 0.000025933985$	$e_9 \approx 0.000010024985$	$e_{10} \approx 0.000007625985$

Table 3.1: Observed frequency and absolute error of p_2 for second order difference equations, using that $p_2 = \arctan(\sqrt{2})/\pi \approx 0.304086723985$.

With the above results, the regression line of $Y = \log(e_m)$ versus $X = \log(M) = m\log(10)$ is $Y = -0.505\,X - 1.260$ with $R^2 = 0.893$. The slope is therefore $-0.505 \approx -1/2$ as was expected a priori since, in practice, the absolute error behaves as $O(M^{-1/2})$ as $M \to \infty$ (see the Step 2 in the Introduction).

A more detailed explanation of the second step, about the improvement of the Monte Carlo estimations using the least squares method, is as follows: the probabilities p_k satisfy some affine relations, like the ones in Lemma 3.1 or the ones in Proposition 7.3 below. Then, if we denote $\mathbf{p} = (p_0, \ldots, p_n)^t \in \mathbb{R}^{n+1}$ it is possible to write $\mathbf{p} = \mathrm{M}\mathbf{q} + b$ where $\mathbf{q} \in \mathbb{R}^k$ with $k \leq n$ is a vector whose components are different elements of $p_0, \ldots, p_n$; $\mathrm{M} \in \mathcal{M}_{n\times k}(\mathbb{R})$; and $b \in \mathbb{R}^k$. Let $\widetilde{\mathbf{p}} = (\widetilde{p}_0, \ldots, \widetilde{p}_n)^t$ be the vector with the estimated probabilities obtained by the observed relative frequencies using the Monte Carlo method. Then, we can find the least squares solution [20, Def. 6.1] of the the system,

$$\widetilde{\mathbf{p}} = \mathrm{M}\widehat{\mathbf{q}} + b, \tag{3.3}$$

which is

$$\widehat{\mathbf{q}} = (\mathrm{M}^t \cdot \mathrm{M})^{-1} \cdot \mathrm{M}^t \cdot (\widetilde{\mathbf{p}} - b), \tag{3.4}$$

see [9, Sect. 5.7, p. 198] or [22, p. 200]. So we can find some improved estimations $\widehat{\mathbf{p}}$, via

$$\widehat{\mathbf{p}} = \mathrm{M}\widehat{\mathbf{q}} + b. \tag{3.5}$$

Some detailed examples are given in Sections 4, 5 and 7.

4 Linear random differential systems

Consider linear differential systems $\dot{\mathbf{x}} = A\,\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, where A is a random matrix whose entries are i.i.d. random variables with $\mathrm{N}(0,1)$ distribution. Let X be the random variable that counts the number of eigenvalues of A with negative real part, $s(A)$.

Proposition 4.1. *With the above notations, set $p_k = P(X = k)$. The following holds:*

(a) $\sum_{k=0}^{n} p_k = 1$.

(b) For all $k \in \{0, 1, \ldots, n\}$, $p_k = p_{n-k}$.

(c) $\sum_{i\,\text{odd}} p_i = \sum_{i\,\text{even}} p_i = \frac{1}{2}$.

Proof. The assertion (a) is trivial. To prove (b) we observe that if a matrix A has k eigenvalues with negative real part, then $B = -A$ has $n - k$ eigenvalues with negative real part. Calling q_m the probability that B has m eigenvalues with negative real part, we get that $p_m = q_m$. This is so, because if $X \sim \mathrm{N}(0, 1)$ then $-X \sim \mathrm{N}(0, 1)$ and as a consequence the entries of A and B have the same distribution. Then, $q_k = p_{n-k}$ and the result follows.

To see (c) we claim that $s(A)$ is even if and only if the determinant of A is positive and, moreover, $P(\det(A) > 0) = 1/2$. From this claim we get the result because $\sum_{i\,\text{even}} p_i$ is the probability of $s(A)$ being even. To prove the first part of the claim notice first that we can assume that $0 \neq \det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n eigenvalues of A. We write $\lambda_1 \lambda_2 \cdots \lambda_n = (\lambda_1 \lambda_2 \cdots \lambda_k)(\lambda_{k+1} \lambda_{k+2} \cdots \lambda_n)$ where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all the real negative eigenvalues. Observe also that for complex eigenvalues $\lambda \bar{\lambda} > 0$. Hence $\lambda_{k+1} \lambda_{k+2} \cdots \lambda_n > 0$, $\mathrm{sign}(\det(A)) = (-1)^k$ and the condition that $s(A)$ is even is characterized by $\det(A) > 0$. To prove that $P(\det(A) > 0) = 1/2$ note that if B is the matrix obtained by changing the sign of one column of A then $\det(A) \cdot \det(B) < 0$ and hence $P(\det(A) < 0) = P(\det(B) > 0)$. Furthermore, since the entries of A and B have the same distribution we have $P(\det(B) > 0) = P(\det(A) > 0)$, thus $P(\det(A) < 0) = P(\det(A) > 0) = 1/2$. □

From the above proposition it easily follows:

Corollary 4.2. *Consider* $\dot{\mathbf{x}} = A\,\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$ *with* $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ *a random matrix with i.i.d.* $\mathrm{N}(0, 1)$ *entries, let X be the random variable defined above and $p_k = P(X = k)$. Then the probabilities p_k satisfy all the consequences of Lemma 3.1. In particular $E(X) = n/2$.*

Now we reproduce some experiments to estimate the probabilities p_k for low dimensional cases. We apply the Monte Carlo method, that is, for each considered dimension n, we have generated 10^8 matrices $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ whose entries are pseudo-random numbers simulating the realizations on n^2 independent random variables with $\mathrm{N}(0, 1)$ distribution. For each matrix A we have computed the characteristic polynomial, and counted the number of eigenvalues with negative real part by using the Routh–Hurwitz zeros counter [10, §15.715, p. 1076]. We are aware that the stability of the calculation of the coefficients of the characteristic polynomial from the entries of a matrix is critical (see [22, pp. 378–379] and references therein); however we have only used this calculation for low dimensions, namely $n \leq 4$. For $n \geq 5$, and in order to decrease the computation time, we have directly computed numerically the eigenvalues of A and counted the number of them with negative real part.

For each considered dimension of the phase space n, and in order to take advantage of the relations stated in Corollary 4.2, we can refine the solutions using the least squares solutions of the inconsistent linear system associated with these relations when using the observed frequencies obtained by the Monte Carlo simulation.

We give details of one example. Set $n = 7$, for instance. By Corollary 4.2 we have $p_3 = p_4 = \frac{1}{2} - p_0 - p_1 - p_2$; $p_5 = p_2$; $p_6 = p_1$ and $p_7 = p_0$. So, using the notation introduced in

Section 3.2, we can write $\mathbf{p} = \mathrm{M}\mathbf{q} + b$, where $\mathbf{p}^t = (p_0, \ldots, p_7)$;

$$\mathrm{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \mathbf{q} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}; \text{ and } b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The observed relative frequencies in our Monte Carlo simulation are

$$\widetilde{\mathbf{p}}^t = \left(\frac{31643}{50000000}, \frac{261137}{12500000}, \frac{7124967}{50000000}, \frac{1344047}{4000000}, \frac{33597117}{100000000}, \frac{14248187}{100000000}, \frac{1043913}{50000000}, \frac{63379}{100000000} \right).$$

By finding the least squares solution of the system (3.3) ([9, Sect. 5.7, p. 198] or [22, p. 200]), given by (3.5), we obtain

$$\widehat{\mathbf{p}}^t = \left(\frac{25333}{40000000}, \frac{2088461}{100000000}, \frac{28498121}{200000000}, \frac{16799573}{50000000}, \frac{16799573}{50000000}, \frac{28498121}{200000000}, \frac{2088461}{100000000}, \frac{25333}{40000000} \right).$$

The other cases follow similarly.

We summarize the results of our experiments in the Table 4.1, where the observed relative frequencies and the estimates are presented only up to the fifth decimal (in the table, and in the whole paper, frequency stands for relative frequency) because as we already explained in the introduction, the predicted absolute error will be of order 10^{-4}. Observe that in the cases $n = 1, 2$ the true probabilities are known. We include the results of the Monte Carlo simulations for completeness, but it makes no sense to apply the least squares method.

Dimension	Observed frequency	Least squares	Relations (Corol. 4.2)
$n = 1$	$\widetilde{p}_0 = 0.49996$ $\widetilde{p}_1 = 0.50004$		$p_0 = 0.5$ $p_1 = 0.5$
$n = 2$	$\widetilde{p}_0 = 0.24999$ $\widetilde{p}_1 = 0.50006$ $\widetilde{p}_2 = 0.24995$		$p_0 = 0.25$ $p_1 = 0.5$ $p_2 = 0.25$
$n = 3$	$\widetilde{p}_0 = 0.10447$ $\widetilde{p}_1 = 0.39542$ $\widetilde{p}_2 = 0.39557$ $\widetilde{p}_3 = 0.10454$	$\widehat{p}_0 = 0.10450$ $\widehat{p}_1 = 0.39550$ $\widehat{p}_2 = 0.39550$ $\widehat{p}_3 = 0.10450$	p_0 $p_1 = \frac{1}{2} - p_0$ $p_2 = \frac{1}{2} - p_0$ $p_3 = p_0$
$n = 4$	$\widetilde{p}_0 = 0.03722$ $\widetilde{p}_1 = 0.25009$ $\widetilde{p}_2 = 0.42556$ $\widetilde{p}_3 = 0.24998$ $\widetilde{p}_4 = 0.03715$	$\widehat{p}_0 = 0.03721$ $\widehat{p}_1 = 0.25000$ $\widehat{p}_2 = 0.42558$ $\widehat{p}_3 = 0.25000$ $\widehat{p}_4 = 0.03721$	p_0 $p_1 = \frac{1}{4}$ $p_2 = \frac{1}{2} - 2p_0$ $p_3 = \frac{1}{4}$ $p_4 = p_0$
$n = 5$	$\widetilde{p}_0 = 0.01126$ $\widetilde{p}_1 = 0.13028$ $\widetilde{p}_2 = 0.35848$ $\widetilde{p}_3 = 0.35852$ $\widetilde{p}_4 = 0.13020$ $\widetilde{p}_5 = 0.01126$	$\widehat{p}_0 = 0.01126$ $\widehat{p}_1 = 0.13024$ $\widehat{p}_2 = 0.35850$ $\widehat{p}_3 = 0.35850$ $\widehat{p}_4 = 0.13024$ $\widehat{p}_5 = 0.01126$	p_0 p_1 $p_2 = \frac{1}{2} - p_0 - p_1$ $p_3 = \frac{1}{2} - p_0 - p_1$ $p_4 = p_1$ $p_5 = p_0$

Dimension	Observed frequency	Least squares	Relations (Corol. 4.2)
$n=6$	$\widetilde{p}_0 = 0.00289$	$\widehat{p}_0 = 0.00288$	p_0
	$\widetilde{p}_1 = 0.05675$	$\widehat{p}_1 = 0.05678$	p_1
	$\widetilde{p}_2 = 0.24710$	$\widehat{p}_2 = 0.24712$	$p_2 = \frac{1}{4} - p_0$
	$\widetilde{p}_3 = 0.38642$	$\widehat{p}_3 = 0.38644$	$p_3 = \frac{1}{2} - 2p_1$
	$\widetilde{p}_4 = 0.24714$	$\widehat{p}_4 = 0.24712$	$p_4 = \frac{1}{4} - p_0$
	$\widetilde{p}_5 = 0.56810$	$\widehat{p}_5 = 0.05678$	$p_5 = p_1$
	$\widetilde{p}_6 = 0.00289$	$\widehat{p}_6 = 0.00288$	$p_6 = p_0$
$n=7$	$\widetilde{p}_0 = 0.00063$	$\widehat{p}_0 = 0.00063$	p_0
	$\widetilde{p}_1 = 0.02089$	$\widehat{p}_1 = 0.02088$	p_1
	$\widetilde{p}_2 = 0.14250$	$\widehat{p}_2 = 0.14249$	p_2
	$\widetilde{p}_3 = 0.33601$	$\widehat{p}_3 = 0.33600$	$p_3 = \frac{1}{2} - p_0 - p_1 - p_2$
	$\widetilde{p}_4 = 0.33597$	$\widehat{p}_4 = 0.33600$	$p_4 = \frac{1}{2} - p_0 - p_1 - p_2$
	$\widetilde{p}_5 = 0.14248$	$\widehat{p}_5 = 0.14249$	$p_5 = p_2$
	$\widetilde{p}_6 = 0.02088$	$\widehat{p}_6 = 0.02088$	$p_6 = p_1$
	$\widetilde{p}_7 = 0.00063$	$\widehat{p}_7 = 0.00063$	$p_7 = p_0$
$n=8$	$\widetilde{p}_0 = 0.00012$	$\widehat{p}_0 = 0.00012$	p_0
	$\widetilde{p}_1 = 0.00651$	$\widehat{p}_1 = 0.00650$	p_1
	$\widetilde{p}_2 = 0.06948$	$\widehat{p}_2 = 0.06948$	p_2
	$\widetilde{p}_3 = 0.24356$	$\widehat{p}_3 = 0.24350$	$p_3 = \frac{1}{4} - p_1$
	$\widetilde{p}_4 = 0.36080$	$\widehat{p}_4 = 0.36080$	$p_4 = \frac{1}{2} - 2p_0 - 2p_2$
	$\widetilde{p}_5 = 0.24346$	$\widehat{p}_5 = 0.24350$	$p_5 = \frac{1}{4} - p_1$
	$\widetilde{p}_6 = 0.06946$	$\widehat{p}_6 = 0.06948$	$p_6 = p_2$
	$\widetilde{p}_7 = 0.00650$	$\widehat{p}_7 = 0.00650$	$p_7 = p_1$
	$\widetilde{p}_8 = 0.00012$	$\widehat{p}_8 = 0.00012$	$p_8 = p_0$
$n=9$	$\widetilde{p}_0 = 0.00002$	$\widehat{p}_0 = 0.00002$	p_0
	$\widetilde{p}_1 = 0.00171$	$\widehat{p}_1 = 0.00171$	p_1
	$\widetilde{p}_2 = 0.02880$	$\widehat{p}_2 = 0.02879$	p_2
	$\widetilde{p}_3 = 0.14952$	$\widehat{p}_3 = 0.14955$	p_3
	$\widetilde{p}_4 = 0.31987$	$\widehat{p}_4 = 0.31993$	$p_4 = \frac{1}{2} - p_0 - p_1 - p_2 - p_3$
	$\widetilde{p}_5 = 0.31999$	$\widehat{p}_5 = 0.31993$	$p_5 = \frac{1}{2} - p_0 - p_1 - p_2 - p_3$
	$\widetilde{p}_6 = 0.14958$	$\widehat{p}_6 = 0.14955$	$p_6 = p_3$
	$\widetilde{p}_7 = 0.02878$	$\widehat{p}_7 = 0.02879$	$p_7 = p_2$
	$\widetilde{p}_8 = 0.00171$	$\widehat{p}_8 = 0.00171$	$p_8 = p_1$
	$\widetilde{p}_9 = 0.00002$	$\widehat{p}_9 = 0.00002$	$p_9 = p_0$
$n=10$	$\widetilde{p}_0 = 0$	$\widehat{p}_0 = 0$	p_0
	$\widetilde{p}_1 = 0.00038$	$\widehat{p}_1 = 0.00038$	p_1
	$\widetilde{p}_2 = 0.01015$	$\widehat{p}_2 = 0.01015$	p_2
	$\widetilde{p}_3 = 0.07850$	$\widehat{p}_3 = 0.07850$	p_3
	$\widetilde{p}_4 = 0.23987$	$\widehat{p}_4 = 0.23985$	$p_4 = \frac{1}{4} - p_0 - p_2$
	$\widetilde{p}_5 = 0.34224$	$\widehat{p}_5 = 0.34224$	$p_5 = \frac{1}{2} - 2p_1 - 2p_3$
	$\widetilde{p}_6 = 0.23984$	$\widehat{p}_6 = 0.23985$	$p_6 = \frac{1}{4} - p_0 - p_2$
	$\widetilde{p}_7 = 0.07849$	$\widehat{p}_7 = 0.07850$	$p_7 = p_3$
	$\widetilde{p}_8 = 0.01015$	$\widehat{p}_8 = 0.01015$	$p_8 = p_2$
	$\widetilde{p}_9 = 0.00038$	$\widehat{p}_9 = 0.00038$	$p_9 = p_1$
	$\widetilde{p}_{10} = 0$	$\widehat{p}_{10} = 0$	$p_{10} = p_0$

Table 4.1: Linear stability indexes for linear random differential systems.

5 Linear random differential equations of order n

In this section we consider linear random homogeneous differential equations of order n

$$A_n x^{(n)} + A_{n-1} x^{(n-1)} + \cdots + A_2 x'' + A_1 x' + A_0 x = 0, \tag{5.1}$$

where $x = x(t)$, the derivatives are taken in respect to t, and A_j are again i.i.d. random variables with $\mathrm{N}(0,1)$ distribution.

To get the stability index for these differential equations we only need to know the probability distributions of the number of roots with negative real part of its associated random characteristic polynomial:

$$Q(\lambda) = A_n \lambda^n + A_{n-1}\lambda^{n-1} + \cdots + A_1 \lambda + A_0.$$

Let X be the random variable that counts the number of roots of $Q(\lambda)$ with negative real parts and define $p_k = P(X = k)$ for $k = 0, 1, \ldots, n$.

Proposition 5.1. *Set $p_k = P(X = k)$, where X is the random variable defined above. Then*

(a) $\sum_{k=0}^{n} p_k = 1$.

(b) *For all $k \in \{0, 1, \ldots, n\}$, $p_k = p_{n-k}$.*

(c) $\sum_{i \text{ odd}} p_i = \sum_{i \text{ even}} p_i = \frac{1}{2}$.

Proof. The proof of (a) is trivial. To prove (b) consider equation (5.1) with its characteristic polynomial $Q(\lambda)$ and also the new differential equation

$$(-1)^n A_n x^{(n)} + (-1)^{n-1} A_{n-1} x^{(n-1)} + \cdots + A_2 x'' - A_1 x' + A_0 x = 0 \tag{5.2}$$

with its characteristic polynomial $Q^*(\lambda) = Q(-\lambda) = (-1)^n A_n \lambda^n + (-1)^{n-1} A_{n-1} \lambda^{n-1} + \cdots - A_1 \lambda + A_0$. Since $Q(\lambda) = 0$ if and only if $Q^*(-\lambda) = 0$ we get that $p_k = p^*_{n-k}$ where p^*_i the probability that $Q^*(\lambda)$ has i roots with negative real part. But also $p_k = p^*_k$ because the coefficients of the equations (5.1) and (5.2) have the same distributions. Hence, the result follows.

Similarly, as in the proof of (c) of Proposition 4.1, we observe that the polynomial $Q(\lambda)$ has an odd number of roots with negative real part if and only if $A_0 \cdot A_n < 0$, because we can neglect the case of having some roots with zero real part. Since the coefficients of (5.1) are symmetric independent random variables, the probability that $Q(\lambda)$ has an odd number of roots with negative real part is

$$P(\{A_0 > 0\} \cap \{A_n < 0\}) + P(\{A_0 < 0\} \cap \{A_n > 0\}) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}. \qquad \square$$

Corollary 5.2. *Consider the linear random homogeneous differential equation of order n (5.1), with all A_i being i.i.d. $\mathrm{N}(0,1)$ random variables, let X be defined above, and set $p_k = P(X = k)$. Then the probabilities p_k satisfy all the conclusions of Lemma 3.1. In particular $E(X) = n/2$.*

For each n, let r_n be the probability of the origin to be a global stable attractor (asymptotically stable equilibrium) for (5.1), that is $r_n = p_n$. By Proposition 5.1(b) this probability coincides with the probability of being a repeller because $p_n = p_0$. Our results in Proposition 5.4 seem to indicate that r_n decreases with n. Before proving this proposition we need a preliminary result.

Lemma 5.3. *Let U, V, S and T be i.i.d. random variables with standard normal distribution. Then $p^+ := P(U > 0; V > 0; S > 0; T > 0; UT - SV > 0) = 1/32$.*

Proof. Set $\mathcal{A}^\pm = \{U > 0; V > 0; S > 0; T > 0; \pm(UT - SV) > 0\}$, and $\mathcal{A}^0 = \{U > 0; V > 0; S > 0; T > 0; UT - SV = 0\}$. Denote by $p^\pm = P(\mathcal{A}^\pm)$ and $p^0 = P(\mathcal{A}^0)$. Then, since $p^0 = 0$ and $\mathcal{A}^- \cup \mathcal{A}^0 \cup \mathcal{A}^+ = \{U > 0; V > 0; S > 0; T > 0\}$ it holds that $p^+ + p^- = (1/2)^4 = 1/16$. To end the proof it suffices to show that $p^+ = p^-$.

Notice first that

$$\begin{aligned}\mathcal{A}^+ &= \{U > 0; V > 0; S > 0; T > 0; UT - SV > 0\} = \{V > 0; S > 0; T > 0; UT - SV > 0\},\\ \mathcal{A}^- &= \{U > 0; V > 0; S > 0; T > 0; UT - SV < 0\} = \{U > 0; S > 0; T > 0; SV - UT > 0\}.\end{aligned}$$

This is so, because for instance in the definition of $\mathcal{A}^+$, the last inequality can also be written as $U > SV/T > 0$ and from it we know that the condition $U > 0$ can be removed. Finally, interchanging U and V and S and T we get the same relations in the definitions of $\mathcal{A}^+$ and $\mathcal{A}^-$. Since all variables are independent $\mathrm{N}(0, 1)$, both sets have the same probability and $p^+ = p^-$, as we wanted to prove. □

Proposition 5.4. *With the above notations, $r_n \leq 1/2^n$, for all $n \geq 1$. Moreover, $r_1 = 1/2$, $r_2 = 1/4$, $r_3 = 1/16$ and $r_4 < r_3/2 = 1/32$.*

Proof of Proposition 5.4. Notice that r_n is the probability that the characteristic polynomial $Q(\lambda)$, associated with the random differential equation (5.1), is a Hurwitz stable polynomial; that is $r_n = P(\text{Every root of } Q(\lambda) \text{ belongs to } \mathfrak{R}^-)$, where $\mathfrak{R}^- = \{z \in \mathbb{C} \text{ such that } \mathrm{Re}(z) < 0\}$. It is well-known that a necessary condition for a polynomial to have every root in $\mathfrak{R}^-$ is that all its coefficients have the same sign. This is so because it holds for polynomials of degree 1 and 2, and this property is preserved when we multiply two polynomials satisfying it. Hence,

$$\{A_0, \dots, A_n \text{ such that all roots of } P(\lambda) \text{ are in } \mathfrak{R}^-\} \subset \left\{\bigcap_{i=0}^{n}\{A_i < 0\}\right\} \bigcup \left\{\bigcap_{i=0}^{n}\{A_i > 0\}\right\}. \quad (5.3)$$

Since the variables A_i are independent and symmetric

$$P\left(\bigcap_{i=0}^{n}\{A_i < 0\}\right) = P\left(\bigcap_{i=0}^{n}\{A_i > 0\}\right) = \frac{1}{2^{n+1}}.$$

As a consequence,

$$r_n \leq P\left(\bigcap_{i=0}^{n}\{A_i < 0\}\right) + P\left(\bigcap_{i=0}^{n}\{A_i > 0\}\right) = \frac{1}{2^n},$$

and the first statement follows.

The equalities $r_1 = 1/2$ and $r_2 = 1/4$ are a simple consequence that for $n = 1, 2$ the inclusion (5.3) is an equality.

Let us prove that $r_3 = p_3 = 1/16$. By using the Routh–Hurwitz criterion [10, §15.715, p. 1076], it can be seen that $a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ has every root in $\mathfrak{R}^-$ if and only if all its coefficients have the same sign and moreover $a_1a_2 - a_0a_3 > 0$. Hence, $p_3 = p_3^- + p_3^+$, where $p_3^- := P(A_0 < 0; A_1 < 0; A_2 < 0; A_3 < 0; A_1A_2 - A_0A_3 > 0)$; and $p_3^+ :=$

$P(A_0 > 0; A_1 > 0; A_2 > 0; A_3 > 0; A_1A_2 - A_0A_3 > 0)$, with all the A_i being $\mathrm{N}(0,1)$ distributed and independent. Due to their symmetry, the random variables A_i and $-A_i$, for $i = 0, \ldots, 3$ have the same distribution and hence $p_3^+ = p_3^-$. Therefore $p_3 = 2p_3^+$. The result follows now by Lemma 5.3, which gives $p_3^+ = 1/32$.

Let us prove that $r_4 < r_3/2$. To compare both probabilities, here it will be more convenient to write the coefficients of the polynomials with subscripts with increasing ordering, that is $q_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. With this notation, which also respects the traditional notation when writing the Hurwitz matrices, and when $a_0 > 0$, the Routh–Hurwitz conditions to have stability index n for $n = 3, 4$ are precisely that the principal minors of the following matrices

$$\begin{pmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{pmatrix},$$

are positive, where the left-hand one corresponds to the case $n = 3$ and the other to the case when $n = 4$. Hence, these conditions when $a_0 > 0$ and for $n = 3$ are: $a_1 > 0, a_1a_2 - a_0a_3 > 0$ and $a_3 > 0$. Similarly, for $n = 4$ the conditions are $a_1 > 0, a_1a_2 - a_0a_3 > 0$, $a_3(a_1a_2 - a_0a_3) - a_4a_1^2 > 0$ and $a_4 > 0$.

Consider now, for $n = 3, 4$, the random polynomials $Q_n(x) = \tilde{A}_0x^n + \tilde{A}_1x^{n-1} + \cdots + \tilde{A}_{n-1}x + \tilde{A}_n$, where $\tilde{A}_i \sim \mathrm{N}(0,1)$ and are independent (notice that with this notation each coefficient $\tilde{A}_i$ is the coefficient A_{n-i} of the characteristic polynomial). For simplicity we denote with the same name the coefficients of Q_3 and Q_4 although they are different random variables. As above, $r_3 = 2p_3^+$ and $r_4 = 2p_4^+$, where $p_k^+ = P(\mathcal{A}_k^+)$, with

$$\begin{aligned} \mathcal{A}_3^+ &= \{\tilde{A}_0 > 0; \tilde{A}_1 > 0; \tilde{A}_3 > 0; \tilde{A}_1\tilde{A}_2 - \tilde{A}_0\tilde{A}_3 > 0\}, \\ \mathcal{A}_4^+ &= \{\tilde{A}_0 > 0; \tilde{A}_1 > 0; \tilde{A}_3 > \tilde{A}_4\tilde{A}_1^2/(\tilde{A}_1\tilde{A}_2 - \tilde{A}_0\tilde{A}_3); \tilde{A}_1\tilde{A}_2 - \tilde{A}_0\tilde{A}_3 > 0, \tilde{A}_4 > 0\}. \end{aligned}$$

Notice that if we define

$$\mathcal{B} = \{\tilde{A}_0 > 0; \tilde{A}_1 > 0; \tilde{A}_3 > 0; \tilde{A}_1\tilde{A}_2 - \tilde{A}_0\tilde{A}_3 > 0; \tilde{A}_4 > 0\}$$

it is clear that $P(\mathcal{B}) = p_3^+/2$ and, moreover $\mathcal{A}_4^+ \subset \mathcal{B}_3$, with the inclusion being strict. Since the joint density is positive and $\mathcal{B} \cap (\mathcal{A}_4^+)^c$ has positive Lebesgue measure, we have $P(\mathcal{B} \cap (\mathcal{A}_4^+)^c) > 0$. Thus $P(\mathcal{A}_4^+) < P(\mathcal{B})$, and hence $p_4^+ = P(\mathcal{A}_4^+) < P(\mathcal{B}) = p_3^+/2$, and $r_4 < r_3/2$, as we wanted to show. □

Corollary 5.5. *Consider a linear random homogeneous differential equation of order $n = 3$ and the random variable X defined above. Then $p_0 = p_3 = 1/16$ and $p_1 = p_2 = 7/16$.*

Proof. By the above proposition, for $n = 3$, $p_0 = p_3 = r_3 = 1/16$. Hence, by Proposition 5.1, $p_1 = p_2 = 7/16$. □

The computations in this case are similar to the ones of the previous section and the obtained results are summarized in Table 5.1. We only give some comments for the cases $n = 8$ and 10, where we have encountered that the vectors $\widehat{\mathbf{p}}$ have negative and very small entries. This has occurred because the observed frequencies obtained by the Monte Carlo approach corresponding to these probabilities are not accurate enough. For this reason, we have made a new optimization step. As before, we use the least squares method to obtain a vector $\widehat{\mathbf{p}}$. However, if negative entries appear in this vector (which is clearly objectionable), we force them to be zero and find a new least squares estimate, which still respects the original linear constraints.

We explain this process for the $n = 8$ order case; the $n = 10$ case follows analogously. The observed relative frequencies vector obtained by the Monte Carlo method is

$$\widetilde{\mathbf{p}}^t = \left(\frac{1}{50000000}, \frac{6599}{50000000}, \frac{1159359}{50000000}, \frac{4996163}{20000000}, \frac{45377377}{100000000}, \frac{4995607}{20000000}, \frac{2318357}{100000000}, \frac{13497}{100000000}, \frac{1}{100000000}\right).$$

The relations stated in Corollary 5.2 are $p_3 = 1/4 - p_1$, $p_4 = 1/2 - 2p_0 - 2p_2$, $p_5 = p_3$, $p_6 = p_2$, $p_7 = p_1$, $p_8 = p_0$. By solving the system (3.3) with

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ -2 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \mathbf{q} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}; \text{ and } b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

we obtain

$$\widehat{\mathbf{q}}^t = \left(-\frac{5779}{200000000}, \frac{13569}{80000000}, \frac{4631293}{200000000}\right).$$

Hence, by (3.5) we get

$$\widehat{\mathbf{p}} = \left(\frac{-5779}{200000000}, \frac{13569}{80000000}, \frac{4631293}{200000000}, \frac{19986431}{80000000}, \frac{22687243}{50000000}, \frac{19986431}{80000000}, \frac{4631293}{200000000}, \frac{13569}{80000000}, \frac{-5779}{200000000}\right).$$

So we impose that $p_0 = p_8 = 0$. Thus we have $p_3 = p_5 = 1/4 - p_1$, $p_4 = 1/2 - 2p_0 - 2p_2 = 1/2 - 2p_2$, $p_6 = p_2$ and $p_7 = p_1$. We find the least squares solution of the system

$$\begin{pmatrix} \widehat{p}_1 \\ \widehat{p}_2 \\ \widehat{p}_3 \\ \widehat{p}_4 \\ \widehat{p}_5 \\ \widehat{p}_6 \\ \widehat{p}_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -2 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \widehat{p}_1^* \\ \widehat{p}_2^* \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix}.$$

Using (3.4) and (3.5) we obtain

$$\begin{aligned} \widehat{\mathbf{p}}^* &= \left(0, \frac{13569}{80000000}, \frac{13882321}{600000000}, \frac{19986431}{80000000}, \frac{136117679}{300000000}, \frac{19986431}{80000000}, \frac{13882321}{600000000}, \frac{13569}{80000000}, 0\right) \\ &\simeq (0, 0.00017, 0.02314, 0.24983, 0.45373, 0.24983, 0.02314, 0.00017, 0). \end{aligned}$$

Dimension	Observed frequency	Least squares	Relations (Corol. 5.2 and 5.5)
$n=1$	$\widetilde{p}_0 = 0.49997$ $\widetilde{p}_1 = 0.50003$		$p_0 = 0.5$ $p_1 = 0.5$
$n=2$	$\widetilde{p}_0 = 0.24994$ $\widetilde{p}_1 = 0.49999$ $\widetilde{p}_2 = 0.25007$		$p_0 = 0.25$ $p_1 = 0.5$ $p_2 = 0.25$
$n=3$	$\widetilde{p}_0 = 0.06252$ $\widetilde{p}_1 = 0.43743$ $\widetilde{p}_2 = 0.43756$ $\widetilde{p}_3 = 0.06249$		$p_0 = \frac{1}{16} = 0.0625$ $p_1 = \frac{7}{16} = 0.4375$ $p_2 = \frac{7}{16} = 0.4375$ $p_3 = \frac{1}{16} = 0.0625$
$n=4$	$\widetilde{p}_0 = 0.00928$ $\widetilde{p}_1 = 0.24998$ $\widetilde{p}_2 = 0.48152$ $\widetilde{p}_3 = 0.24994$ $\widetilde{p}_4 = 0.00929$	$\widehat{p}_0 = 0.00925$ $\widehat{p}_1 = 0.25$ $\widehat{p}_2 = 0.48150$ $\widehat{p}_3 = 0.25$ $\widehat{p}_4 = 0.00925$	p_0 $p_1 = \frac{1}{4}$ $p_2 = \frac{1}{2} - 2p_0$ $p_3 = \frac{1}{4}$ $p_4 = p_0$
$n=5$	$\widetilde{p}_0 = 0.00071$ $\widetilde{p}_1 = 0.08405$ $\widetilde{p}_2 = 0.41526$ $\widetilde{p}_3 = 0.41523$ $\widetilde{p}_4 = 0.08404$ $\widetilde{p}_5 = 0.00071$	$\widehat{p}_0 = 0.00071$ $\widehat{p}_1 = 0.08404$ $\widehat{p}_2 = 0.41525$ $\widehat{p}_3 = 0.41525$ $\widehat{p}_4 = 0.08404$ $\widehat{p}_5 = 0.00071$	p_0 p_1 $p_2 = \frac{1}{2} - p_0 - p_1$ $p_3 = \frac{1}{2} - p_0 - p_1$ $p_4 = p_1$ $p_5 = p_0$
$n=6$	$\widetilde{p}_0 = 0.00003$ $\widetilde{p}_1 = 0.01723$ $\widetilde{p}_2 = 0.24994$ $\widetilde{p}_3 = 0.46562$ $\widetilde{p}_4 = 0.24993$ $\widetilde{p}_5 = 0.01723$ $\widetilde{p}_6 = 0.00003$	$\widehat{p}_0 = 0.00005$ $\widehat{p}_1 = 0.01720$ $\widehat{p}_2 = 0.24995$ $\widehat{p}_3 = 0.46560$ $\widehat{p}_4 = 0.24995$ $\widehat{p}_5 = 0.01720$ $\widehat{p}_6 = 0.00005$	p_0 p_1 $p_2 = \frac{1}{4} - p_0$ $p_3 = \frac{1}{2} - 2p_1$ $p_4 = \frac{1}{4} - p_0$ $p_5 = p_1$ $p_6 = p_0$
$n=7$	$\widetilde{p}_0 = 0$ $\widetilde{p}_1 = 0.00200$ $\widetilde{p}_2 = 0.09571$ $\widetilde{p}_3 = 0.40224$ $\widetilde{p}_4 = 0.40233$ $\widetilde{p}_5 = 0.09573$ $\widetilde{p}_6 = 0.00199$ $\widetilde{p}_7 = 0$	$\widehat{p}_0 = 0$ $\widehat{p}_1 = 0.00200$ $\widehat{p}_2 = 0.09572$ $\widehat{p}_3 = 0.40228$ $\widehat{p}_4 = 0.40228$ $\widehat{p}_5 = 0.09572$ $\widehat{p}_6 = 0.00200$ $\widehat{p}_7 = 0$	p_0 p_1 p_2 $p_3 = \frac{1}{2} - p_0 - p_1 - p_2$ $p_4 = \frac{1}{2} - p_0 - p_1 - p_2$ $p_5 = p_2$ $p_6 = p_1$ $p_7 = p_0$
$n=8$	$\widetilde{p}_0 = 0$ $\widetilde{p}_1 = 0.00013$ $\widetilde{p}_2 = 0.02319$ $\widetilde{p}_3 = 0.24981$ $\widetilde{p}_4 = 0.45377$ $\widetilde{p}_5 = 0.24978$ $\widetilde{p}_6 = 0.02318$ $\widetilde{p}_7 = 0.00013$ $\widetilde{p}_8 = 0$	$\widehat{p}_0^* = 0$ $\widehat{p}_1^* = 0.00017$ $\widehat{p}_2^* = 0.02314$ $\widehat{p}_3^* = 0.24983$ $\widehat{p}_4^* = 0.45372$ $\widehat{p}_5^* = 0.24983$ $\widehat{p}_6^* = 0.02314$ $\widehat{p}_7^* = 0.00017$ $\widehat{p}_8^* = 0$	p_0 p_1 p_2 $p_3 = \frac{1}{4} - p_1$ $p_4 = \frac{1}{2} - 2p_0 - 2p_2$ $p_5 = \frac{1}{4} - p_1$ $p_6 = p_2$ $p_7 = p_1$ $p_8 = p_0$

Dimension	Observed frequency	Least squares	Relations (Corol. 5.2 and 5.5)
$n=9$	$\widetilde{p}_0 = 0$	$\widehat{p}_0 = 0$	p_0
	$\widetilde{p}_1 = 0.00001$	$\widehat{p}_1 = 0.00005$	p_1
	$\widetilde{p}_2 = 0.00336$	$\widehat{p}_2 = 0.00337$	p_2
	$\widetilde{p}_3 = 0.10337$	$\widehat{p}_3 = 0.10335$	p_3
	$\widetilde{p}_4 = 0.39328$	$\widehat{p}_4 = 0.39328$	$p_4 = \frac{1}{2} - p_0 - p_1 - p_2 - p_3$
	$\widetilde{p}_5 = 0.39328$	$\widehat{p}_5 = 0.39328$	$p_5 = \frac{1}{2} - p_0 - p_1 - p_2 - p_3$
	$\widetilde{p}_6 = 0.10332$	$\widehat{p}_6 = 0.10335$	$p_6 = p_3$
	$\widetilde{p}_7 = 0.00338$	$\widehat{p}_7 = 0.00337$	$p_7 = p_2$
	$\widetilde{p}_8 = 0$	$\widehat{p}_8 = 0.00005$	$p_8 = p_1$
	$\widetilde{p}_9 = 0$	$\widehat{p}_9 = 0$	$p_9 = p_0$
$n=10$	$\widetilde{p}_0 = 0$	$\widehat{p}_0^* = 0$	p_0
	$\widetilde{p}_1 = 0$	$\widehat{p}_1^* = 0.00002$	p_1
	$\widetilde{p}_2 = 0.00030$	$\widehat{p}_2^* = 0.00028$	p_2
	$\widetilde{p}_3 = 0.02784$	$\widehat{p}_3^* = 0.02787$	p_3
	$\widetilde{p}_4 = 0.24976$	$\widehat{p}_4^* = 0.24972$	$p_4 = \frac{1}{4} - p_0 - p_2$
	$\widetilde{p}_5 = 0.44421$	$\widehat{p}_5^* = 0.44422$	$p_5 = \frac{1}{2} - 2p_1 - 2p_3$
	$\widetilde{p}_6 = 0.24973$	$\widehat{p}_6^* = 0.24972$	$p_6 = \frac{1}{4} - p_0 - p_2$
	$\widetilde{p}_7 = 0.02787$	$\widehat{p}_7^* = 0.02787$	$p_7 = p_3$
	$\widetilde{p}_8 = 0.00029$	$\widehat{p}_8^* = 0.00028$	$p_8 = p_2$
	$\widetilde{p}_9 = 0$	$\widehat{p}_9^* = 0.00002$	$p_9 = p_1$
	$\widetilde{p}_{10} = 0$	$\widehat{p}_{10}^* = 0$	$p_{10} = p_0$

Table 5.1: Stability indexes for order n linear random homogeneous differential equations.

6 Linear random maps

In order to keep the approach of the preceding sections, we suggest to consider random linear discrete dynamical systems of the form

$$\mathcal{B}\,\mathbf{x}_{k+1} = A\,\mathbf{x}_k \text{ where } \mathbf{x} \in \mathbb{R}^n, \tag{6.1}$$

where $\mathcal{B}$ and each of the n^2 entries of the random matrix A are i.i.d. $\mathrm{N}(0,1)$ random variables. Observe that to ensure that the results are invariant under time-scaling, is necessary to add the term $\mathcal{B}$ in the left-hand side of Equation (6.1). Then, given a linear discrete random system (6.1), its characteristic random polynomial associated with the matrix $\frac{1}{\mathcal{B}}A$ is

$$Q(\lambda) = Q_n\lambda^n + Q_{n-1}\lambda^{n-1} + \cdots + Q_1\lambda + Q_0$$

where each random variable Q_j is a polynomial in the variables $1/\mathcal{B}, A_{1,1}, \ldots, A_{n,n}$ which has a complicated distribution function. We denote by X the random variables that assigns to each Q its number of roots with modulus smaller than 1, that is, the stability index of the matrix $\frac{1}{\mathcal{B}}A$. Also p_k denotes the probabilities that X takes the value k.

As we will see in the examples, in this case the condition $p_k = p_{n-k}$ is no longer satisfied. Among other reasons it happens that the entries of A^{-1} have complicated distributions. Since we do not know other relations on the probabilities p_k apart from the trivial one $\sum_{k=0}^{n} p_k = 1$, and this is directly fulfilled by the observed relative frequencies, in this case we do not perform the least squares refinement.

The case $n = 1$ is the only one that we have been able to solve analytically. Notice that in this situation the only solution of $Q(\lambda) = 0$ is $\lambda = A/B$, with A and B independent and $N(0,1)$. Hence $p_0 = P(|A/B| > 1)$ and $p_1 = P(|A/B| < 1) = P(|B/A| > 1)$. Since A/B and B/A have the same distribution it holds that $p_0 = p_1 = 1/2$. The results obtained for $n \leq 10$ are shown in Table 6.1.

Dimension	Observed frequency
$n = 1$	$\widetilde{p}_0 = 0.49994$
	$\widetilde{p}_1 = 0.50006$
$n = 2$	$\widetilde{p}_0 = 0.46348$
	$\widetilde{p}_1 = 0.27705$
	$\widetilde{p}_2 = 0.25947$
$n = 3$	$\widetilde{p}_0 = 0.45261$
	$\widetilde{p}_1 = 0.25828$
	$\widetilde{p}_2 = 0.15351$
	$\widetilde{p}_3 = 0.13560$
$n = 4$	$\widetilde{p}_0 = 0.45040$
	$\widetilde{p}_1 = 0.24732$
	$\widetilde{p}_2 = 0.14799$
	$\widetilde{p}_3 = 0.08127$
	$\widetilde{p}_4 = 0.07302$
$n = 5$	$\widetilde{p}_0 = 0.44957$
	$\widetilde{p}_1 = 0.24536$
	$\widetilde{p}_2 = 0.13956$
	$\widetilde{p}_3 = 0.08116$
	$\widetilde{p}_4 = 0.04443$
	$\widetilde{p}_5 = 0.03992$
$n = 6$	$\widetilde{p}_0 = 0.44944$
	$\widetilde{p}_1 = 0.24419$
	$\widetilde{p}_2 = 0.13838$
	$\widetilde{p}_3 = 0.07536$
	$\widetilde{p}_4 = 0.04606$
	$\widetilde{p}_5 = 0.02449$
	$\widetilde{p}_6 = 0.02209$
$n = 7$	$\widetilde{p}_0 = 0.44937$
	$\widetilde{p}_1 = 0.24394$
	$\widetilde{p}_2 = 0.13723$
	$\widetilde{p}_3 = 0.07480$
	$\widetilde{p}_4 = 0.04226$
	$\widetilde{p}_5 = 0.02636$
	$\widetilde{p}_6 = 0.01367$
	$\widetilde{p}_7 = 0.01236$

Dimension	Observed frequency
$n = 8$	$\widetilde{p}_0 = 0.44937$
	$\widetilde{p}_1 = 0.24381$
	$\widetilde{p}_2 = 0.13702$
	$\widetilde{p}_3 = 0.07388$
	$\widetilde{p}_4 = 0.04207$
	$\widetilde{p}_5 = 0.02394$
	$\widetilde{p}_6 = 0.01526$
	$\widetilde{p}_7 = 0.00768$
	$\widetilde{p}_8 = 0.00698$
$n = 9$	$\widetilde{p}_0 = 0.44941$
	$\widetilde{p}_1 = 0.24374$
	$\widetilde{p}_2 = 0.13680$
	$\widetilde{p}_3 = 0.07371$
	$\widetilde{p}_4 = 0.04139$
	$\widetilde{p}_5 = 0.02400$
	$\widetilde{p}_6 = 0.01374$
	$\widetilde{p}_7 = 0.00889$
	$\widetilde{p}_8 = 0.00434$
	$\widetilde{p}_9 = 0.00397$
$n = 10$	$\widetilde{p}_0 = 0.44934$
	$\widetilde{p}_1 = 0.24371$
	$\widetilde{p}_2 = 0.13687$
	$\widetilde{p}_3 = 0.07358$
	$\widetilde{p}_4 = 0.04129$
	$\widetilde{p}_5 = 0.02348$
	$\widetilde{p}_6 = 0.01388$
	$\widetilde{p}_7 = 0.00792$
	$\widetilde{p}_8 = 0.00520$
	$\widetilde{p}_9 = 0.00247$
	$\widetilde{p}_{10} = 0.00226$

Table 6.1: Stability indexes for linear random maps.

As in the other models, for each dimension $n \leq 10$, we generate 10^8 discrete systems of the form (6.1). For each matrix $\frac{1}{B}A$ we have computed the characteristic polynomial Q and its associated polynomial $Q^\star$ (see Equation (3.2)) and have counted the number of roots of this last polynomial by using the Routh–Hurwitz zero counter. For $n \geq 5$ and in order to decrease the computation time we have directly numerically computed the eigenvalues of the matrix and counted the number of them with modulus less than one.

7 Linear random difference equations of order n

Finally we consider difference equations of order n of type

$$A_n x_{k+n} + A_{n-1} x_{k+n-1} + \cdots + A_1 x_{k+1} + A_0 x_k = 0,$$

where all the coefficients are i.i.d. random variables with $\mathrm{N}(0,1)$ distribution. In this situation, the stability index is given by the number of zeros with modulus smaller than 1 of the random characteristic polynomial $Q(\lambda) = A_n\lambda^n + A_{n-1}\lambda^{n-1} + \cdots + A_1\lambda + A_0$. As in the preceding sections let X be the random variable that counts the number of roots of $Q(\lambda)$ with modulus smaller than 1 and set $p_k = P(X = k)$ for $k = 0, 1, \ldots, n$.

Before proving some relations among the probabilities p_k, we give two preliminary lemmas. Let $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-u^2} \mathrm{d}u$ be the error function. The following result is stated in [4]. We prove it for the sake of completeness.

Lemma 7.1. *For $\alpha > 0$ and $\beta \in \mathbb{R}$,*

$$F(\alpha,\beta) := \int_0^\infty \mathrm{e}^{-\alpha^2x^2}\,\mathrm{erf}(\beta x)\,\mathrm{d}x = \frac{\arctan(\beta/\alpha)}{\alpha\sqrt{\pi}}.$$

Proof. Fixed $\alpha > 0$, the function that defines F is absolutely integrable because $|\mathrm{erf}(x)| \leq 1$. Moreover its partial derivative with respect to β is also absolutely integrable. Hence $\lim_{\beta\to 0} F(\alpha,\beta) = F(\alpha,0) = 0$ and

$$\begin{aligned}\frac{\partial F(\alpha,\beta)}{\partial\beta} &= \int_0^\infty \frac{\partial}{\partial\beta}\left(\mathrm{e}^{-\alpha^2x^2}\,\mathrm{erf}(\beta x)\right)\mathrm{d}x = \frac{2}{\sqrt{\pi}}\int_0^\infty x\,\mathrm{e}^{-\alpha^2x^2}\mathrm{e}^{-\beta^2x^2}\,\mathrm{d}x \\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty x\,\mathrm{e}^{-(\alpha^2+\beta^2)\,x^2}\,\mathrm{d}x = \frac{1}{(\alpha^2+\beta^2)\sqrt{\pi}}.\end{aligned}$$

Therefore

$$F(\alpha,\beta) = F(\alpha,0) + \int_0^\beta \frac{\partial F(\alpha,t)}{\partial t}\,\mathrm{d}t = \int_0^\beta \frac{1}{(\alpha^2+t^2)\sqrt{\pi}}\,\mathrm{d}t = \frac{\arctan(\beta/\alpha)}{\alpha\sqrt{\pi}},$$

as we wanted to prove. □

The next result is a consequence of the previous lemma.

Lemma 7.2. *Let $U \sim \mathrm{N}(0,\sigma^2)$ and $V \sim \mathrm{N}(0,\rho^2)$ be independent normal random variables. Then $P(U^2 - V^2 > 0) = \frac{2}{\pi}\arctan(\sigma/\rho)$.*

Proof. The joint density function of the random vector (U,V) is $f_\sigma(u)f_\rho(v)$, where $f_s(u) = \mathrm{e}^{-u^2/(2s^2)}/(\sqrt{2\pi}s)$. Observe that the points $(u,v) \in \mathbb{R}^2$ such that $u^2 - v^2 > 0$ is the region where $-|u| < v < |u|$, hence by symmetry,

$$\begin{aligned}P(U^2 - V^2 > 0) &= 4\int_0^\infty f_\sigma(u)\int_0^u f_\rho(v)\,\mathrm{d}v\,\mathrm{d}u = \frac{4}{2\pi\sigma\rho}\int_0^\infty \mathrm{e}^{-u^2/(2\sigma^2)}\int_0^u \mathrm{e}^{-v^2/(2\rho^2)}\,\mathrm{d}v\,\mathrm{d}u \\ &= \frac{2}{\pi\sigma\rho}\int_0^\infty \mathrm{e}^{-u^2/(2\sigma^2)}\,\mathrm{erf}\left(\frac{u}{\sqrt{2}\rho}\right)\sqrt{\frac{\pi}{2}}\rho\,\mathrm{d}u = \frac{2}{\pi}\arctan\left(\frac{\sigma}{\rho}\right),\end{aligned}$$

where in the last equality we have used Lemma 7.1. □

Notice that with the notation of the above lemma, $P(U^2 - V^2 > 0) + P(U^2 - V^2 < 0) = 1$. Hence

$$P(U^2 - V^2 < 0) = 1 - \frac{2}{\pi}\arctan\left(\frac{\sigma}{\rho}\right) = \frac{2}{\pi}\arctan\left(\frac{\rho}{\sigma}\right), \tag{7.1}$$

where we have used the fact that $\arctan(x) + \arctan(1/x) = \pi/2$ or, simply, the same lemma interchanging U and V. Observe also that when $\sigma = \rho$, $P(U^2 - V^2 > 0) = P(U^2 - V^2 < 0) = 1/2$, a result that, in fact, is a straightforward consequence that in this situation $U^2 - V^2$ and $V^2 - U^2$ have the same distribution.

Proposition 7.3. *With the above notation:*

(a) $\sum_{k=0}^{n} p_k = 1$.

(b) For all $k \in \{0, 1, \ldots, n\}$, $p_k = p_{n-k}$.

(c) When n is odd, $\sum_{i \text{ even}} p_i = \sum_{i \text{ odd}} p_i = \frac{1}{2}$.

(d) When $n = 2k$ is even,

$$\sum_{i \text{ even}} p_i = \frac{2}{\pi}\arctan\left(\sqrt{\frac{k+1}{k}}\right) \quad \text{and} \quad \sum_{i \text{ odd}} p_i = \frac{2}{\pi}\arctan\left(\sqrt{\frac{k}{k+1}}\right). \tag{7.2}$$

Proof. The first assertion is obvious. To see the second one we compare the difference equation $a_n x_{k+n} + a_{n-1}x_{k+n-1} + \cdots + a_1 x_{k+1} + a_0 x_k = 0$, with $a_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, with characteristic polynomial $Q(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$ with the difference equation $a_n x_k + a_{n-1}x_{k+1} + \cdots + a_0 x_{k+n} = 0$ with characteristic polynomial $Q^*(\lambda) = a_n + a_{n-1}\lambda + \cdots + a_1\lambda^{n-1} + a_0\lambda^n$. Notice that if $Q(\lambda)$ has m non-zero roots with modulus smaller than 1 and $n - m$ with modulus bigger than 1, then the converse follows for $Q^*(\lambda)$ because $Q(\lambda) = 0$ if and only if $Q^*(\frac{1}{\lambda}) = 0$. From this result applied to the corresponding random polynomials we get that $p_k = p_{n-k}$, because both have identically distributed coefficients. So we have proved statement (b). To prove items (c) and (d) recall first that it was proved in item (c) of Proposition 5.1 that a polynomial $Q(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$, without roots with zero real part, has an even number of roots with negative real part if and only if $a_n a_0 > 0$. By using the polynomial

$$\begin{aligned} Q^\star(z) &= a_n(z+1)^n + a_{n-1}(z+1)^{n-1}(z-1) + \ldots + a_0(z-1)^n \\ &= (a_n + a_{n-1} + \cdots + a_1 + a_0)z^n + \cdots + (a_n - a_{n-1} + a_{n-2} - \cdots + (-1)^n a_0), \end{aligned}$$

introduced in Section 3 (Equation (3.2)) we get that $Q(\lambda)$, without roots of modulus 1, has an even number $(2m)$ of roots with modulus smaller than 1 if and only if $Q^\star(z)$ has exactly $2m$ roots with negative real part and this happens if and only if $(a_n + a_{n-1} + \cdots + a_1 + a_0) \cdot (a_n - a_{n-1} + a_{n-2} - \cdots + (-1)^n a_0) > 0$. Hence, considering the corresponding random polynomials, we have that

$$\begin{aligned} \sum_{i \text{ even}} p_i &= P\big((A_n + A_{n-1} + \cdots + A_0) \cdot (A_n - A_{n-1} + \cdots + (-1)^n A_0) > 0\big) \\ &= P(U^2 - V^2 > 0), \end{aligned}$$

where $U = A_n + A_{n-2} + A_{n-4} + \cdots$ and $V = A_{n-1} + A_{n-3} + A_{n-5} + \cdots$ and the sums end either at A_0 or A_1 according the parity of n. Since $A_j \sim \mathrm{N}(0,1)$ and all A_j are independent we get that when $n = 2k$ (resp. $n = 2k-1$) then $U \sim \mathrm{N}(0, k+1)$ (resp. $U \sim \mathrm{N}(0.k)$) and $V \sim \mathrm{N}(0, k)$ and U and V are independent. Hence, by using Lemma 7.2, we obtain that when $n = 2k-1$, $P(U^2 - V^2 > 0) = 1/2$ and that when $n = 2k$,

$$\sum_{i \text{ even}} p_i = P(U^2 - V^2 > 0) = \frac{2}{\pi} \arctan\left(\sqrt{\frac{k+1}{k}}\right).$$

The sum of all p_i when i is odd can be obtained from the above formula, see also (7.1). □

Corollary 7.4.

(i) Consider the linear random homogeneous difference equation of order n, let X be the random variable defined above and $p_k = P(X = k)$. Then the probabilities p_k satisfy all the conclusions of Lemma 3.1. In particular $E(X) = n/2$.

(ii) Moreover the new affine relations given in Equations (7.2) hold. In particular, for $n = 2$, $p_0 = p_2 = \frac{1}{\pi}\arctan(\sqrt{2})$ and $p_1 = \frac{2}{\pi}\arctan(1/\sqrt{2})$; and for $n = 4$, $p_1 = p_3 = \frac{1}{\pi}\arctan(\sqrt{2/3})$.

In this case, and for the situations where we have not been able to obtain the exact probabilities we have done similar computations than in the previous section, first with the Monte Carlo method, generating for each order $n = 0, \ldots, 10$, 10^8 random vectors $(A_0, \ldots, A_n) \in \mathbb{R}^{n+1}$ whose components are pseudo-random numbers with $\mathrm{N}(0,1)$ distribution. Then, by using the relations in Proposition 7.3 and Corollary 7.4 we have performed a least squares refinement.

For instance for $n = 4$, by Corollary 7.4 we have $p_1 = p_3 = \arctan(\sqrt{2/3})/\pi \simeq 0.217953$; $p_2 = 2\arctan(\sqrt{3/2})/\pi - 2p_0$ and $p_4 = p_0$. Hence, we fix the values $\widehat{p}_1 = p_1$ and $\widehat{p}_3 = p_3$ and system (3.3) can be written in the form

$$\mathrm{M}\widehat{\mathbf{q}} + b = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot (\widehat{p}_0) + \begin{pmatrix} 0 \\ \frac{2}{\pi}\arctan\left(\sqrt{\frac{3}{2}}\right) \\ 0 \end{pmatrix} = \begin{pmatrix} \widetilde{p}_0 \\ \widetilde{p}_2 \\ \widetilde{p}_4 \end{pmatrix}.$$

Hence we can easily find the least squares solution of the above incompatible linear system:

$$(1, -2, 1) \cdot \left[\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot (\widehat{p}_0) + \begin{pmatrix} 0 \\ \frac{2}{\pi}\arctan\left(\sqrt{\frac{3}{2}}\right) \\ 0 \end{pmatrix} \right] = (1, -2, 1) \cdot \begin{pmatrix} \widetilde{p}_0 \\ \widetilde{p}_2 \\ \widetilde{p}_4 \end{pmatrix},$$

and thus we get the result in Equation (3.4): $\widehat{p}_0 = \frac{1}{6}\left(\widetilde{p}_0 - 2\widetilde{p}_2 + \widetilde{p}_4\right) + \frac{2}{3\pi}\arctan\left(\sqrt{3/2}\right)$, and therefore $\widehat{p}_4 = \widehat{p}_0$ and $\widehat{p}_2 = 2\arctan(\sqrt{3/2})/\pi - 2\widehat{p}_0$. Since our Monte Carlo simulations give

$$(\widetilde{p}_0, \widetilde{p}_2, \widetilde{p}_4) = \left(\frac{2056203}{20000000}, \frac{7169499}{20000000}, \frac{10285619}{100000000}\right) \simeq (0.10281, 0.35847, 0.10286),$$

the above relations show that $(\widehat{p}_0, \widehat{p}_2, \widehat{p}_4) \simeq (0.10282, 0.35846, 0.10282)$.

All our results are collected in Table 7.1.

Order	Observed frequency	Least squares	Relations (Prop. 7.3 and Cor. 7.4))
$n=1$	$\widetilde{p}_0=0.49991$ $\widetilde{p}_1=0.50009$		$p_0=0.5$ $p_1=0.5$
$n=2$	$\widetilde{p}_0=0.30410$ $\widetilde{p}_1=0.39184$ $\widetilde{p}_2=0.30406$		$p_0=\frac{1}{\pi}\arctan(\sqrt{2})\simeq 0.304087$ $p_1=\frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{2}}\right)\simeq 0.391826$ $p_2=\frac{1}{\pi}\arctan(\sqrt{2})\simeq 0.304087$
$n=3$	$\widetilde{p}_0=0.17251$ $\widetilde{p}_1=0.32752$ $\widetilde{p}_2=0.32753$ $\widetilde{p}_3=0.17244$	$\widehat{p}_0=0.17248$ $\widehat{p}_1=0.32752$ $\widehat{p}_2=0.32752$ $\widehat{p}_3=0.17248$	p_0 $p_1=\frac{1}{2}-p_0$ $p_2=\frac{1}{2}-p_0$ $p_3=p_0$
$n=4$	$\widetilde{p}_0=0.10281$ $\widetilde{p}_1=0.21792$ $\widetilde{p}_2=0.35847$ $\widetilde{p}_3=0.21794$ $\widetilde{p}_4=0.10286$	$\widehat{p}_0=0.10282$ $\widehat{p}_1=0.21795$ $\widehat{p}_2=0.35846$ $\widehat{p}_3=0.21795$ $\widehat{p}_4=0.10282$	p_0 $p_1=\frac{1}{\pi}\arctan\left(\sqrt{\frac{2}{3}}\right)\simeq 0.217953$ $p_2=\frac{2}{\pi}\arctan\left(\sqrt{\frac{3}{2}}\right)-2p_0$ $p_3=\frac{1}{\pi}\arctan\left(\sqrt{\frac{2}{3}}\right)\simeq 0.217953$ $p_4=p_0$
$n=5$	$\widetilde{p}_0=0.05909$ $\widetilde{p}_1=0.15331$ $\widetilde{p}_2=0.28760$ $\widetilde{p}_3=0.28756$ $\widetilde{p}_4=0.15335$ $\widetilde{p}_5=0.05908$	$\widehat{p}_0=0.05909$ $\widehat{p}_1=0.15333$ $\widehat{p}_2=0.28758$ $\widehat{p}_3=0.28758$ $\widehat{p}_4=0.15333$ $\widehat{p}_5=0.05909$	p_0 p_1 $p_2=\frac{1}{2}-p_0-p_1$ $p_3=\frac{1}{2}-p_0-p_1$ $p_4=p_1$ $p_5=p_0$
$n=6$	$\widetilde{p}_0=0.03501$ $\widetilde{p}_1=0.09726$ $\widetilde{p}_2=0.23777$ $\widetilde{p}_3=0.25985$ $\widetilde{p}_4=0.23781$ $\widetilde{p}_5=0.09724$ $\widetilde{p}_6=0.03505$	$\widehat{p}_0=0.03502$ $\widehat{p}_1=0.09726$ $\widehat{p}_2=0.23779$ $\widehat{p}_3=0.25986$ $\widehat{p}_4=0.23779$ $\widehat{p}_5=0.09726$ $\widehat{p}_6=0.03502$	p_0 p_1 $p_2=\frac{1}{\pi}\arctan\left(\sqrt{\frac{4}{3}}\right)-p_0$ $p_3=\frac{2}{\pi}\arctan\left(\sqrt{\frac{3}{4}}\right)-2p_1$ $p_4=\frac{1}{\pi}\arctan\left(\sqrt{\frac{4}{3}}\right)-p_0$ $p_5=p_1$ $p_6=p_0$
$n=7$	$\widetilde{p}_0=0.02025$ $\widetilde{p}_1=0.06432$ $\widetilde{p}_2=0.17174$ $\widetilde{p}_3=0.24376$ $\widetilde{p}_4=0.24361$ $\widetilde{p}_5=0.17177$ $\widetilde{p}_6=0.06428$ $\widetilde{p}_7=0.02025$	$\widehat{p}_0=0.02025$ $\widehat{p}_1=0.06430$ $\widehat{p}_2=0.17176$ $\widehat{p}_3=0.24369$ $\widehat{p}_4=0.24369$ $\widehat{p}_5=0.17176$ $\widehat{p}_6=0.06430$ $\widehat{p}_7=0.02025$	p_0 p_1 p_2 $p_3=\frac{1}{2}-p_0-p_1-p_2$ $p_4=\frac{1}{2}-p_0-p_1-p_2$ $p_5=p_2$ $p_6=p_1$ $p_7=p_0$
$n=8$	$\widetilde{p}_0=0.01194$ $\widetilde{p}_1=0.03994$ $\widetilde{p}_2=0.12272$ $\widetilde{p}_3=0.19230$ $\widetilde{p}_4=0.25701$ $\widetilde{p}_5=0.19238$ $\widetilde{p}_6=0.12724$ $\widetilde{p}_7=0.03994$ $\widetilde{p}_8=0.01197$	$\widehat{p}_0=0.01196$ $\widehat{p}_1=0.03994$ $\widehat{p}_2=0.12726$ $\widehat{p}_3=0.19234$ $\widehat{p}_4=0.25700$ $\widehat{p}_5=0.19234$ $\widehat{p}_6=0.12726$ $\widehat{p}_7=0.03994$ $\widehat{p}_8=0.01196$	p_0 p_1 p_2 $p_3=\frac{1}{\pi}\arctan\left(\sqrt{\frac{4}{5}}\right)-p_1$ $p_4=\frac{2}{\pi}\arctan\left(\sqrt{\frac{5}{4}}\right)-2p_0-2p_2$ $p_5=\frac{1}{\pi}\arctan\left(\sqrt{\frac{4}{5}}\right)-p_1$ $p_6=p_2$ $p_7=p_1$ $p_8=p_0$

Order	Observed frequency	Least squares	Relations (Prop. 7.3 and Cor. 7.4)
$n=9$	$\widetilde{p}_0 = 0.00693$	$\widehat{p}_0 = 0.00693$	p_0
	$\widetilde{p}_1 = 0.02556$	$\widehat{p}_1 = 0.02556$	p_1
	$\widetilde{p}_2 = 0.08711$	$\widehat{p}_2 = 0.08711$	p_2
	$\widetilde{p}_3 = 0.15653$	$\widehat{p}_3 = 0.15653$	p_3
	$\widetilde{p}_4 = 0.22389$	$\widehat{p}_4 = 0.22386$	$p_4 = \frac{1}{2} - p_0 - p_1 - p_2 - p_3$
	$\widetilde{p}_5 = 0.22382$	$\widehat{p}_5 = 0.22386$	$p_5 = \frac{1}{2} - p_0 - p_1 - p_2 - p_3$
	$\widetilde{p}_6 = 0.15654$	$\widehat{p}_6 = 0.15653$	$p_6 = p_3$
	$\widetilde{p}_7 = 0.08712$	$\widehat{p}_7 = 0.08711$	$p_7 = p_2$
	$\widetilde{p}_8 = 0.02557$	$\widehat{p}_8 = 0.02556$	$p_8 = p_1$
	$\widetilde{p}_9 = 0.00693$	$\widehat{p}_9 = 0.00693$	$p_9 = p_0$
$n=10$	$\widetilde{p}_0 = 0.00409$	$\widehat{p}_0 = 0.00411$	p_0
	$\widetilde{p}_1 = 0.01567$	$\widehat{p}_1 = 0.01566$	p_1
	$\widetilde{p}_2 = 0.06089$	$\widehat{p}_2 = 0.06091$	p_2
	$\widetilde{p}_3 = 0.11500$	$\widehat{p}_3 = 0.11497$	p_3
	$\widetilde{p}_4 = 0.19950$	$\widehat{p}_4 = 0.19947$	$p_4 = \frac{1}{\pi}\arctan\left(\sqrt{\frac{6}{5}}\right) - p_0 - p_2$
	$\widetilde{p}_5 = 0.20978$	$\widehat{p}_5 = 0.20976$	$p_5 = \frac{2}{\pi}\arctan\left(\sqrt{\frac{5}{6}}\right) - 2p_1 - 2p_3$
	$\widetilde{p}_6 = 0.19941$	$\widehat{p}_6 = 0.19947$	$p_6 = \frac{1}{\pi}\arctan\left(\sqrt{\frac{6}{5}}\right) - p_0 - p_2$
	$\widetilde{p}_7 = 0.11499$	$\widehat{p}_7 = 0.11497$	$p_7 = p_3$
	$\widetilde{p}_8 = 0.06088$	$\widehat{p}_8 = 0.06091$	$p_8 = p_2$
	$\widetilde{p}_9 = 0.01570$	$\widehat{p}_9 = 0.01566$	$p_9 = p_1$
	$\widetilde{p}_{10} = 0.00408$	$\widehat{p}_{10} = 0.00411$	$p_{10} = p_0$

Table 7.1: Stability indexes for order n linear random homogeneous difference equations.

Acknowledgments

We want to thank Professors Maria Jolis and Joan Torregrosa for their helpful indications. We also acknowledge the anonymous reviewer for the careful reading and for addressing to us very interesting suggestions that have allowed us to improve our work.

The authors are supported by Ministry of Science and Innovation–State Research Agency of the Spanish Government through grants PID2019-104658GB-I00 (MICINN/AEI, first and second authors) and DPI2016-77407-P (MICINN/AEI/FEDER, UE, third author). The first and second authors are also supported by the grant 2017-SGR-1617 from AGAUR, Generalitat de Catalunya. The third author acknowledges the group's research recognition 2017-SGR-388 from AGAUR, Generalitat de Catalunya.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **16**, 1–21; https://doi.org/10.14232/ejqtde.2021.1.16 www.math.u-szeged.hu/ejqtde/

Period function of planar turning points

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Received 17 November 2020, appeared 22 March 2021
Communicated by Armengol Gasull

Abstract. This paper is devoted to the study of the period function of planar generic and non-generic turning points. In the generic case (resp. non-generic) a non-degenerate (resp. degenerate) center disappears in the limit $\epsilon \to 0$, where $\epsilon \geq 0$ is the singular perturbation parameter. We show that, for each $\epsilon > 0$ and $\epsilon \sim 0$, the period function is monotonously increasing (resp. has exactly one minimum). The result is valid in an ϵ-uniform neighborhood of the turning points. We also solve a part of the conjecture about a uniform upper bound for the number of critical periods inside classical Liénard systems of fixed degree, formulated by De Maesschalck and Dumortier in 2007. We use singular perturbation theory and the family blow-up.

Keywords: critical periods, family blow-up, period function, slow-fast systems.

2020 Mathematics Subject Classification: 34E15, 34E17.

1 Introduction

We consider slow-fast polynomial Liénard equations of center type

$$X_{\epsilon,\eta} : \begin{cases} \dot{x} = y - \left(x^{2n} + \displaystyle\sum_{k=1}^{l} a_k x^{2n+2k} \right), \\ \dot{y} = \epsilon^{2n} \left(-x^{2n-1} + \displaystyle\sum_{k=1}^{m} b_k x^{2n+2k-1} \right), \end{cases} \tag{1.1}$$

where $l, m, n \geq 1$, $\eta := (a_1, \ldots, a_l, b_1, \ldots, b_m)$ is kept in a compact set K of $\mathbb{R}^{l+m}$ and $\epsilon \geq 0$ is the singular perturbation parameter kept small. System $X_{\epsilon,\eta}$ is invariant under the symmetry $(x,t) \to (-x,-t)$ and has a center at the origin for all $\epsilon > 0$, $\epsilon \sim 0$, and for all $\eta \in K$. The center is non-degenerate when $n = 1$ or nilpotent when $n > 1$. In the limit $\epsilon = 0$, we encounter drastic changes in the dynamics of (1.1): the system has a curve of singular points, given by $\{y = x^{2n} + \sum_{k=1}^{l} a_k x^{2n+2k}\}$, passing through the origin, and horizontal regular orbits (see Figure 1.1).

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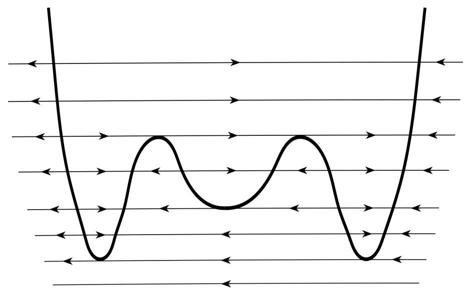

Figure 1.1: Dynamics of $X_{0,\eta}$ with 5 contact points.

A portion of the curve of singularities near the origin consists of the normally attracting part $\{x > 0\}$, the normally repelling part $\{x < 0\}$ and the contact point $(x, y) = (0, 0)$ between them. We call the contact point a turning point because closed orbits surrounding the center, for $\epsilon > 0$ small, pass from the attracting part to the repelling part of the curve of singularities. When $n = 1$, the turning point at the origin is generic (sometimes called simple). When $n > 1$, we deal with a non-generic or degenerate turning point.

The period function of a center assigns to each periodic orbit its minimal period. Isolated critical points of the period function are called critical periods (or critical periodic orbits) and are central in the qualitative study of the period function. One can note that critical periods do not depend on the parametrization of the set of periodic orbits used. Indeed, if $\{\gamma_s\}_{s\in(0,1)}$ is such a parametrization and $s \mapsto T(s)$ is the period of the periodic orbit γ_s, for any diffeomorphism $s \mapsto \xi = \xi(s)$, $\frac{d}{ds}T(\xi(s)) = \frac{d}{d\xi}T(\xi(s))\frac{d}{ds}\xi(s)$. Therefore the number of isolated zeros of $\frac{d}{ds}(T \circ \xi)$ and $\frac{d}{ds}T$ are the same.

The main purpose of this paper is to give a complete local study of the period function of $X_{\epsilon,\eta}$, near the center at the origin, in both the generic and non-generic case. The study is valid in a small fixed neighborhood of the turning point that is independent of (ϵ, η). Thus, the neighborhood does not shrink to the origin as $\epsilon \to 0$. In the generic (resp. non-generic) case, the period function of the center in $X_{\epsilon,\eta}$ is strictly monotonous increasing (resp. has exactly one critical period which is a minimum). More precisely, let us denote by $T(y; \epsilon)$ the period function of the center at the origin of system (1.1) with $\epsilon > 0$, $\epsilon \sim 0$, parametrized by the positive y-axis. Then we have:

Theorem 1.1. *Let $l, m \geq 1$ and $n = 1$ (resp. $n > 1$) be fixed. For any compact $K \subset \mathbb{R}^{l+m}$ there exist $\epsilon_0 > 0$ and $y_0 > 0$ small enough such that the period function $T(y; \epsilon)$ of the center of system (1.1) is strictly monotonous increasing (resp. has a global minimum) in the interval $]0, y_0]$, for all $\epsilon \in]0, \epsilon_0]$ and $\eta \in K$.*

We prove Theorem 1.1 in Section 3.4. To prove Theorem 1.1, we use a blow-up at the origin in the (x, y, ϵ)-space to desingularize system (1.1). Roughly speaking, after the blow-up we distinguish between "very small", "small" and "intermediate" closed orbits surrounding the center $(x, y) = (0, 0)$. The period function of the center of system (1.1) cannot be studied uniformly in these three regions and we have to use different techniques for each type of closed orbits. To treat the period function near the very small closed orbits (the ones closest to the center), we use Chicone and Jacobs [2], in the generic case, and generalized polar coordinates, in the non-generic case. The small closed orbits can be treated using the monotonicity criterion due to Schaaf [11], in the generic case, and a result due to Sabatini [10], in the non-generic case. The size of the very small and small closed orbits tends to zero as $\epsilon \to 0$. In order to obtain the result in an (ϵ, η)-uniform neighborhood of $(x, y) = (0, 0)$, the period function near the passage from the small closed orbits to large closed orbits of size $O(1)$ in the (x, y)-space has to be studied. In this passage, we encounter the so-called intermediate closed orbits. The period function near such intermediate closed orbits, in both the generic and non-generic case, will be studied using techniques from [6, 8], where small-amplitude limit cycles in an ϵ-uniform neighborhood of slow-fast Hopf points have been investigated (the slow-fast Hopf points correspond to the generic case). For more details we refer to Section 2.

We point out that Theorem 1.1 can be proved in a more general framework of smooth Liénard systems. More precisely, the same local result is true if we replace (1.1) with $\{\dot{x} = y - x^{2n} + O(x^{2n+2}), \dot{y} = \epsilon^{2n}\left(-x^{2n-1} + O(x^{2n+1})\right)\}$ where $O(x^{2n+2})$ (resp. $O(x^{2n+1})$) is an even (resp. odd) C^∞-perturbation term that may depend on parameters kept in a compact set. The proof in this more general setting is analogous to the proof for polynomial Liénard equations presented in this paper.

Theorem 1.1, in the generic case $n = 1$, can be used to solve a part of the following conjecture formulated in [4]: there exists a uniform upper bound on the number of critical periods of classical Liénard equations $\{\dot{x} = y - G(x), \dot{y} = -x\}$ where G is an even polynomial of degree $2N$, $N \geq 1$, and $G(0) = 0$. Moreover, this upper bound is conjectured to be $2N - 2$. Following Theorem 5 in [4], this can be reduced to the following problem: there exist a small $\epsilon_0 > 0$ and an integer $M > 0$ such that the slow-fast system

$$\begin{cases} \dot{x} = y - \left(x^{2N} + \sum_{k=1}^{N-1} c_{2k} x^{2k}\right), \\ \dot{y} = -\epsilon x, \end{cases} \tag{1.2}$$

has at most M critical periods, for all $\epsilon \in]0, \epsilon_0]$ and $(c_2, c_4, \dots, c_{2(N-1)}) \in \mathbb{S}^{N-2}$. The following result covers the case where the curve of singularities of (1.2), at level $\epsilon = 0$, has only one contact point, the one at the origin $(x, y) = (0, 0)$.

Theorem 1.2. *Let $c_2^0 > 0$ be small and fixed and let $N \geq 1$ be a fixed integer. Denote by C the set of all values $(c_2, c_4, \dots, c_{2(N-1)}) \in \mathbb{S}^{N-2}$ such that $c_2 \geq c_2^0$ and $\frac{G'(x)}{x} > 0$ for all $x \in \mathbb{R}$, where $G(x) = x^{2N} + \sum_{k=1}^{N-1} c_{2k} x^{2k}$. For any compact set $\tilde{C}$, with $\tilde{C} \subset C$, there exists a small $\epsilon_0 > 0$ such that system (1.2) has no critical periods for all $\epsilon \in]0, \epsilon_0]$ and $(c_2, c_4, \dots, c_{2(N-1)}) \in \tilde{C}$.*

We prove Theorem 1.2 in Section 3.5. Note that keeping the parameter in a compact set $\tilde{C}$ ensures that the critical curve has no contact points other than the origin. The compact set

$$\tilde{C} = \{(c_2, c_4, \dots, c_{2(N-1)}) \in \mathbb{S}^{N-2} \mid c_2 \geq c_2^0 \text{ and } c_i \geq 0 \text{ for } i = 4, \dots, 2(N-1)\}$$

is always contained in the set C defined in Theorem 1.2. When $N = 1$, Theorem 1.2 implies that $\{\dot{x} = y - x^2, \dot{y} = -\epsilon x\}$ has no critical periods for all $\epsilon \in]0, \epsilon_0]$, for some small $\epsilon_0 > 0$. When $N = 2$, we have to deal with the slow-fast systems $\{\dot{x} = y - x^4 \pm x^2, \dot{y} = -\epsilon x\}$. From Theorem 1.2 follows that the system with the negative sign in front of x^2 has no critical periods. The system with the positive sign in front of x^2 is conjectured to have at most 2 critical periods (see [4]). As explained in [4], it is more difficult to deal with the part of the conjecture when the curve of singularities of (1.2) has more contact points.

When c_2 is uniformly nonzero, Theorem 1.1 in the generic case implies that system (1.2) has no critical periods in an ϵ-uniform neighborhood of the origin in the (x, y)-space. It suffices to notice that the change of coordinates $(x, y) \to (c_2 x, c_2 y)$ transforms (1.2) into (1.1). See Section 3.5.

2 Blow-up and statement of results

2.1 Family blow-up at the origin in the (x, y, ϵ)-space

To be able to study the period function near the turning point, uniformly in (ϵ, η) with $\epsilon > 0$ small, we have to desingularize the system $X_{\epsilon,\eta}$ near $(x, y, \epsilon) = (0, 0, 0)$ using the so-called family blow-up. The family blow-up is the following "singular" coordinate change with $n \geq 1$:

$$\Psi : \mathbb{R}^+ \times \mathbb{S}^2_+ \to \mathbb{R}^3 : (r, (\bar{x}, \bar{y}, \bar{\epsilon})) \mapsto (x, y, \epsilon) = (r\bar{x}, r^{2n}\bar{y}, r\bar{\epsilon}),\ \bar{\epsilon} \geq 0.$$

We define the blown-up vector field as the pullback of $X_{\epsilon,\eta} + 0\frac{\partial}{\partial\epsilon}$ divided by r^{2n-1}: $\bar{X}_\eta := \frac{1}{r^{2n-1}}\Psi^*\left(X_{\epsilon,\eta} + 0\frac{\partial}{\partial\epsilon}\right)$. To study the blown-up vector field $\bar{X}_\eta$ (or $r^{2n-1}\bar{X}_\eta$) near the blow-up locus $\{0\} \times \mathbb{S}^2_+$, it is convenient to use different charts with "rectified" coordinates, instead of the spherical coordinates. For our purposes, only the family chart $\{\bar{\epsilon} = 1\}$ and the phase directional chart $\{\bar{y} = 1\}$ are relevant for the study of the period function since all closed orbits near the center $(x, y) = (0, 0)$ are visible therein (see Figure 2.1).

In the family chart $\{\bar{\epsilon} = 1\}$, we have

$$(x, y, \epsilon) = (r\bar{x}, r^{2n}\bar{y}, r)$$

with $(\bar{x}, \bar{y})$ kept in an arbitrary but fixed compact set. In this chart, $r = \epsilon$ and system (1.1) becomes $X_F := \epsilon^{2n-1}\bar{X}_F$, where

$$\bar{X}_F : \begin{cases} \dot{\bar{x}} = \bar{y} - \left(\bar{x}^{2n} + \sum_{k=1}^{l} a_k\epsilon^{2k}\bar{x}^{2n+2k}\right), \\ \dot{\bar{y}} = -\bar{x}^{2n-1} + \sum_{k=1}^{m} b_k\epsilon^{2k}\bar{x}^{2n+2k-1}. \end{cases} \tag{2.1}$$

System (2.1) is invariant under the symmetry $(\bar{x}, t) \to (-\bar{x}, -t)$ and has a center at the origin, for all $\epsilon \geq 0$, $\epsilon \sim 0$ and $\eta \in K$. When $\epsilon = 0$, we are located on the blow-up locus and the vector field (2.1) becomes

$$\begin{cases} \dot{\bar{x}} = \bar{y} - \bar{x}^{2n}, \\ \dot{\bar{y}} = -\bar{x}^{2n-1}. \end{cases} \tag{2.2}$$

A first integral of (2.2) is given by

$$H(\bar{x}, \bar{y}) = e^{-2n\bar{y}}\left(\bar{y} - \bar{x}^{2n} + \frac{1}{2n}\right). \tag{2.3}$$

Note that the invariant curve $\{\bar{y} = \bar{x}^{2n} - \frac{1}{2n}\}$ is the boundary of the period annulus (see Figure 2.1). The main advantage of the family blow-up is that the blown-up vector field (2.1) has no curves of singularities.

The $(\bar{x}, \bar{y})$-compact sets in which we will study system (2.1) (see Section 2.2) shrink to the origin in the (x, y)-space as $\epsilon \to 0$. To obtain (ϵ, η)-uniform results, we also have to study $X_{\epsilon,\eta}$ in the phase directional chart $\{\bar{y} = 1\}$. In the chart $\{\bar{y} = 1\}$, we deal with the coordinate change

$$(x, y, \epsilon) = (RX, R^{2n}, RE),$$

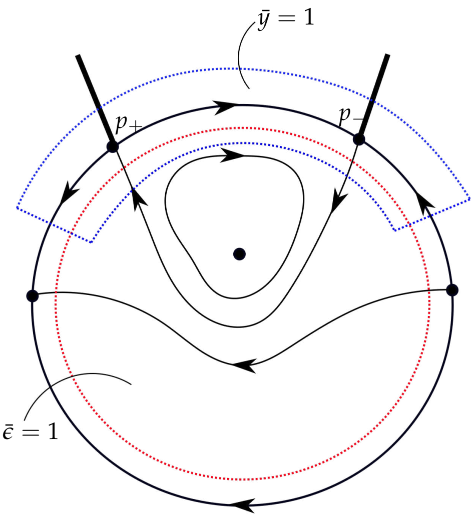

Figure 2.1: The family blow-up at the origin $(x, y, \epsilon) = (0, 0, 0)$ and dynamics on the blow-up locus. To study the period function of (1.1) in an (ϵ, η)-uniform neighborhood of $(x, y) = (0, 0)$, it suffices to use the charts $\{\bar{\epsilon} = 1\}$ and $\{\bar{y} = 1\}$.

where $R \geq 0$ and $E \geq 0$ are small and X is kept in any compact set. System (1.1) becomes $X_D := R^{2n-1}\bar{X}_D$, where

$$\bar{X}_D : \begin{cases} \dot{X} = 1 - \left(X^{2n} + \sum_{k=1}^{l} a_k R^{2k} X^{2n+2k} \right) + \frac{1}{2n} X E^{2n} F(X, R, \eta), \\ \dot{R} = -\frac{1}{2n} R E^{2n} F(X, R, \eta), \\ \dot{E} = \frac{1}{2n} E^{2n+1} F(X, R, \eta), \end{cases} \tag{2.4}$$

with $F(X, R, \eta) = X^{2n-1} - \sum_{k=1}^{m} b_k R^{2k} X^{2n+2k-1}$. For $R = E = 0$, the system has semi-hyperbolic singularities at $X = -1$ (denoted by p_+) and $X = 1$ (denoted by p_-). The singularity p_+ (resp. p_-) has the X-axis as unstable (resp. stable) manifold and a two-dimensional center manifold, transverse to the X-axis.

Using (2.1) and (2.4) we easily detect the singular polycycle Γ on the blow-up locus consisting of singularities p_+ and p_- and the regular orbits that are heteroclinic to them (see Figure 2.1). Note that $p_\pm$ are the end points of the regular curve $\{\bar{y} = \bar{x}^{2n} - \frac{1}{2n}\}$.

It is clear now that the study of the period function of the center in (1.1), in a small ϵ-uniform neighborhood of $(x, y) = (0, 0)$, can be divided into three parts: the study near the center $(\bar{x}, \bar{y}) = (0, 0)$ of (2.1), the study of the interior of the period annulus inside the family (2.1), away from $(\bar{x}, \bar{y}) = (0, 0)$ and Γ, and the study near Γ, combining systems (2.1) and (2.4). The results related to the first two parts (resp. the third part) are stated in Section 2.2 (resp. Section 2.3).

2.2 Statement of results inside the vector field $\bar{X}_F$

Let $l, m \geq 1$ be fixed. For the vector field $\bar{X}_F$ given in (2.1) we define by $T_F(\bar{y}; \epsilon)$ the period function of the center at the origin parametrized by the $\bar{y}$-axis. As we will see in Sections 3.1 and 3.2, the function $T_F(\bar{y}; \epsilon)$ is well defined in any compact interval $[\bar{y}_1, \bar{y}_2]$ and when the turning point is generic it can be extended analytically to $\bar{y} = 0$. We prove the following two results concerning the period function of system $\bar{X}_F$. The first one states the behaviour of the period function of the center of system (2.1) close to the equilibrium at the origin, whereas the second one is a global statement in the interior of the period annulus.

Theorem 2.1. *For any compact $K \subset \mathbb{R}^{l+m}$ there exist $\bar{y}_0 > 0$ small enough and $\epsilon_0 > 0$ small enough such that $\frac{d}{d\bar{y}} T_F(\bar{y};\epsilon) > 0$ (resp. $\frac{d}{d\bar{y}} T_F(\bar{y};\epsilon) < 0$) for $n = 1$ (resp. $n > 1$) for all $\bar{y} \in]0, \bar{y}_0]$, $\epsilon \in [0, \epsilon_0]$ and $\eta \in K$. Moreover, $T_F(\bar{y};\epsilon) \to 2\pi$ (resp. $+\infty$) as $\bar{y} \to 0^+$ for $n = 1$ (resp. $n > 1$).*

Theorem 2.2. *For any compact $K \subset \mathbb{R}^{l+m}$ and any $0 < \bar{y}_1 < \bar{y}_2 < +\infty$ there exists $\epsilon_0 > 0$ small enough such that $\frac{d}{d\bar{y}} T_F(\bar{y};\epsilon) > 0$ (resp. $\frac{d^2}{d\bar{y}^2} T_F(\bar{y};\epsilon) > 0$) for $n = 1$ (resp. $n > 1$) for all $\bar{y} \in [\bar{y}_1, \bar{y}_2]$, $\epsilon \in [0, \epsilon_0]$ and $\eta \in K$.*

We prove Theorem 2.1 in Section 3.1 and Theorem 2.2 in Section 3.2. A key fact in the proof of the previous results is that, when $\epsilon = 0$, the vector field (2.1) becomes (2.2) with a first integral given by (2.3). As we will see, the period function of (2.1) is an ϵ-perturbation of the period function of (2.2).

2.3 Statement of results near Γ

In both the generic and non-generic case, we have the following result about the period function of the center of the vector field $r^{2n-1}\bar{X}_\eta$, with $r > 0$, in an η-uniform neighborhood of Γ.

Theorem 2.3. *Let $l, m \geq 1$ be fixed. For any compact $K \subset \mathbb{R}^{l+m}$ there exists $\epsilon_0 > 0$ small enough such that the period function of the center of system $r^{2n-1}\bar{X}_\eta$, with $r > 0$, near the polycycle Γ is monotonous increasing for all $\epsilon \in]0, \epsilon_0]$ and $\eta \in K$.*

We prove Theorem 2.3 in Section 3.3. For a precise definition of a neighborhood of Γ in the family blow-up coordinates and the period function near Γ see Section 3.3.

3 Proof of Theorem 1.1–Theorem 2.3

First we prove Theorem 2.1, Theorem 2.2 and Theorem 2.3. Then we glue them together and prove Theorem 1.1 (see Section 3.4). Theorem 1.2 is proved in Section 3.5.

3.1 Proof of Theorem 2.1

Let us start considering the case $n = 1$. We define by $\mathcal{T}_F(\bar{x};\epsilon)$ the period function of system (2.1) parametrized by the $\bar{x}$-axis. Notice that, since $n = 1$, the center at the origin is non-degenerate and therefore the period function can be extended analytically to $\bar{x} = 0$. For $\epsilon \geq 0$ small system (2.1) is an analytic perturbation of the quadratic system

$$\begin{cases} \dot{\bar{x}} = \bar{y} - \bar{x}^2, \\ \dot{\bar{y}} = -\bar{x}. \end{cases} \tag{3.1}$$

Therefore we can consider the Taylor's series development at $\epsilon = 0$ of $\mathcal{T}_F(\bar{x};\epsilon)$,

$$\mathcal{T}_F(\bar{x};\epsilon) = \mathcal{T}_0(\bar{x}) + O(\epsilon),$$

where $\mathcal{T}_0(\bar{x})$ is the period function of system (3.1) parametrized by the $\bar{x}$-axis. In particular, if $\frac{d}{d\bar{x}}\mathcal{T}_0(\bar{x}) > 0$ then $\frac{d}{d\bar{x}}\mathcal{T}_F(\bar{x};\epsilon) > 0$ for every $\epsilon \geq 0$ small enough. In consequence, the assertion concerning $n = 1$ in Theorem 2.1 will follow once we show that the period function $\mathcal{T}_0(\bar{x})$ of the quadratic system (3.1) is monotonous increasing near the origin.

To do so we use Chicone and Jacobs [2] result on quadratic centers to deduce that, in a neighborhood of the origin,

$$\mathcal{T}_0(\bar{x}) = 2\pi + p_2(\lambda)\bar{x}^2 + O(\bar{x}^3),$$

where $p_2(\lambda) = \frac{\pi}{12}(16\lambda_2^2 + 8\lambda_2\lambda_5 + \lambda_5^2 + 18\lambda_3^2 - 12\lambda_3\lambda_6 + 9\lambda_3\lambda_4 + 10\lambda_6^2 - \lambda_4\lambda_6 + \lambda_4^2)$, $\lambda = (\lambda_i)_{i=2}^6$, and λ_i stand for the coefficients of the Bautin's normal form for quadratic systems

$$\begin{cases} \dot{x} = -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} = x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{cases}$$

In our case system (3.1) can be brought to the Bautin's normal form with the change of variable $\{\bar{y} \mapsto -\bar{y}\}$ and corresponds to the parameters $\lambda_2 = \lambda_5 = \lambda_6 = 0$, $\lambda_3 = 1$ and $\lambda_4 = -2$. Consequently, for system (3.1) the period function near the origin can be written as

$$\mathcal{T}_0(\bar{x}) = 2\pi + \frac{\pi}{3}\bar{x}^2 + O(\bar{x}^3).$$

This fact, together with the discussion at the beginning of the section, shows that there exist $\epsilon_0, \bar{x}_0 > 0$ small such that $\frac{d}{d\bar{x}}\mathcal{T}_F(\bar{x};\epsilon) > 0$ for $\bar{x} \in]0, \bar{x}_0]$ and $\epsilon \in [0, \epsilon_0]$. Since monotonicity is unaltered by parametrization, this finishes the proof of Theorem 2.1 for the case $n = 1$.

For $n > 1$ the center at the origin becomes degenerate and Chicone–Jacobs procedure do not apply. With the aim of studying the period function of system (2.1) near the origin $(\bar{x}, \bar{y}) = (0, 0)$ for $n > 1$ we consider the change to generalized polar coordinates

$$(\bar{x}, \bar{y}) = (r\cos\theta, r^n \sin\theta)$$

with $r \geq 0$ and $\theta \in \mathbb{T}$. After this change system (2.1) is written as

$$\begin{cases} \dot{r} = \dfrac{r^n}{\cos^2\theta + n\sin^2\theta}\big(\cos\theta\sin\theta - \cos^{2n-1}\theta\sin\theta + O(r)\big), \\ \dot{\theta} = \dfrac{r^{n-1}}{\cos^2\theta + n\sin^2\theta}\big(-n\sin^2\theta - \cos^{2n}\theta + O(r)\big). \end{cases}$$

We note that terms with ϵ small are inside $O(r)$ so the forthcoming arguments are uniform with respect to $\epsilon \in [0, \epsilon_0]$.

For $r > 0$ small enough we have $\dot{\theta} < 0$. Therefore we can parametrize the orbits near the origin by $\varphi := -\theta$. We denote by $\mathscr{T}_F(s;\epsilon)$ the period of the solution $r(\varphi, s)$ and for the sake of simplicity we write $f(\varphi) := \cos^2\varphi + n\sin^2\varphi$, $\alpha(\varphi) := \cos^{2n-1}\varphi\sin\varphi - \cos\varphi\sin\varphi$, $\beta(\varphi) := n\sin^2\varphi + \cos^{2n}\varphi$. Note that $\beta(\varphi) > 0$. Due to the symmetry of system (2.1) the function $\mathscr{T}_F(s;\epsilon)$ writes

$$\mathscr{T}_F(s;\epsilon) = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\varphi}{\dot{\varphi}} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\varphi)d\varphi}{r(\varphi,s)^{n-1}\big(\beta(\varphi) + O(r(\varphi,s))\big)}.$$

Moreover,

$$\frac{\frac{d}{d\varphi}r(\varphi,s)}{r(\varphi,s)} = \frac{\alpha(\varphi)}{\beta(\varphi)} + O(r(\varphi,s)) = \frac{\alpha(\varphi)}{\beta(\varphi)} + O(s),$$

where in the second equality we use $r(\varphi, s) = O(s)$. Therefore,

$$r(\varphi, s) = r(0, s)e^{\int_0^{\varphi}\left(\frac{\alpha(\phi)}{\beta(\phi)}+O(s)\right)d\phi} = s\left(e^{\int_0^{\varphi}\frac{\alpha(\phi)}{\beta(\phi)}d\phi} + O(s)\right).$$

We denote $\rho(\varphi) := e^{\int_0^{\varphi}\frac{\alpha(\phi)}{\beta(\phi)}d\phi} > 0$. Substituting the previous equality in the expression of $\mathscr{T}_F(s;\epsilon)$ and taking into account that $O(r(\varphi, s)) = O(s)$ we get

$$\begin{aligned}
\mathscr{T}_F(s;\epsilon) &= \frac{2}{s^{n-1}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\varphi)d\varphi}{(\rho(\varphi)+O(s))^{n-1}(\beta(\varphi)+O(s))} \\
&= \frac{2}{s^{n-1}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{f(\varphi)}{\rho(\varphi)^{n-1}\beta(\varphi)} + O(s)\right)d\varphi \\
&= \frac{2}{s^{n-1}}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\varphi)d\varphi}{\rho(\varphi)^{n-1}\beta(\varphi)} + O(s)\right).
\end{aligned}$$

Since f, ρ and β are positive, the last equality shows that $\mathscr{T}_F(s;\epsilon) \to +\infty$ as $s \to 0^+$ for $n > 1$. Moreover,

$$\begin{aligned}
\frac{d}{ds}\mathscr{T}_F(s;\epsilon) &= -\frac{2(n-1)}{s^n}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\varphi)d\varphi}{\rho(\varphi)^{n-1}\beta(\varphi)} + O(s)\right) + \frac{2}{s^{n-1}}O(1) \\
&= \frac{1}{s^n}\left(-2(n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\varphi)d\varphi}{\rho(\varphi)^{n-1}\beta(\varphi)} + O(s)\right).
\end{aligned}$$

The last equality shows that $\frac{d}{ds}\mathscr{T}_F(s;\epsilon) \to -\infty$ as $s \to 0^+$. This ends the proof of Theorem 2.1 for $n > 1$.

Remark 3.1. We could also use the following generalized polar coordinates

$$(\bar{x}, \bar{y}) = (r\rho_1(\theta), r^n\rho_2(\theta))$$

where $(\rho_1(\theta), \rho_2(\theta))$ is the solution of $\{\dot{x} = -y, \dot{y} = x^{2n-1}\}$ with initial condition $(x(0), y(0)) = (1, 0)$. Using this coordinate change the above expressions become simpler (e.g. $\beta(\varphi) = 1$ for all φ, with $\varphi = -\theta$).

3.2 Proof of Theorem 2.2

In order to study the global behaviour of the period function of system (2.1) uniformly on $\epsilon \geq 0$ small in a compact set inside the period annulus it is enough to study the period function of the system (2.2), that is when $\epsilon = 0$. We denote by $T_0(\bar{y})$ the period function of system (2.2) parametrized by the positive $\bar{y}$-axis, and we consider $\bar{y}$ inside an arbitrary compact interval $[\bar{y}_1, \bar{y}_2]$ with $0 < \bar{y}_1 < \bar{y}_2 < +\infty$. By continuity with respect to the small parameter ϵ of system (2.1), taking ϵ small enough the $\bar{y}$-axis is also transversal to all orbits of (2.1), which are also periodic for $\bar{y} \in [\bar{y}_1, \bar{y}_2]$. We can then define $T_F(\bar{y};\epsilon)$ as the period function of system (2.1) parametrized by the same $\bar{y}$ as T_0. The function $T_F(\bar{y};\epsilon)$ is analytic for $\epsilon \geq 0$, $\epsilon \sim 0$, and so we can consider its Taylor's series development at $\epsilon = 0$,

$$T_F(\bar{y};\epsilon) = T_0(\bar{y}) + O(\epsilon).$$

Then, since the center of system (2.2) is not isochronous, properties of the period function of system for $\epsilon = 0$ are reflected for $\epsilon \geq 0$ small enough. In particular, $\frac{d}{d\bar{y}}T_0(\bar{y}) > 0$ and $\frac{d^2}{d\bar{y}^2}T_0(\bar{y}) > 0$ for all $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ will imply $\frac{d}{d\bar{y}}T_F(\bar{y};\epsilon) > 0$ and $\frac{d^2}{d\bar{y}^2}T_F(\bar{y};\epsilon) > 0$ for all $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ and $\epsilon \geq 0$ small, respectively. For this reason, Theorem 2.2 is a consequence of the following result.

Proposition 3.2. *The period function of the center (2.2) is strictly monotone increasing for $n = 1$ and it is strictly convex for $n > 1$.*

The first part of the proof of Proposition 3.2 relies on the application of the following monotonicity criterion due to Schaaf [11].

Theorem 3.3. *Consider a Hamiltonian system of the form $\dot{u} = -v$, $\dot{v} = g(u)$, where function g satisfy the following assumptions:*

1. *$g : \mathbb{R} \to \mathbb{R}$ is three times continuously differentiable with $g(0) = 0$ and $g'(0) > 0$.*
2. *For all $u \in \mathbb{R}$ where $g'(u) > 0$: $\big(5(g'')^2 - 3g'g'''\big)(u) > 0$.*
3. *If $g'(u) = 0$ then $g(u)g''(u) < 0$.*

Then the origin is a center and the period function is strictly increasing in the whole period annulus.

One of the key elements to prove the second part of Proposition 3.2 is to show that at most one critical period can exist in the interior of the period annulus. To do so we use the following result due to Sabatini [10]. For the sake of shortness in the statement, we define the following operator for smooth functions defined on an interval I:

$$\mathcal{K}[g] := \frac{3g^2g''^2 - 3gg'^2g'' - g^2g'g'''}{g'^4}.$$

Theorem 3.4. *Consider a Hamiltonian of the form $H(u,v) = G(u) + F(v)$, where $G(u) = \alpha u^{2k} + o(u^{2k}) \in C^\infty(I_G)$, $F(v) = \beta v^{2\ell} + o(v^{2\ell}) \in C^\infty(I_F)$, $0 \in I_G \cap I_F$, $0 < k, \ell \in \mathbb{N}$, $\alpha, \beta > 0$. Here I_G and I_F denote the maximal interval of definition of G and F, respectively. Then the origin is a center and if*

$$\mu_{s2} := 4\left(1 + 2\frac{GG''}{G'^2}\frac{FF''}{F'^2} + \mathcal{K}[G] + \mathcal{K}[F]\right) > 0$$

then the period function is strictly convex in the whole period annulus.

Proof of Proposition 3.2. The change of variables $\{u = \ln(1 + 2n(\bar{y} - \bar{x}^{2n})), v = \bar{x}\}$ transforms (2.2) into the Hamiltonian system with separable variables

$$\begin{cases} \dot{u} = -2nv^{2n-1}, \\ \dot{v} = V_n'(u), \end{cases} \tag{3.2}$$

where $V_n(u) = \frac{1}{2n}(e^u - u - 1)$. We notice that both periodic functions of system (3.2) and (2.2) are the same through the change of variable. We shall prove the results for (3.2).

Let us prove the first assertion of the statement. To do so, we apply Schaaf's criterion to system (3.2) with $n = 1$. After a positive constant rescaling of time and taking $g = V_1'$ we have that the assumptions in Theorem 3.3 are fulfilled since $g'(u) = V_1''(u) = \frac{1}{2}e^u > 0$ for all $u \in \mathbb{R}$ and $\big(5(g'')^2 - 3g'g'''\big)(u) = \frac{1}{2}e^{2u} > 0$ for all $u \in \mathbb{R}$. Therefore the period function of system (3.2) is strictly increasing and so $\frac{d}{d\bar{y}}T_0(\bar{y}) > 0$ for all $\bar{y} > 0$. This proves the assertion concerning $n = 1$.

Let us consider $n > 1$. With the aim of applying Theorem 3.4 we denote $G(u) = V_n(u) = \frac{1}{2n}(e^u - u - 1)$ and $F(v) = v^{2n}$. Clearly the first part of the assumptions of the theorem are fulfilled since $V_n(u) = \frac{1}{4n}u^2 + o(u^2)$. We claim that $\mu_{s2} \geq \frac{1}{n^2} > 0$ for all $n \geq 2$. After showing the inequality, the result follows by direct application of Theorem 3.4.

Using the expressions of F and G we have that

$$\hat{\mu}_{s2}(u,n) := \mu_{s2}(u,n) - \frac{1}{n^2} = \frac{4e^u}{n(e^u-1)^4}\eta(u,n),$$

where $\eta(u,n) = nu^2 + (1+3n)u + 2n + 1 + (2nu^2 - 2u - 3n - 3)e^u + ((1-3n)u+3)e^{2u} + (n-1)e^{3u}$. A direct computation shows that

$$\frac{d}{dn}\hat{\mu}_{s2}(u) = \frac{4(e^u-u-1)e^u}{n^2(e^u-1)^2} > 0$$

for all $u \in \mathbb{R}$. Therefore to prove the claim it is enough to show that $\eta(u,2) \geq 0$.

We perform a derivation-division procedure with respect to e^u achieving the following equality:

$$e^{-u}\frac{d^3}{du^3}\left(e^{-u}\frac{d^3}{du^3}\eta(u,2)\right) = 216e^u - 40u - 156.$$

The previous expression has exactly two simple negative zeros. Indeed, its derivative is zero only at $u = \ln(5/27)$, the image at $u = 0$ is positive and the limits $u \to \pm\infty$ are both $+\infty$. A sequence of simple arguments of continuity, number of zeros of the derivative, the values at $u = 0$ and the values of the limits at $\pm\infty$ yields to show that $\eta(u,2) \geq 0$ for all $u \in \mathbb{R}$. This finishes the proof of the claim. □

3.3 Proof of Theorem 2.3

We define a section $\Sigma_1 \subset \{X = 0\}$ parametrized by $(R_1, E_1) \in [0, R_1^0] \times [0, E_1^0]$ for some small $R_1^0, E_1^0 > 0$. The section Σ_1 is defined using the coordinates (X, R, E) of (2.4) (we write (R_1, E_1) instead of (R, E) to avoid confusion later). Similarly, we define $\Sigma_4 \subset \{\bar{x} = 0\}$ parametrized by $(\bar{y}, \epsilon)$, with $\epsilon \in [0, R_1^0 E_1^0]$, where $(\bar{x}, \bar{y}, \epsilon)$ are the coordinates of (2.1). The sections Σ_1, Σ_4 are transverse to the blown-up vector field $\bar{X}_\eta$ and located near the polycycle Γ (see Figure 3.1).

Since system (2.1) (resp. (2.4)) is invariant under the symmetry $(\bar{x}, t) \to (-\bar{x}, -t)$ (resp. $(X, t) \to (-X, -t)$), it suffices to study the time spent between Σ_1 and Σ_4, i.e. the half time period function of $r^{2n-1}\bar{X}_\eta$, denoted by H. Our goal is to prove that $\mathcal{L}H > 0$ on Σ_1 (for $R_1^0, E_1^0 > 0$ small enough but fixed), with $\epsilon > 0$, where $\mathcal{L}$ is the Lie-derivative along the vector field $R\frac{\partial}{\partial R} - E\frac{\partial}{\partial E}$ (see Section 3.3.5). This implies that $r^{2n-1}\bar{X}_\eta$ $(r > 0)$ has no critical periods near Γ and the period function is monotonous increasing there. When $\epsilon = 0$, system (1.1) has no center.

We aligned up H in three parts: the time $H_{1,2}$ spent between Σ_1 and Σ_2 (Section 3.3.2), the time $H_{2,3}$ spent between Σ_2 and Σ_3, near the semi-hyperbolic singularity p_- (Section 3.3.1), and the time $H_{3,4}$ between Σ_3 and Σ_4 (Section 3.3.3). In Section 3.3.4 we glue the local results together. Section 3.3.5 is devoted to the study of the Lie-derivative $\mathcal{L}H$.

3.3.1 The study of $H_{2,3}$

In this section we study the time $H_{2,3}$ inside the family X_D, i.e. $\bar{X}_D$ multiplied by R^{2n-1}. First, we bring $\widetilde{X}_D := F(X, R, \eta)^{-1}\bar{X}_D$, locally near $p_- = (1, 0, 0)$, to a normal form which simplifies the study of $H_{2,3}$ (transverse sections $\Sigma_{2,3}$ will be defined in the normal form coordinates). Since p_- is partially hyperbolic for all $\eta \in K$, there exists a C^k η-family of center manifolds at p_-, given as a graph of $X = 1 + \psi(R, E, \eta)$ with $\psi(0, 0, \eta) \equiv 0$. Following [5] in the generic

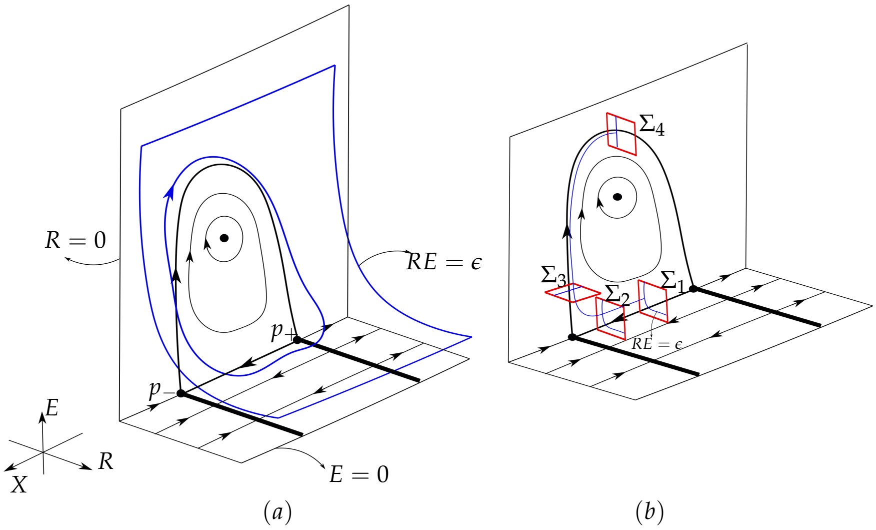

Figure 3.1: (a) Closed orbits near the polycycle Γ, inside $RE = \epsilon$, for a fixed $\epsilon > 0$. Γ is located on the blow-up locus $\{r = 0\}$ (it corresponds to $\{R = 0\}$ in the phase directional chart). The center is visible in the family chart. (b) The study of the time spent inside $\{x \geq 0\}$ is divided into three parts: $\Sigma_1 \to \Sigma_2$, $\Sigma_2 \to \Sigma_3$ and $\Sigma_3 \to \Sigma_4$. In the first two parts, we use the vector field X_D, and in the last part we use X_F.

case or [3] in the non-generic case, an η-family of center manifolds can be chosen to be C^∞ (i.e. ψ can be C^∞). We fix such ψ.

Using the coordinate change $Z = X - (1 + \psi(R, E, \eta))$, the fixed family of center manifolds becomes $\{Z = 0\}$ and the vector field $\widetilde{X}_D$ changes to

$$\begin{cases} \dot{Z} = -\big(\Phi(R, E, \eta) + O(Z)\big)Z, \\ \dot{R} = -\dfrac{1}{2n}RE^{2n}, \\ \dot{E} = \dfrac{1}{2n}E^{2n+1}, \end{cases} \tag{3.3}$$

where Φ is a smooth function with $\Phi(0, 0, \eta) = 2n$. We used the fact that the family of center manifolds is invariant for $\widetilde{X}_D$. Now, we can normally linearize the vector field (3.3) using Theorem 1.1 of [7].

Theorem 3.5. *There is a smooth family $\Pi_\eta : (Z, R, E) \to (\bar{Z}, R, E)$ of local diffeomorphisms, defined in an η-uniform neighborhood of the origin in the (Z, R, E)-space, which brings (3.3) into the normally linearized vector field*

$$\hat{X}_D : \begin{cases} \dot{Z} = -\Phi(R, E, \eta)Z, \\ \dot{R} = -\dfrac{1}{2n}RE^{2n}, \\ \dot{E} = \dfrac{1}{2n}E^{2n+1}, \end{cases} \tag{3.4}$$

where Φ is defined in (3.3) and where we denote $\bar{Z}$ again by Z. The diffeomorphisms Π_η preserve $\{RE = \text{const}\}$: $\Pi_\eta(Z, R, E) = (Z(1 + Z\pi_\eta(Z, R, E)), R, E)$ with a smooth family π_η.

Remark 3.6. The coordinate change in the normal linearization theorem from [7] is C^∞-smooth and preserves the parameter η and the leaves of the foliation $\{RE = \text{const}\}$ (the center variables R, E are preserved). In [6], this normal linearization theorem has been used in the generic case (see also Remark 1.2 in [7]). In the same way we apply it to the non-generic case. We point out that we could also use C^k center manifolds and the normal linearization theorem of [1] with a C^k-coordinate change that preserves η and $\{RE = \text{const}\}$. The size of the domain of the coordinate change may tend to zero as $k \to \infty$. The finite smoothness is not a problem in our proof.

We conclude that, in the normal form coordinates (Z, R, E) of (3.4), X_D can be written as

$$R^{2n-1}\kappa(Z, R, E, \eta)\hat{X}_D, \tag{3.5}$$

where $\kappa(Z, R, E, \eta) = F(Z(1 + O(Z)) + 1 + \psi, R, \eta)$ and $\kappa(0, 0, 0, \eta) = 1$.

In the normal form coordinates, we define $\Sigma_2 \subset \{Z = -Z_0\}$, parametrized by $(R_2, E_2) \in [0, R_2^0] \times [0, E_2^0]$ for some small constants $Z_0, R_2^0, E_2^0 > 0$, and $\Sigma_3 \subset \{E = E_3\}$, parametrized by (Z, R) with $Z \sim 0$ and $R \in [0, R_3]$ for some small constants $R_3, E_3 > 0$. All the constants are chosen such that the transverse sections $\Sigma_{2,3}$ are located in the domain of Π_η^{-1} and such that the passage w.r.t. $\hat{X}_D$ between Σ_2 and Σ_3 is well-defined.

We can now find the time $H_{2,3}(R_2, E_2)$ in (3.5), spent between Σ_2 and Σ_3. Note that the orbit of $\hat{X}_D$ (or (3.5) with $R > 0$) with the initial point $(R_2, E_2) > (0, 0)$ on Σ_2 has the form

$$\left(Z(E, R_2, E_2), \frac{R_2E_2}{E}, E\right)$$

with $Z(E, R_2, E_2) = -Z_0 \exp\left(-2n \int_{E_2}^{E} \frac{\Phi(\frac{R_2E_2}{s}, s, \eta)}{s^{2n+1}} ds\right)$. Using this, the time $H_{2,3}$ can be written as

$$H_{2,3}(R_2, E_2) = \frac{2n}{(R_2E_2)^{2n-1}} \int_{E_2}^{E_3} \frac{dE}{E^2\kappa(Z(E, R_2, E_2), \frac{R_2E_2}{E}, E, \eta)}. \tag{3.6}$$

Since $|Z(E, R_2, E_2)| \le Z_0$ for $E \ge E_2$ and κ is positive and bounded for $(Z, R, E) \sim (0, 0, 0)$ and $\eta \in K$, it is clear that (3.6) tends to $+\infty$ as $\epsilon = R_2E_2 \to 0$, uniformly in η. (Note that the integral in (3.6) is of order $O(\frac{1}{E_2})$.) We will use the expression (3.6) in Section 3.3.4.

We conclude this section with a result about the transition map of $\hat{X}_D$ between Σ_2 and Σ_3.

Proposition 3.7. *There is a C^∞-function J in $(R_2, E_2, E_2^2 \ln E_2, \eta)$ such that the transition map $(R_2, E_2) \to (Z, R)$ along the trajectories of (3.4) between Σ_2 and Σ_3 is given by $R = \frac{R_2E_2}{E_3}$ and*

$$Z = -Z_0 \exp\left(-\frac{1}{E_2^{2n}} J(R_2, E_2, E_2^2 \ln E_2, \eta)\right)$$

with $J(0, 0, 0, \eta) = 2n$.

Proof. When $n = 1$, the proof of the proposition can be found in [6] (Proposition 4.9). The proof of the case "$n > 1$" is analogous to the proof of the case "$n = 1$". □

Proposition 3.7 implies that the transition map between Σ_2 and Σ_3 is C^∞-smooth in (R_2, E_2, η). This will be used in the gluing process in Section 3.3.4.

3.3.2 The study of $H_{1,2}$

In this section we deal with the time $H_{1,2}$, spent between Σ_1 and Σ_2, inside the vector field X_D. The smooth sections $\Sigma_{1,2}$ are defined above. Note that the system (2.4) has no singularities between Σ_1 and Σ_2 (since the section Σ_2 is located uniformly away from the singularity p_-, the X-component of (2.4) is strictly positive between Σ_1 and Σ_2, for all $(R, E) \sim (0, 0)$ and for all η kept in the compact set K). Since X_D is (2.4), multiplied by R^{2n-1}, it can be seen that

$$H_{1,2}(R_1, E_1) = \frac{1}{R_1^{2n-1}} I_1 \left(R_1, E_1, \eta\right), \tag{3.7}$$

where I_1 is a strictly positive C^∞-function. We conclude this section with

Proposition 3.8. *There exists a C^∞-function $J(R_1, E_1, \eta)$ such that the transition map $(R_1, E_1) \to (R_2, E_2)$ along the trajectories of (2.4) between Σ_1 and Σ_2 is given by*

$$(R_2, E_2) = \left(R_1(1 + E_1^{2n} J(R_1, E_1, \eta)), E_1(1 + E_1^{2n} J(R_1, E_1, \eta))^{-1}\right).$$

Proof. In the generic case ($n = 1$), the proof of the proposition is given in [6, Proposition 5.1]. The proof of the non-generic case ($n > 1$) is analogous to the proof of the generic case. □

We use (3.7) and Proposition 3.8 in Section 3.3.4.

3.3.3 The study of $H_{3,4}$

In this section we deal with the time $H_{3,4}$, spent between Σ_3 and Σ_4, inside the vector field X_F (X_F is equal to (2.1), multiplied by a constant $\epsilon^{2n-1} = (RE)^{2n-1}$). The smooth sections $\Sigma_{3,4}$ are defined above. If we parametrize Σ_3 with $(\bar{x}, \epsilon)$ ($(\bar{x}, \bar{y}, \epsilon)$ are the coordinates of (2.1)), then we can write $H_{3,4}$ as

$$H_{3,4}(\bar{x}, \epsilon) = \frac{1}{\epsilon^{2n-1}} I_3 \left(\bar{x}, \epsilon, \eta\right), \tag{3.8}$$

where I_3 is a strictly positive C^∞-function. This follows from the fact that the vector field (2.1) is regular along Γ on the blow-up locus, between Σ_3 and Σ_4 (see Figure 3.1).

3.3.4 The study of H

In this section we glue together the local results obtained in Sections 3.3.1–3.3.3 and find an expression for the half time period function H. We know that

$$H(R_1, E_1) = H_{1,2}(R_1, E_1) + H_{2,3}(R_2, E_2) + H_{3,4}(\bar{x}, \epsilon),$$

where the orbit of $r^{2n-1}\bar{X}_\eta$ ($\bar{X}_\eta$ is the blown-up vector field defined in Section 2) with the initial point $(R_1, E_1) \in \Sigma_1$ intersects section Σ_2 at the point (R_2, E_2) and section Σ_3 at the point $(\bar{x}, \epsilon)$. From (3.6) follows that

$$H_{2,3}(R_2, E_2) = \frac{2n}{(R_1 E_1)^{2n-1}} \int_{E_2}^{E_3} \frac{dE}{E^2 \kappa(Z(E, R_2, E_2), \frac{R_1 E_1}{E}, E, \eta)}, \tag{3.9}$$

where R_2 and E_2 are the C^∞-functions of (R_1, E_1, η) given in Proposition 3.8. Here we used that $\epsilon = R_1 E_1 = R_2 E_2$. Now, we want express $H_{3,4}(\bar{x}, \epsilon)$ in terms of (R_1, E_1). Let us recall that the constant $E_3 > 0$ comes from the definition of Σ_3. Using $\bar{x} = \frac{X}{E_3}$ and $X = Z(1 + O(Z)) +$

$1+\psi(\frac{R_1E_1}{E_3},E_3,\eta)$ on Σ_3 ($O(Z)$ is a C^∞-function, see Section 3.3.1), and the fact that Z is a C^∞-function in (R_1,E_1,η) (we combine Proposition 3.7 and Proposition 3.8), we see that $\bar{x}$ is a C^∞-function of (R_1,E_1,η) and $\epsilon=R_1E_1$. This and (3.8) imply that

$$H_{3,4}(\bar{x},\epsilon)=\frac{1}{(R_1E_1)^{2n-1}}\tilde{I}_3\left(R_1,E_1,\eta\right) \tag{3.10}$$

where $\tilde{I}_3$ is a strictly positive C^∞-function. Combining (3.7), (3.9) and (3.10), we finally get

$$H(R_1,E_1)=\frac{2n}{(R_1E_1)^{2n-1}}\left(\int_{E_2}^{E_3}\frac{dE}{E^2\kappa(Z(E,R_2,E_2),\frac{R_1E_1}{E},E,\eta)}+I(R_1,E_1,\eta)\right), \tag{3.11}$$

where I is a C^∞-function (thus, bounded). Note that the $H_{2,3}$-contribution is dominant in (3.11) and that $H(R_1,E_1)$ tends to $+\infty$ as $\epsilon=R_1E_1\to 0$, uniformly in η. We know that $R_2=R_1\left(1+o(1)\right)$ and $E_2=E_1\left(1+o(1)\right)$ where the $o(1)$-terms are C^∞-functions of (R_1,E_1,η), equal to 0 when $E_1=0$. In Section 3.3.5 we show that the Lie-derivative of the integral in (3.11) is of order $O(\frac{1}{E_1})$.

3.3.5 Lie-derivative of H

When we fix any value of (ϵ,η), with $\epsilon>0$ small, H is 1-variable function defined on interval $\{(R_1,E_1)\in\Sigma_1 \mid R_1E_1=\epsilon\}$ (see Figure 3.1(b)). To study critical periods of H on such intervals, we define the Lie-derivative of H along the vector field $R_1\frac{\partial}{\partial R_1}-E_1\frac{\partial}{\partial E_1}$ (it is tangent to the intervals and without singularities there):

$$\mathcal{L}H:=R_1\frac{\partial H}{\partial R_1}-E_1\frac{\partial H}{\partial E_1}.$$

It can be easily seen that the Lie-derivative of a C^∞-function in (R_1,E_1,η) (e.g. $\widetilde{I}$ in (3.11)) is a C^∞-function in (R_1,E_1,η), equal to zero when $(R_1,E_1)=(0,0)$. We also have $\mathcal{L}\left(R_1E_1\right)=0$ and $\mathcal{L}(R_1^{l_1}E_1^{l_2})=(l_1-l_2)R_1^{l_1}E_1^{l_2}$ for $l_1,l_2\in\mathbb{Z}$. For more details about the Lie-derivative we refer the reader to [6,9].

The Lie-derivative of the time (3.11) can be written as

$$\begin{aligned}(\mathcal{L}H)(R_1,E_1)=\frac{2n}{(R_1E_1)^{2n-1}}\Bigg(&\frac{1+o(1)}{E_1\kappa(-Z_0,R_1(1+o(1)),E_1(1+o(1)),\eta)}\\&+\int_{E_2}^{E_3}\frac{-\frac{\partial\kappa}{\partial Z}(Z(E,R_2,E_2),\frac{R_1E_1}{E},E,\eta)}{E^2\left(\kappa(Z(E,R_2,E_2),\frac{R_1E_1}{E},E,\eta)\right)^2}(\mathcal{L}Z)(E,R_1,E_1)dE+o(1)\Bigg),\end{aligned} \tag{3.12}$$

where $o(1)$-terms are C^∞-functions of (R_1,E_1,η), equal to zero when $(R_1,E_1)=(0,0)$, and $\mathcal{L}Z$ is given by

$$(\mathcal{L}Z)(E,R_1,E_1)=\frac{2nZ_0\left(\Phi(R_1,E_1,\eta)+o(1)\right)}{E_1^{2n}}\exp\left(-2n\int_{E_2}^{E}\frac{\Phi(\frac{R_1E_1}{s},s,\eta)}{s^{2n+1}}ds\right), \tag{3.13}$$

where $o(1)$-terms have the same property as above. We show that the first term in (3.12) is dominant.

Since κ is uniformly positive near the origin ($\kappa(0,0,0,\eta)=1$) and $\frac{\partial\kappa}{\partial Z}$ and Φ are bounded, we find an upper bound for the integral in (3.12):

$$\left|\int_{E_1(1+o(1))}^{E_3}\right| \leq \frac{\alpha Z_0}{E_1^{2n}}\int_{E_1(1+o(1))}^{E_3}\frac{1}{E^2}\exp\left(-2n\int_{E_1(1+o(1))}^{E}\frac{\Phi(\frac{R_1E_1}{s},s,\eta)}{s^{2n+1}}ds\right)dE \tag{3.14}$$

for some constant $\alpha>0$ independent of Z_0. (We write $E_2=E_1(1+o(1))$.) For $E_1>0$ and $E_1\sim 0$, we aligned up the integral on the right-hand side of (3.14) in two parts:

$$\int_{E_1(1+o(1))}^{E_3}=\int_{E_1(1+o(1))}^{2E_1}+\int_{2E_1}^{E_3}.$$

We denote the first integral by J_1 and the second by J_2. We make in J_1 the change of variable $E=E_1\tau$, getting

$$\begin{aligned}
J_1 &= \frac{1}{E_1}\int_{1+o(1)}^{2}\frac{1}{\tau^2}\exp\left(-2n\int_{E_1(1+o(1))}^{E_1\tau}\frac{\Phi(\frac{R_1E_1}{s},s,\eta)}{s^{2n+1}}ds\right)d\tau\\
&= \frac{1}{E_1}\int_{1+o(1)}^{2}\frac{1}{\tau^2}\exp\left(-\frac{2n}{E_1^{2n}}\int_{1+o(1)}^{\tau}\frac{\Phi(\frac{R_1}{u},E_1u,\eta)}{u^{2n+1}}du\right)d\tau\\
&\leq \frac{1}{E_1}\int_{1+o(1)}^{2}\frac{1}{\tau^2}\exp\left(-\frac{\beta}{E_1^{2n}}\left(\tau-1-o(1)\right)\right)d\tau\\
&\leq \gamma E_1^{2n-1}
\end{aligned} \tag{3.15}$$

for some constants $\beta,\gamma>0$ independent of Z_0. In the second step we used the change of variable $s=E_1u$ and in the third step we used the fact that the integrand function in $\int_{1+o(1)}^{\tau}$ is uniformly positive ($\Phi(0,0,\eta)=2n$). In the last step the term $\frac{1}{\tau^2}$ is bounded on the segment $[1+o(1),2]$ and the integral of the exponential function is bounded by E_1^{2n}, multiplied by a positive constant. Note also that the $o(1)$-terms in the last step are equal.

Concerning the integral J_2 we get

$$\begin{aligned}
J_2 &= \int_{2E_1}^{E_3}\frac{1}{E^2}\exp\left(-2n\int_{E_1(1+o(1))}^{E}\frac{\Phi(\frac{R_1E_1}{s},s,\eta)}{s^{2n+1}}ds\right)dE\\
&\leq \int_{2E_1}^{E_3}\frac{1}{E^2}\exp\left(-2n\int_{E_1(1+o(1))}^{2E_1}\frac{\Phi(\frac{R_1E_1}{s},s,\eta)}{s^{2n+1}}ds\right)dE\\
&= \int_{2E_1}^{E_3}\frac{1}{E^2}\exp\left(-\frac{2n}{E_1^{2n}}\int_{(1+o(1))}^{2}\frac{\Phi(\frac{R_1}{u},E_1u,\eta)}{u^{2n+1}}du\right)dE\\
&\leq \exp\left(-\frac{\beta}{E_1^{2n}}\right)\int_{2E_1}^{E_3}\frac{1}{E^2}dE \leq \frac{\gamma}{E_1}\exp\left(-\frac{\beta}{E_1^{2n}}\right)
\end{aligned} \tag{3.16}$$

for some new constants $\beta,\gamma>0$. Finally, combining inequalities (3.15) and (3.16) we obtain

$$\left|\int_{E_1(1+o(1))}^{E_3}\right| \leq \frac{\alpha Z_0}{E_1^{2n}}(J_1+J_2)\leq \frac{\alpha_1 Z_0}{E_1}+\frac{\alpha_2}{E_1^{2n+1}}\exp\left(-\frac{\beta}{E_1^{2n}}\right)$$

for some constants $\alpha_1,\alpha_2,\beta>0$. It is clear now that the first term in (3.12) is the leading term since $\frac{1}{\kappa}>\alpha_1 Z_0$ ($Z_0>0$ is as small as we want but fixed).

We conclude that there are no critical periods for any fixed level $\epsilon > 0$ on Σ_1 with $R_1^0, E_1^0 > 0$ small enough and fixed, uniformly in η. The Lie-derivative $\mathcal{L}H$ tends to $+\infty$ as $\epsilon \to 0$, uniformly in η. Since $\mathcal{L}H > 0$ and $2n\bar{y}\frac{\partial}{\partial \bar{y}} + 0\frac{\partial}{\partial \epsilon} = R_1\frac{\partial}{\partial R_1} - E_1\frac{\partial}{\partial E_1}$ the period function is monotonous increasing (as large $\bar{y}$ increases, i.e. as we go away from the center $(\bar{x}, \bar{y}) = (0,0)$, the period function increases). This completes the proof of Theorem 2.3.

3.4 Proof of Theorem 1.1

Let $n \geq 1$ and $T(y;\epsilon)$ be the period function of the center at the origin of system (1.1) with $\epsilon > 0$, $\epsilon \sim 0$, parametrized by the positive y-axis, with $y \sim 0$. We have the following relation between the (x, y, ϵ)-coordinates, the family directional coordinates and the phase directional coordinates defined in Section 2.1:

$$x = \epsilon\bar{x} = RX, \quad y = \epsilon^{2n}\bar{y} = R^{2n}, \quad \epsilon = RE.$$

Note that the positive y-axis is given by $\{x = 0\}$. In the family chart (resp. the phase directional chart), it corresponds to $\{\bar{x} = 0\}$ (resp. $\{X = 0\}$).

For each $\epsilon > 0$ and $\epsilon \sim 0$, we consider T in the following intervals: $]0, \epsilon^{2n}\bar{y}_0]$, $[\epsilon^{2n}\bar{y}_1, \epsilon^{2n}\bar{y}_2]$ and $[\epsilon^{2n}\bar{y}_3, y_0]$ where $\bar{y}_0, \bar{y}_1, y_0 > 0$ are small and independent of ϵ and $\bar{y}_2, \bar{y}_3 > 0$ are large and independent of ϵ. For $\bar{y}_0, y_0$ small and $\bar{y}_3$ large, it suffices to decrease $\bar{y}_1$ and increase $\bar{y}_2$ to cover the interval $]0, y_0]$. In the interval $]0, \epsilon^{2n}\bar{y}_0]$ (resp. $[\epsilon^{2n}\bar{y}_1, \epsilon^{2n}\bar{y}_2]$ and $[\epsilon^{2n}\bar{y}_3, y_0]$) we use Theorem 2.1 (resp. Theorem 2.2 and Theorem 2.3). Let us recall that the results of Theorem 2.3 are valid in a section $\Sigma_1 \subset \{X = 0\}$ parametrized by $(R, E) \in]0, R_1^0] \times]0, E_1^0]$ where $R_1^0, E_1^0 > 0$ are small enough and fixed (see Section 3.3). The interval $]0, R_1^0] \times \{E_1^0\}$ corresponds to $y = \epsilon^{2n}(E_1^0)^{-2n}$ and $\epsilon \in]0, R_1^0 E_1^0]$ (we denote $(E_1^0)^{-2n}$ by $\bar{y}_3$). The interval $\{R_1^0\} \times]0, E_1^0]$ is given by $y = (R_1^0)^{2n}$ (we denote $(R_1^0)^{2n}$ by y_0) and $\epsilon \in]0, R_1^0 E_1^0]$. Theorem 2.1 is valid for $\bar{y} \in]0, \bar{y}_0]$ and $\epsilon \in]0, \epsilon_0]$ where $\bar{y}_0, \epsilon_0 > 0$ are small enough. In the (y, ϵ)-coordinates, it corresponds to $y \in]0, \epsilon^{2n}\bar{y}_0]$ and $\epsilon \in]0, \epsilon_0]$. Finally, for any small $\bar{y}_1 > 0$ and any large $\bar{y}_2 > 0$, Theorem 2.2 is valid for $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ and $\epsilon \in]0, \epsilon_1]$ where $\epsilon_1 > 0$ is small enough. It corresponds to $y \in [\epsilon^{2n}\bar{y}_1, \epsilon^{2n}\bar{y}_2]$ and $\epsilon \in]0, \epsilon_1]$ in the original coordinates.

Note that the notion of critical period is independent of the chosen coordinates and the chosen transverse section (for example, if we work with the polar coordinates (r, θ) instead of $(\bar{x}, \bar{y})$, we have the same number of critical periods, counting multiplicity).

We consider two cases: $n = 1$ and $n > 1$. Suppose first that $n = 1$. Following Theorem 2.1, we have that $\frac{\partial T}{\partial y}(y;\epsilon) > 0$ for all $y \in]0, \epsilon^2\bar{y}_0]$ and $\epsilon \in]0, \epsilon_0]$. Indeed, we know that $T(y;\epsilon) = \frac{1}{\epsilon}T_F(\frac{y}{\epsilon^2};\epsilon)$ where $T_F(\bar{y};\epsilon)$ is the period function of the center of (2.1), parametrized by the positive $\bar{y}$-axis. Now, it suffices to see that

$$\frac{\partial T}{\partial y}(y;\epsilon) = \frac{1}{\epsilon^3}\frac{\partial T_F}{\partial \bar{y}}\left(\frac{y}{\epsilon^2};\epsilon\right) \tag{3.17}$$

and that $\frac{\partial T_F}{\partial \bar{y}}(\bar{y};\epsilon) > 0$ for all $\bar{y} \in]0, \bar{y}_0]$ and $\epsilon \in]0, \epsilon_0]$ (Theorem 2.1). On the other hand, we know that $T(y;\epsilon) = T_D(\sqrt{y}, \frac{\epsilon}{\sqrt{y}})$ where $T_D(R, E)$ is the period function of $r\bar{X}_\eta$ near the polycycle Γ ($\bar{X}_\eta$ is the blown-up vector field). Note that

$$\frac{\partial T}{\partial y}(y;\epsilon) = \frac{1}{2y}(\mathcal{L}T_D)\left(\sqrt{y}, \frac{\epsilon}{\sqrt{y}}\right)$$

and that $\mathcal{L}T_D > 0$ for all $(R, E) \in]0, R_1^0] \times]0, E_1^0]$ (see Theorem 2.3). Thus, $\frac{\partial T}{\partial y}(y;\epsilon) > 0$ for all $y \in [\epsilon^2\bar{y}_3, y_0]$ and $\epsilon > 0$ small. Finally, by taking $\bar{y}_1 < \bar{y}_0$ and $\bar{y}_2 > \bar{y}_3$, we have that

$\frac{\partial T_F}{\partial \bar{y}}(\bar{y};\epsilon) > 0$ for all $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ and $\epsilon > 0$ small (see Theorem 2.2) and thus $\frac{\partial T}{\partial y}(y;\epsilon) > 0$ for all $y \in [\epsilon^2\bar{y}_1, \epsilon^2\bar{y}_2]$ and $\epsilon > 0$ small (see (3.17)). This ends the proof of Theorem 1.1 in the generic case.

Suppose now that $n > 1$. The study of the non-generic case is similar to the study of the generic case. We have $\frac{\partial T}{\partial y}(y;\epsilon) < 0$ for all $y \in]0, \epsilon^2\bar{y}_0]$ and for all $\epsilon > 0$ small (see Theorem 2.1), and $\frac{\partial T}{\partial y}(y;\epsilon) > 0$ for all $y \in [\epsilon^2\bar{y}_3, y_0]$ and $\epsilon > 0$ small (see Theorem 2.3). Using Theorem 2.2 we find that $\frac{\partial^2 T}{\partial y^2}(y;\epsilon) > 0$ for all $y \in [\epsilon^2\bar{y}_1, \epsilon^2\bar{y}_2]$ and $\epsilon > 0$ small. This implies that at most one critical period can exist in $]0, y_0]$. Since $\frac{\partial T}{\partial y}$ goes from $-$ to $+$, we conclude that precisely one critical period exists in $]0, y_0]$. This completes the proof of Theorem 1.1 in the non-generic case.

3.5 Proof of Theorem 1.2

We consider system (1.2) with $N \geq 1$ and denote $c := (c_2, c_4, \ldots, c_{2(N-1)}) \in \mathbb{S}^{N-2}$ (when $N = 1$, we don't have the parameter c). When $N \geq 2$, we assume that $c_2 \geq c_2^0$ for some arbitrarily small and fixed $c_2^0 > 0$. Let C and G be as defined in Theorem 1.2 and let $\tilde{C}$ be an arbitrary and fixed compact subset of C. Let $c \in \tilde{C}$. We replace ϵ in (1.2) by ϵ^2. It is clear that, if we can prove the result in a small interval in the new ϵ-space, then we have proved it in a small interval in the old ϵ-space.

If we apply the scaling $(x, y) = \left(\frac{\tilde{x}}{c_2}, \frac{\tilde{y}}{c_2}\right)$ to (1.2), we get

$$\begin{cases} \dot{x} = y - \left(x^2 + \sum_{k=2}^{N-1} \bar{c}_{2k}x^{2k} + \bar{c}_{2N}x^{2N}\right), \\ \dot{y} = -\epsilon^2 x, \end{cases} \tag{3.18}$$

where $\bar{c}_{2k} = c_{2k}c_2^{1-2k}$, for $k = 2, \ldots, N-1$, and $\bar{c}_{2N} = c_2^{1-2N}$. (We use the old notation (x, y) instead of $(\tilde{x}, \tilde{y})$ for the sake of simplicity.) Since c is kept in the compact set $\tilde{C}$, it is clear that $\bar{c} = (\bar{c}_4, \ldots, \bar{c}_{2N})$ is also contained in a compact set, denoted by $\bar{C}$, and that

$$\frac{\bar{G}'(x)}{x} > 0, \tag{3.19}$$

for all $x \in \mathbb{R}$ and $\bar{c} \in \bar{C}$, where $\bar{G}$ denotes the polynomial in x in the first equation of (3.18). Note that $\bar{G}(x) = c_2 G(\frac{x}{c_2})$ and that system (3.18) is of type (1.1) with $n = 1$.

It suffices to show that there exists $\epsilon_0 > 0$ small such that system (3.18) has no critical periods for all $\epsilon \in]0, \epsilon_0]$ and $\bar{c} \in \bar{C}$. Let $T(y;\epsilon)$ be the period function of the center at the origin of system (3.18) with $\epsilon > 0$ and $\epsilon \sim 0$, parametrized by the positive y-axis. In the rest of this section we prove that $\frac{d}{dy}T(y;\epsilon) > 0$ on $\{y > 0\}$, for all $\epsilon \in]0, \epsilon_0]$ and $\bar{c} \in \bar{C}$, for some $\epsilon_0 > 0$. This will imply that there are no critical periods uniformly in $\epsilon \sim 0$. We study the period function T in the following intervals: $]0, y_0]$, $[\rho, \frac{1}{\rho}]$ and $[y_1, \infty[$, where $y_0 > 0$ is small enough, $y_1 > 0$ is large enough and $\rho > 0$ is arbitrarily small (see Figure 3.2). When we find y_0 and y_1, we decrease ρ (i.e., increase the segment $[\rho, \frac{1}{\rho}]$) to cover the entire $\{y > 0\}$.

Following Theorem 1.1, there exist $\epsilon_0 > 0$ and $y_0 > 0$ such that $\frac{d}{dy}T(y;\epsilon) > 0$ for all $y \in]0, y_0]$ and $(\epsilon, \bar{c}) \in]0, \epsilon_0] \times \bar{C}$.

Consider now the period function T in the segment $[\rho, \frac{1}{\rho}]$, for any small and fixed $\rho > 0$. The reduced flow (sometimes called the slow system) of (3.18) along the critical curve $\{y =$

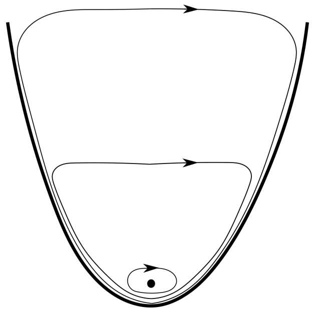

Figure 3.2: Dynamics of (3.18). The curve of singularities, at level $\epsilon = 0$, with indication of small-amplitude, detectable and large closed orbits, for $\epsilon > 0$ and $\epsilon \sim 0$.

$\bar{G}(x)\}$, away from the contact point $(x, y) = (0, 0)$, is given by

$$x' = -\frac{x}{\bar{G}'(x)} \text{ (or } y' = -x).$$

Note that the reduced flow is well-defined and uniformly negative for all x kept in large compact sets and $\bar{c} \in \bar{C}$. Here we use (3.19). The orbit through the point $y \in [\rho, \frac{1}{\rho}]$ is attracted to the curve of singularities $\{x > 0\}$, follows the reduced flow directed towards the turning point at the origin and then goes back to the point y due to the symmetry. This implies that T is well-defined for $y \in [\rho, \frac{1}{\rho}]$. Following [3, Theorem 2.1] or [4], the period function T and its derivative, restricted to the segment $[\rho, \frac{1}{\rho}]$, are given by

$$T(y; \epsilon) = 2\frac{1}{\epsilon^2}\big(T_0(y) + o(1)\big) \text{ and } \frac{d}{dy}T(y; \epsilon) = 2\frac{1}{\epsilon^2}\left(\frac{d}{dy}T_0(y) + o(1)\right),$$

with $\epsilon > 0$ small enough, where $T_0(y)$ is the transition time (at level $\epsilon = 0$) of the reduced flow along the attracting part of the curve of singularities between the ω-limit of the point $y \in [\rho, \frac{1}{\rho}]$ and the turning point. Using the expressions for the reduced flow we have for $y \in [\rho, \frac{1}{\rho}]$

$$T_0(y) = -\int_y^0 \frac{d\tilde{y}}{\tilde{x}} = -\int_y^0 \frac{d\tilde{y}}{g(\tilde{y})},$$

where $\tilde{x} = g(\tilde{y})$ represents the attracting part of the critical curve, i.e. $\tilde{y} = \bar{G}(g(\tilde{y}))$. Finally, we get

$$\frac{d}{dy}T_0(y) = \frac{1}{g(y)} > 0$$

for all $y \in [\rho, \frac{1}{\rho}]$ and $\bar{c} \in \bar{C}$. We conclude that $\frac{d}{dy}T(y; \epsilon) > 0$ for all $y \in [\rho, \frac{1}{\rho}]$, $\bar{c} \in \bar{C}$ and $\epsilon \in]0, \epsilon_0]$ for some small $\epsilon_0 > 0$. We point out that we are allowed to use the results of [3] because the reduced flow has no singularities.

It remains to show that $\frac{d}{dy}T(y; \epsilon) > 0$ for $y \in [y_1, \infty[$, $\bar{c} \in \bar{C}$ and $\epsilon \in]0, \epsilon_0]$ for $y_1 > 0$ large enough and $\epsilon_0 > 0$ small enough. To investigate the period function when $y \to \infty$, we apply the coordinate change $(x, y) = \left(\frac{\tilde{x}}{q}, \frac{1}{q^{2N}}\right)$ to (3.18), where $q > 0$ is small and $\tilde{x}$ is kept in a compact set. In the new coordinates (3.18) becomes $\frac{1}{q^{2N-1}}X_\infty$ where the vector field X_∞ is given by

$$\begin{cases} \dot{\tilde{x}} = 1 - \left(q^{2N-2}\tilde{x}^2 + \sum_{k=2}^{N-1} \bar{c}_{2k} q^{2N-2k}\tilde{x}^{2k} + \bar{c}_{2N}\tilde{x}^{2N}\right) + \frac{1}{2N}\epsilon^2 q^{4N-2}\tilde{x}^2, \\ \dot{q} = \frac{1}{2N}\epsilon^2 q^{4N-1}\tilde{x}. \end{cases} \tag{3.20}$$

On the line $\{q = 0\}$ (it represents infinity in the (x, y)-phase space), system (3.20) has two semi-hyperbolic singularities $\tilde{x} = \pm\left(\frac{1}{\bar{c}_{2N}}\right)^{\frac{1}{2N}}$ (resp. $\tilde{x} = \pm 1$) when $N \geq 2$ (resp. $N = 1$). Note that $\bar{c}_{2N}$ is uniformly positive and bounded. It suffices to look at the positive sign. When $\epsilon = 0$, we have the curve of (semi-hyperbolic) singularities $\tilde{x} = \left(\frac{1}{\bar{c}_{2N}}\right)^{\frac{1}{2N}} + O(q)$ (resp. $\tilde{x} = 1$). The reduced flow is given by

$$q' = \frac{1}{2N}q^{4N-1}\left(\left(\frac{1}{\bar{c}_{2N}}\right)^{\frac{1}{2N}} + O(q)\right) \quad \left(\text{resp. } q' = \frac{1}{2}q^3\right).$$

Using a Takens normal form for C^k-equivalence (see e.g. [5]), system X_∞ near the semi-hyperbolic singularity on the line $\{q = 0\}$ is C^k-equivalent to

$$\begin{cases} \dot{\hat{x}} = -\hat{x}, \\ \dot{\hat{q}} = \epsilon^2 \hat{q}^{4N-1} h(\hat{q}, \epsilon, \bar{c}), \end{cases} \tag{3.21}$$

where h is a positive C^k-function. We denote system (3.21) by $\hat{X}_\infty$. We conclude that in the normal form coordinates $(\hat{x}, \hat{q})$ the vector field $\frac{1}{q^{2N-1}}X_\infty$ can be written as

$$\frac{1}{\hat{q}^{2N-1}\hat{h}(\hat{x}, \hat{q}, \epsilon, \bar{c})}\hat{X}_\infty, \tag{3.22}$$

where $\hat{h}$ is a positive C^k-function. We choose two transverse sections $\Sigma_- \subset \{\hat{x} = \hat{x}_0\}$, parametrized by $\hat{q}$, and $\Sigma_+ \subset \{\hat{q} = \hat{q}_0\}$, parametrized by $\hat{x}$, for some small and fixed $\hat{x}_0, \hat{q}_0 > 0$. We compute the time of (3.22) spent between Σ_- and Σ_+, near $(\hat{x}, \hat{q}) = (0, 0)$. The orbit of (3.21) or (3.22) starting at $\hat{q}_1 \in \Sigma_-$, with $\hat{q}_1 > 0$, is given by

$$\hat{x}(\hat{q}, \hat{q}_1) = \hat{x}_0 \exp\left(-\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}} \frac{dz}{z^{4N-1}h(z, \epsilon, \bar{c})}\right).$$

Now is the time spent by the orbit given by

$$\mathcal{T}(\hat{q}_1; \epsilon) = \frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}_0} \frac{\bar{h}(\hat{x}(z, \hat{q}_1), z, \epsilon, \bar{c})dz}{z^{2N}}$$

with a positive C^k-function $\bar{h}$. The derivative is given by

$$\frac{d}{d\hat{q}_1}\mathcal{T}(\hat{q}_1; \epsilon) = -\frac{\bar{h}(\hat{x}_0, \hat{q}_1, \epsilon, \bar{c})}{\epsilon^2 \hat{q}_1^{2N}} + \frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}_0} \frac{\frac{\partial \bar{h}}{\partial \hat{x}}(\hat{x}(z, \hat{q}_1), z, \epsilon, \bar{c})\frac{\partial \hat{x}}{\partial \hat{q}_1}(z, \hat{q}_1)}{z^{2N}}dz. \tag{3.23}$$

Now, we proceed exactly as in Section 3.3.5. The first term in (3.23) tends to $-\infty$ as $\epsilon^2\hat{q}_1^{2N} \to 0$ and we show that it is a dominant term. We have

$$\left|\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}_0}\right| \leq \frac{\alpha \hat{x}_0}{\epsilon^4 \hat{q}_1^{4N-1}}\int_{\hat{q}_1}^{\hat{q}_0} \frac{\exp\left(-\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{z}\frac{ds}{s^{4N-1}h(s,\epsilon,\bar{c})}\right)}{z^{2N}}dz \tag{3.24}$$

with a positive constant α. We used the fact that $\frac{\partial \bar{h}}{\partial \hat{x}}$ is bounded and h is uniformly positive. For the $[\hat{q}_1, 2\hat{q}_1]$-part of the integral on the right hand side of (3.24), we get

$$\begin{aligned}
\int_{\hat{q}_1}^{2\hat{q}_1} &= \frac{1}{\hat{q}_1^{2N-1}} \int_1^2 \frac{\exp\left(-\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}_1\widetilde{z}} \frac{ds}{s^{4N-1}h(s,\epsilon,\bar{c})}\right)}{\widetilde{z}^{2N}} d\widetilde{z} \\
&= \frac{1}{\hat{q}_1^{2N-1}} \int_1^2 \frac{\exp\left(-\frac{1}{\epsilon^2\hat{q}_1^{4N-2}}\int_1^{\widetilde{z}} \frac{d\widetilde{s}}{\widetilde{s}^{4N-1}h(\hat{q}_1\widetilde{s},\epsilon,\bar{c})}\right)}{\widetilde{z}^{2N}} d\widetilde{z} \\
&\leq \frac{1}{\hat{q}_1^{2N-1}} \int_1^2 \frac{\exp\left(-\frac{\beta(\widetilde{z}-1)}{\epsilon^2\hat{q}_1^{4N-2}}\right)}{\widetilde{z}^{2N}} d\widetilde{z} \\
&\leq \gamma\epsilon^2\hat{q}_1^{2N-1},
\end{aligned} \tag{3.25}$$

where $\beta, \gamma > 0$ are constants. (See Section 3.3.5 for each step.) On the other hand, we have

$$\begin{aligned}
\int_{2\hat{q}_1}^{\hat{q}_0} &\leq \int_{2\hat{q}_1}^{\hat{q}_0} \frac{\exp\left(-\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{2\hat{q}_1} \frac{ds}{s^{4N-1}h(s,\epsilon,\bar{c})}\right)}{z^{2N}} dz \\
&= \int_{2\hat{q}_1}^{\hat{q}_0} \frac{\exp\left(-\frac{1}{\epsilon^2\hat{q}_1^{4N-2}}\int_1^{2} \frac{d\widetilde{s}}{\widetilde{s}^{4N-1}h(\hat{q}_1\widetilde{s},\epsilon,\bar{c})}\right)}{z^{2N}} dz \\
&\leq \exp\left(-\frac{\beta}{\epsilon^2\hat{q}_1^{4N-2}}\right) \int_{2\hat{q}_1}^{\hat{q}_0} \frac{dz}{z^{2N}} \\
&\leq \frac{\gamma}{\hat{q}_1^{2N-1}} \exp\left(-\frac{\beta}{\epsilon^2\hat{q}_1^{4N-2}}\right)
\end{aligned} \tag{3.26}$$

for some new constants $\beta, \gamma > 0$. Combining (3.24), (3.25) and (3.26) we finally have

$$\left|\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}_0}\right| \leq \frac{\alpha_1\hat{x}_0}{\epsilon^2\hat{q}_1^{2N}} + \frac{\alpha_2}{\epsilon^4\hat{q}_1^{6N-2}}\exp\left(-\frac{\beta}{\epsilon^2\hat{q}_1^{4N-2}}\right)$$

for positive constants $\alpha_1, \alpha_2, \beta$. Now, it suffices to notice that $\hat{x}_0 > 0$ can be arbitrarily small but fixed.

The time of $\frac{1}{q^{2N-1}}X_\infty$ spent between $\{\widetilde{x} = 0\}$ and Σ_- is of order $O(q^{2N-1})$ (X_∞ is regular in this region). Following [3], the time spent between Σ_- and the turning point and its derivative are of order $O(\frac{1}{\epsilon^2})$. This implies that the contribution (3.23) is dominant. Thus, $\frac{d}{dy}T(y;\epsilon) > 0$ for large y. This ends the proof of Theorem 1.2.

Acknowledgements

D.R. has been partially supported by the Agencia Estatal de Investigación and Ministerio de Ciencia, Innovación y Universidades Grant No. MTM2017-86795-C3-1-P. D.R. is member of the Consolidated Research Group 2017 SGR 1617 funded by AGAUR (Generalitat de Catalunya). D.R is a Serra Húnter Fellow.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 17, 1–7; https://doi.org/10.14232/ejqtde.2021.1.17 www.math.u-szeged.hu/ejqtde/

The decay rates of solutions to a chemotaxis-shallow water system

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Received 12 October 2020, appeared 24 March 2021
Communicated by Bo Zhang

Abstract. In this paper, we consider the large time behavior of solution for the chemotaxis-shallow water system in $\mathbb{R}^2$. The lower bound for time decay rates of the bacterial density and the chemoattractant concentration are proved by the method of energy estimates, which implies these two variables tend to zero at the L^2-rate $(1+t)^{-\frac{1}{2}}$. Furthermore, by the Fourier splitting method, we also show the first order spatial derivatives of the bacterial density tends to zero at the L^2-rate $(1+t)^{-1}$.

Keywords: chemotaxis, shallow water system, decay rates.

2020 Mathematics Subject Classification: 35B40, 35Q35, 35Q92, 76N10, 92C17.

1 Introduction

In this paper, we are interested in two-dimensional chemotaxis-shallow water system

$$\begin{cases} n_t + \operatorname{div}(nu) = D_n \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t + \operatorname{div}(cu) = D_c \Delta c - nf(c), \\ h_t + \operatorname{div}(hu) = 0, \\ hu_t + hu \cdot \nabla u + h^2 \nabla n + \dfrac{1}{2}(1+n)\nabla h^2 = \mu \Delta u + (\mu + \lambda)\nabla(\operatorname{div} u), \end{cases} \tag{1.1}$$

which was proposed in [2] to describe the dynamics of the oxygen and aerobic bacteria in the incompressible fluids with free surface. Here n, c, h, u denote the bacterial density, the chemoattractant concentration, the fluid height and the fluid velocity field, respectively. The constants D_n and D_c are the corresponding diffusion coefficients for the cells and substrate. The chemotactic sensitivity $\chi(c)$ and the consumption rate of the substrate by the cells $f(c)$ are supposed to be given smooth functions. The constants μ and λ are the shear viscosity and the bulk viscosity coefficients respectively with the following physical restrictions: $\mu > 0, \mu + \lambda \geq 0$. In order to complete system (1.1), the initial conditions are given by

$$(n, c, h, u)(x, t)|_{t=0} = (n_0(x), c_0(x), h_0(x), u_0(x)), \quad \text{for } x \in \mathbb{R}^2. \tag{1.2}$$

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As the space variable tends to infinity, we assume

$$\lim_{|x|\to\infty}(n_0, c_0, h_0 - 1, u_0)(x) = 0. \tag{1.3}$$

Chemotaxis exists widely in the nature. The bacteria or microorganisms often live in a viscous fluid with chemical stimulation and like to move towards a chemically more advantageous circumstance for better survival known as chemotaxis. To describe the dynamics of swimming bacteria, Tuval et al. [16] proposed a coupled system of the chemotaxis model and the viscous incompressible fluid. Since then, there has been many results in literature on the solvability and stability of this chemotaxis-fluid system. The local weak solution was proved by Lorz [9] and the local smooth solution was showed by Chae–Kang–Lee [1]. Liu–Lorz [8] and Winkler [22] established the global weak solutions. The global classical and strong solution was proved by Winkler [19] and Duan–Lorz–Markowich [4], respectively. The stability problem was studied in [3,11,20,23] and the small-convection limit was investigated by Wang et al.[18]. We also would like refer to [5–7,12,13,15,21,24] and the references therein for more related works on the chemotaxis-fluid system with nonlinear diffusion.

Considering the fact that the surface of the fluid is a free boundary, the modified shallow water type chemotactic model (1.1) is derived in [2]. For large initial data allowing vacuum, i.e. the bacterial density n is allowed to vanish, the authors in [2] established the local existence of strong solutions and the blow-up criterion. In [14], we proved the global well-posedness of strong solution and studied the upper bound decay rates of the global solution with the initial data far from vacuum. Recently, Wang–Wang [17] showed the upper bound decay estimates of the global solutions in L^p space with the initial bacterial density allowing vacuum.

In this paper, based on the previous works [14, 17], we are interested in the large time behavior of the global solution for the chemotaxis-shallow water system with the bacterial density n being allowed to vanish. The lower bound decay rates for the chemoattractant concentration c, the bacterial density n and its one order spatial derivatives will be given.

In what follows, for simplicity, let $D_n = D_c = 1, \chi(c) \equiv 1, f(c) = c$. Furthermore, throughout this paper, we use $H^k(\mathbb{R}^2)(k \in \mathbb{R})$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^k}$ and $L^p(\mathbb{R}^2)(1 \le p \le \infty)$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. C denotes constant independent of time t. For the sake of simplicity, $\|(A, B)\|_X := \|A\|_X + \|B\|_X$.

Now, we first recall the following result obtained in [17].

Theorem 1.1. *Assume that the initial data $(n_0, c_0, h_0 - 1, u_0) \in H^4 \cap L^1$ satisfies $n_0, c_0 \ge 0$ and $h_0 > 0$ and there exists a small positive constant δ_0 such that $\|(n_0, c_0, h_0 - 1, u_0)\|_{H^4\cap L^1} \le \delta_0$, then the system (1.1)–(1.3) has a unique global classical solution which satisfies*

$$\|\nabla^k(n, c, h - 1, u)(t)\|_{L^2} \le C(1 + t)^{-\frac{1+k}{2}}, \quad \text{for } k = 0, 1, 2. \tag{1.4}$$

The main result in this paper can be stated as follows.

Theorem 1.2. *Assume that the assumptions of Theorem 1.1 hold and the Fourier transform $\mathcal{F}(n_0) = \widehat{n}_0$ and $\mathcal{F}(c_0) = \widehat{c}_0$ satisfy $|\widehat{n}_0| \ge \bar{n} > 0$ and $|\widehat{c}_0| \ge \bar{c} > 0$ for $0 \le |\xi| \ll 1$, with $\bar{n}$ and $\bar{c}$ are small constants. Then, the bacterial density n and the chemoattractant concentration c of global solution to the system (1.1)–(1.3) has the lower bound for time decay rates for all $t \ge T_1$*

$$\|(n, c)(t)\|_{L^2} \ge C(1 + t)^{-\frac{1}{2}} \quad \text{and} \quad \|\nabla n(t)\|_{L^2} \ge C(1 + t)^{-1},$$

where T_1 is a positive large time.

Remark 1.3. By combining the results in Theorem 1.1 and Theorem 1.2, one can find that the bacterial density and the chemoattractant concentration tend to zero at the L^2-rate $(1+t)^{-\frac{1}{2}}$ and the first order spatial derivatives of the bacterial density tends to zero at the L^2-rate $(1+t)^{-1}$.

Remark 1.4. From the structure of the system (1.1), we can find the fluid height and the fluid velocity field satisfy the hyperbolic and parabolic coupled system with linear term ∇n. This means that the method in this paper will no longer be valid for the lower bound decay rates of the fluid height and the fluid velocity field.

Remark 1.5. It is worth mentioning that many functions, for example $\delta_0 e^{-|x|}$ or $\delta_0 e^{-|x|^2}$, can fulfill the hypotheses in Theorem 1.1 and Theorem 1.2 simultaneously.

2 The lower bound for time decay rates

Let us first consider the following linearized system of $(1.1)_1$ and $(1.1)_2$.

$$\begin{cases} \partial_t n_l - \Delta n_l = 0, \\ \partial_t c_l - \Delta c_l = 0, \end{cases} \tag{2.1}$$

with the initial data $(n_l, c_l)(x, 0) = (n_0, c_0)(x)$.

Lemma 2.1. *Assume that the Fourier transform $\mathcal{F}(n_0) = \widehat{n}_0$ and $\mathcal{F}(c_0) = \widehat{c}_0$ satisfy $|\widehat{n}_0| \geq \bar{n} > 0$ and $|\widehat{c}_0| \geq \bar{c} > 0$ for $0 \leq |\xi| \ll 1$, with $\bar{n}$ and $\bar{c}$ are small constants. Then, n_l and c_l in (2.1) have the decay rates*

$$\|(n_l, c_l)(t)\|_{L^2} \geq C(1+t)^{-\frac{1}{2}} \quad \text{and} \quad \|\nabla(n_l, c_l)(t)\|_{L^2} \geq C(1+t)^{-1}. \tag{2.2}$$

Proof. Since n_l satisfies a heat equation, with the help of semigroup method, we have $n_l(x,t) = e^{-\Delta t} n_0(x)$. Thus, using the Fourier transform, we have

$$\int_{\mathbb{R}^2} |n_l|^2 dx = \int_{\mathbb{R}^2} |\widehat{n}_0|^2 e^{-2|\xi|^2 t} d\xi \geq \bar{n}^2 \int_{|\xi| \ll 1} e^{-2|\xi|^2 t} d\xi \geq C(1+t)^{-1},$$
$$\int_{\mathbb{R}^2} |\nabla n_l|^2 dx = \int_{\mathbb{R}^2} |\widehat{n}_0|^2 \xi^2 e^{-2|\xi|^2 t} d\xi \geq C(1+t)^{-2}.$$

Similarly, we can also obtain the lower bounds for c_l. Therefore, we complete the proof of this lemma. $\square$

Next, we recall a known result which will be used later (see [3, 17]).

Lemma 2.2. *Assume that the assumptions of Theorem 1.1 hold. Then the global strong solution (n, c, h, u) to the Cauchy problem of system (1.1)–(1.3) satisfies*

$$n(t,x) \geq 0, \qquad c(t,x) \geq 0 \qquad \text{a.e. in } (0, +\infty) \times \mathbb{R}^2. \tag{2.3}$$

Now, we are ready to deal with the nonlinear part of $(1.1)_1$ and $(1.1)_2$. Set $n_r = n - n_l$ and $c_r = c - c_l$, then n_r and c_r satisfy

$$\begin{cases} \partial_t n_r - \Delta n_r = -\operatorname{div}(nu) - \nabla \cdot (n \nabla c), \\ \partial_t c_r - \Delta c_r = -\operatorname{div}(cu) - nc, \end{cases} \tag{2.4}$$

with the initial data $(n_r, c_r)(x, 0) = (0, 0)$. Here, (2.4) is a non-homogeneous linear heat equations.

Remark 2.3. It is worth mentioning that the method in our paper can be extended to parabolic equations with other different types of nonlinear sources to get the lower bound for time decay rates. However, these nonlinear sources can not contain linear part in it. More precisely, taking the logistic source term in the equation for n as an example, we consider

$$\partial_t n_r - \Delta n_r = -\mathrm{div}(nu) - \nabla\cdot(n\nabla c) + \rho n - \mu n^2,$$

where ρ and μ are constants. It follows from (1.4), the linear term $\|\rho n(t)\|_{L^2}$ only gives us $(1+t)^{-\frac{1}{2}}$ decay rate. Thus, we can not get the the lower bound for time decay rates with $\|n_r(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}$.

Lemma 2.4. *Assume that the assumptions of Theorem 1.1 hold. Then, n_r and c_r in (2.4) have the decay rates*

$$\|(n_r, c_r)(t)\|_{L^2} \le C(1+t)^{-1} \quad \text{and} \quad \|\nabla n_r(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}}. \tag{2.5}$$

Proof. Define $S_1 = \mathrm{div}(nu) + \nabla\cdot(n\nabla c)$ and $S_2 = \mathrm{div}(cu)$. By virtue of the semigroup method, Duhamel's principle and Lemma 2.2, from (2.4) we have

$$\begin{aligned}
&\|(n_r, c_r)(t)\|_{L^2}\\
&\le \int_0^t \left(\int_{\mathbb{R}^2} e^{-2|\xi|^2(t-\tau)}(|(\widehat{S}_1,\widehat{S}_2)|^2)d\xi\right)^{\frac{1}{2}} d\tau\\
&\le \int_0^t \left(\int_{|\xi|\le 1} e^{-2|\xi|^2(t-\tau)}(|(\widehat{S}_1,\widehat{S}_2)|^2)d\xi + \int_{|\xi|\ge 1} e^{-2|\xi|^2(t-\tau)}(|(\widehat{S}_1,\widehat{S}_2)|^2)d\xi\right)^{\frac{1}{2}} d\tau \qquad (2.6)\\
&\le C\int_0^t (1+t-\tau)^{-1}(\||\xi|^{-1}(\widehat{S}_1,\widehat{S}_2)\|_{L^\infty} + \|(S_1,S_2)\|_{L^2})d\tau\\
&\le C\int_0^t (1+t-\tau)^{-1}(\|(n,c,u,\nabla c)\|_{L^2}^2 + \|(S_1,S_2)\|_{L^2})d\tau.
\end{aligned}$$

It follows from the Sobolev inequality and (1.4) that

$$\begin{aligned}
\|(S_1,S_2)\|_{L^2} &\le \|\nabla u\|_{L^4}\|(n,c)\|_{L^4} + \|u\|_{L^4}\|\nabla(n,c)\|_{L^4} + \|\nabla n\|_{L^4}\|\nabla c\|_{L^4} + \|n\|_{L^\infty}\|\nabla^2 c\|_{L^2}\\
&\le C(1+t)^{-2}.
\end{aligned} \tag{2.7}$$

Thus, using (1.4) again, we obtain

$$\begin{aligned}
\int_0^t (1+t-\tau^{-1}\|(n,c,u,\nabla c)\|_{L^2}^2\|_{L^2}d\tau &\le \int_0^t (1+t-\tau)^{-1}(1+\tau)^{-1}d\tau \le (1+t)^{-1},\\
\int_0^t (1+t-\tau)^{-1}\|(S_1,S_2)\|_{L^2}d\tau &\le \int_0^t (1+t-\tau)^{-1}(1+\tau)^{-2}d\tau \le (1+t)^{-1}.
\end{aligned}$$

This, together with (2.6), implies

$$\|(n_r, c_r)(t)\|_{L^2} \le C(1+t)^{-1}. \tag{2.8}$$

Next, applying ∇ to $(2.4)_1$, then multiplying by ∇n, integrating over $\mathbb{R}^2$, after integration by parts and using (2.7), it infers that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla n_r|^2dx + \int_{\mathbb{R}^2}|\nabla^2 n_r|^2dx = \int_{\mathbb{R}^2} S_1\cdot\nabla^2 n_r dx \le \frac{1}{2}\int_{\mathbb{R}^2}|\nabla^2 n_r|^2dx + C(1+t)^{-4},$$

which gives

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla n_r|^2dx+\int_{\mathbb{R}^2}|\nabla^2 n_r|^2dx\le C(1+t)^{-4}. \tag{2.9}$$

Denoting the time sphere S_0 (see [10]) as follows

$$S_0:=\left\{\xi\in\mathbb{R}^2 \mid |\xi|\le\left(\frac{R}{1+t}\right)^{\frac{1}{2}}\right\},$$

where R is a constant defined below. Then, we can get

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla^2 n_r|^2dx&\ge\int_{\mathbb{R}^2\setminus S_0}|\xi|^4|\widehat{n}_r|^2d\xi\\&\ge\frac{R}{1+t}\int_{\mathbb{R}^2\setminus S_0}|\xi|^2|\widehat{n}_r|^2d\xi\\&\ge\frac{R}{1+t}\int_{\mathbb{R}^2}|\xi|^2|\widehat{n}_r|^2d\xi-\frac{R^2}{(1+t)^2}\int_{S_0}|\widehat{n}_r|^2d\xi.\end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.9) and then applying (2.8), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla n_r|^2dx+\frac{R}{1+t}\int_{\mathbb{R}^2}|\nabla n_r|^2dx\\&\qquad\le\frac{R^2}{(1+t)^2}\int_{\mathbb{R}^2}|n_r|^2dx+C(1+t)^{-4}\le CR^2(1+t)^{-4}.\end{aligned} \tag{2.11}$$

Choosing $R=\frac{7}{2}$, multiplying (2.11) by $(1+t)^{\frac{7}{2}}$ and integrating over $[0,t]$, it holds that

$$\|\nabla n_r(t)\|_{L^2}^2\le C(1+t)^{-3},$$

which, together with (2.8) completes the proof of this lemma. □

Proof of Theorem 1.2. It follows from Lemma 2.1 and Lemma 2.4 that

$$\begin{aligned}\|(n,c)\|_{L^2}&\ge\|(n_l,c_l)\|_{L^2}-\|(n_r,c_r)\|_{L^2}\\&\ge C(1+t)^{-\frac{1}{2}}-C(1+t)^{-1}\\&\ge C(1+t)^{-\frac{1}{2}}-\frac{C}{(1+t)^{\frac{1}{2}}}(1+t)^{-\frac{1}{2}},\\\|\nabla n\|_{L^2}&\ge\|\nabla n_l\|_{L^2}-\|\nabla n_r\|_{L^2}\\&\ge C(1+t)^{-1}-C(1+t)^{-\frac{3}{2}}\\&\ge C(1+t)^{-1}-\frac{C}{(1+t)^{\frac{1}{2}}}(1+t)^{-1}.\end{aligned}$$

Obviously, we can choose a $T_1>0$ large enough such that for $t\ge T_1$, we have the lower bound for time decay rates

$$\|(n,c)(t)\|_{L^2}\ge C(1+t)^{-\frac{1}{2}}\quad\text{and}\quad\|\nabla n(t)\|_{L^2}\ge C(1+t)^{-1}.$$

Therefore, we complete the proof of Theorem 1.2. □

Acknowledgements

The authors would like to thank the anonymous referees for the valuable comments in improving the presentation of the paper. This work was supported in part by Guangdong Basic and Applied Basic Research Foundation (2020A1515010530), the National Science Foundation of China (11971320).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **18**, 1–23; https://doi.org/10.14232/ejqtde.2021.1.18 www.math.u-szeged.hu/ejqtde/

Rectifiability of orbits for two-dimensional nonautonomous differential systems

Dedicated to Professor Hiroyuki Usami on the occasion of his sixtieth birthday

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Received 28 September 2020, appeared 24 March 2021
Communicated by Mihály Pituk

Abstract. The present study is concerned with the rectifiability of orbits for the two-dimensional nonautonomous differential systems. Criteria are given whether the orbit has a finite length (rectifiable) or not (nonrectifiable). The global attractivity of the zero solution is also discussed. In the linear case, a necessary and sufficient condition can be obtained. Some examples and numerical simulations are presented to explain the results.

Keywords: rectifiability, global attractivity, two-dimensional nonautonomous system.

2020 Mathematics Subject Classification: 34A34, 34D20, 26B15.

1 Introduction

We consider the two-dimensional nonautonomous differential system

$$\begin{aligned} x' &= -e(t)x + f(t)y - p(t)x\left(x^2+y^2\right)^{\lambda}, \\ y' &= -g(t)x - h(t)y - q(t)y\left(x^2+y^2\right)^{\lambda}, \end{aligned} \tag{1.1}$$

where e, f, g, h, p and q are continuous for $t \ge t_0$, and $\lambda > 0$. Since the right hand side of this system is continuously differentiable with respect to (x, y), so it satisfies the Lipschitz condition. Therefore, the local existence and uniqueness of solutions of (1.1) are guaranteed for the initial-value problem. We can show that, for each $t_0 \in \mathbf{R}$ and $(x_0, y_0) \in \mathbf{R}^2$, the initial value problem (1.1) with $(x(t_0), y(t_0)) = (x_0, y_0)$ has a unique solution on $[t_0, \infty)$ under some conditions (this fact will be shown in Lemma 3.4.). We denote it by $(x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$. Clearly, (1.1) has the zero solution $(x(t), y(t)) \equiv (0, 0)$. Throughout this paper, $\|(x, y)\|$ means

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the Euclidean norm of (x,y); that is, $\|(x,y)\| := \sqrt{x^2+y^2}$. Here, let us give a definition about the zero solution of (1.1). The zero solution of (1.1) is said to be *globally attractive* if

$$\lim_{t\to\infty} \|(x(t;t_1,x_0,y_0),y(t;t_1,x_0,y_0))\| = 0$$

for any $t_1 \in [t_0,\infty)$ and any $(x_0,y_0) \in \mathbf{R}^2$. Now rewrite $(x(t;t_0,x_0,y_0),y(t;t_0,x_0,y_0))$ by $(x(t),y(t))$. We define the *orbit* of $(x(t),y(t))$ by

$$\Gamma_{(t_0,x,y)} := \{(x(t),y(t)) \in \mathbf{R}^2 : t \geq t_0\}.$$

The orbit $\Gamma_{(t_0,x,y)}$ is said to be *simple* if $(x(t_1),y(t_1)) \neq (x(t_2),y(t_2))$ for any t_1, $t_2 \in [t_0,\infty)$ with $t_1 \neq t_2$. Now, we assume that the zero solution of (1.1) is globally attractive. The simple orbit $\Gamma_{(t_0,x,y)}$ is said to be *rectifiable* if the length of $\Gamma_{(t_0,x,y)}$ is finite, that is,

$$\lim_{t\to\infty} \int_{t_0}^{t} \|(x'(s),y'(s))\| ds < \infty.$$

Otherwise, it is said to be *nonrectifiable*.

When $\lambda = 1$ and $e(t) = h(t) = a_0$, $f(t) = g(t) = 1$, $p(t) = q(t) = 1$ for all $t \geq t_0$, system (1.1) reduces to the planar nonlinear differential system

$$\begin{aligned} x' &= y - x\left(x^2+y^2+a_0\right), \\ y' &= -x - y\left(x^2+y^2+a_0\right). \end{aligned} \tag{1.2}$$

For every solution $(x(t),y(t))$ of (1.2), using the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$, then we have

$$\begin{aligned} r' &= -r\left(r^2+a_0\right), \\ \theta' &= -1. \end{aligned}$$

From $\theta' = -1$, every orbit $\Gamma_{(t_0,x,y)}$ of (1.2) is rotating in a clockwise direction. Moreover, if we suppose $a_0 \geq 0$, then $r' \leq -r^3$, so that

$$r(t) \leq \frac{1}{\sqrt{2(t-t_0)+r^{-2}(t_0)}} \leq \frac{1}{\sqrt{2(t-t_0)}}$$

for $t \geq t_0$. This says that $a_0 \geq 0$ implies that the zero solution of (1.2) is globally attractive. Hence, every orbit $\Gamma_{(t_0,x,y)}$ of (1.2) is a spiral.

Remark 1.1. Since (1.2) is an autonomous system and $r' \leq -r^3$, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to any nontrivial solution $(x(t),y(t))$ of (1.2) is simple.

Milišić, Žubrinić and Županović [10] studied rectifiability for more general autonomous differential systems based on planar system (1.2). Theorem 8 given in [10] and the above mentioned facts imply the following.

Theorem A. *Let $(x(t),y(t))$ be any nontrivial solution of (1.2). Suppose that $a_0 \geq 0$ holds. Then the zero solution of (1.2) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t),y(t))$ is simple, and (i) and (ii) below hold:*

(i) if $a_0 > 0$, then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if $a_0 = 0$, then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

Remark 1.2. Milišić, Žubrinić and Županović [10] and Žubrinić and Županović [23, 24] dealt with the rectifiability and the fractal analysis of spiral orbits (or trajectories) of some autonomous systems including (1.2). They dealt with more general, but autonomous systems. This study focuses on the rectifiability of the nonautonomous systems.

For simplicity, we denote

$$\begin{aligned} \alpha_1(t) &:= \min\{e(t), h(t)\} - \frac{|f(t)-g(t)|}{2}, \quad \beta_1(t) := \min\{p(t), q(t)\}, \\ \alpha_2(t) &:= \max\{e(t), h(t)\} + \frac{|f(t)-g(t)|}{2}, \quad \beta_2(t) := \max\{p(t), q(t)\}, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \gamma_1(t) &:= -\max\{f(t), g(t)\} - \frac{|e(t)-h(t)|}{2} - \frac{|p(t)-q(t)|}{2}, \\ \gamma_2(t) &:= -\min\{f(t), g(t)\} + \frac{|e(t)-h(t)|}{2} + \frac{|p(t)-q(t)|}{2}. \end{aligned} \tag{1.4}$$

If $e(t) \equiv h(t)$, $f(t) \equiv g(t)$ and $p(t) \equiv q(t)$, then

$$\alpha_1(t) = \alpha_2(t) = e(t), \quad \beta_1(t) = \beta_2(t) = p(t) \quad \text{and} \quad \gamma_1(t) = \gamma_2(t) = -f(t)$$

for $t \geq t_0$. Moreover, for each $c > 0$, we denote

$$\rho_i(t;c) := \exp\left(2\lambda \int_{t_0}^{t} \alpha_i(s)ds\right) \left(c + 2\lambda \int_{t_0}^{t} \beta_i(s) \exp\left(-2\lambda \int_{t_0}^{s} \alpha_i(\tau)d\tau\right) ds\right), \quad i = 1, 2. \tag{1.5}$$

The first main result in this paper is as follows.

Theorem 1.3. *Let $(x(t), y(t))$ be any nontrivial solution of (1.1). Suppose that*

$$\alpha_1(t) \geq 0, \ \beta_1(t) \geq 0 \quad \textit{for} \ \ t \geq t_0, \tag{1.6}$$

$$\alpha_1(t) + \beta_1(t) > 0 \quad \textit{for} \ \ t \geq t_0, \tag{1.7}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \alpha_1(s)ds = \infty \quad \textit{or} \quad \lim_{t\to\infty} \int_{t_0}^{t} \beta_1(s)ds = \infty. \tag{1.8}$$

Then the zero solution of (1.1) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple, and (i), (ii) and (iii) below hold:

(i) if $\alpha_1(t) > 0$ for $t \geq t_0$, and

$$\limsup_{t\to\infty} \frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)} < \infty, \tag{1.9}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if $0 < \lambda < 1/2$ and

$$\limsup_{t\to\infty} \frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t;c) + \beta_1(t)} < \infty \quad \textit{for each} \ \ c > 0, \tag{1.10}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(iii) if $\lambda \geq 1/2$ and

$$\liminf_{t\to\infty} \frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t)\rho_2(t;c) + \beta_2(t)} > 0 \quad \text{for each} \;\; c > 0, \tag{1.11}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

Using Theorem 1.3 we get the following result, immediately.

Corollary 1.4. *Let $(x(t), y(t))$ be any nontrivial solution of (1.1). Let $(x(t), y(t))$ be any nontrivial solution of (1.1). Suppose that (1.6), (1.7) and (1.8) hold. Then the zero solution of (1.1) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple, and (i), (ii) and (iii) below hold:*

(i) if $\alpha_1(t) > 0$ for $t \geq t_0$, and (1.9), then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if $0 < \lambda < 1/2$ and $\beta_1(t) > 0$ for $t \geq t_0$, and

$$\limsup_{t\to\infty} \frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\beta_1(t)} < \infty, \tag{1.12}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(iii) if $\lambda \geq 1/2$ and $\alpha_2(t) = 0$ for $t \geq t_0$, and

$$\liminf_{t\to\infty} \frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\beta_2(t)} > 0, \tag{1.13}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

Corollary 1.4 is expressed in a form which does not include the functions ρ_1 and ρ_2.

If $e(t) = h(t) = a_0 \geq 0$, $f(t) = g(t) = 1$, $p(t) = q(t) = 1$ for all $t \geq t_0$, then system (1.1) reduces to the planar system

$$\begin{aligned} x' &= -a_0 x + y - x\left(x^2 + y^2\right)^{\lambda}, \\ y' &= -x - a_0 y - y\left(x^2 + y^2\right)^{\lambda}. \end{aligned} \tag{1.14}$$

In this case, we know that $\alpha_1(t) = \alpha_2(t) = a_0$, $\beta_1(t) = \beta_2(t) = 1$ and $\gamma_1(t) = \gamma_2(t) = -1$ for all $t \geq t_0$. Then (1.6), (1.7), (1.8), (1.12) and (1.13) hold. If $a_0 > 0$ then (i) in Corollary 1.4 holds. Hence, we get the following result, immediately.

Corollary 1.5. *Let $(x(t), y(t))$ be any nontrivial solution of (1.14). Suppose that $a_0 \geq 0$ holds. Then the zero solution of (1.14) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple, and (i), (ii) and (iii) below hold:*

(i) if $a_0 > 0$, then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if $a_0 = 0$ and $0 < \lambda < 1/2$, then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(iii) if $a_0 = 0$ and $\lambda \geq 1/2$, then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

Remark 1.6. From Corollary 1.5, Theorem A is easily obtained.

Figures 1.1–1.4 below show that the orbits corresponding to the nontrivial solution $(x(t), y(t))$ of (1.14) with $(x(0), y(0)) = (0.9, 0)$. We choose a_0 and λ as follows: $a_0 = 0.1$ and $\lambda = 1$ in Fig. 1.1; $a_0 = 0$ and $\lambda = 0.1$ in Fig. 1.2; $a_0 = 0$ and $\lambda = 0.5$ in Fig. 1.3; $a_0 = 0$ and $\lambda = 0.9$ in Fig. 1.4.

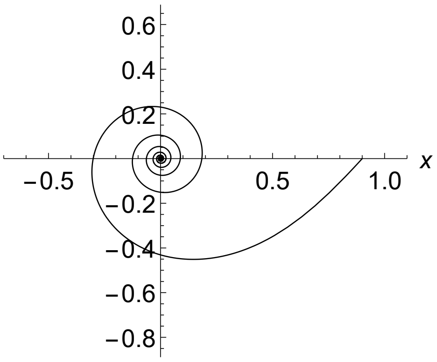

Figure 1.1: $a_0 = 0.1$, $\lambda = 1$; rectifiable.

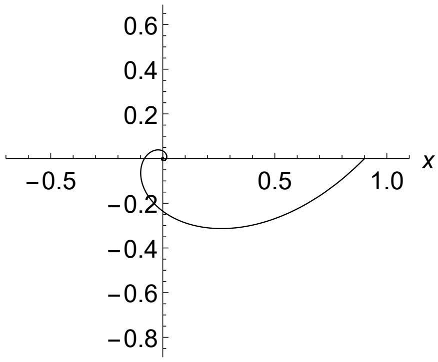

Figure 1.2: $a_0 = 0$, $\lambda = 0.1$; rectifiable.

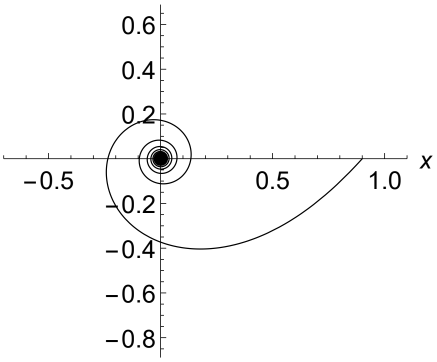

Figure 1.3: $a_0 = 0$, $\lambda = 0.5$; nonrectifiable.

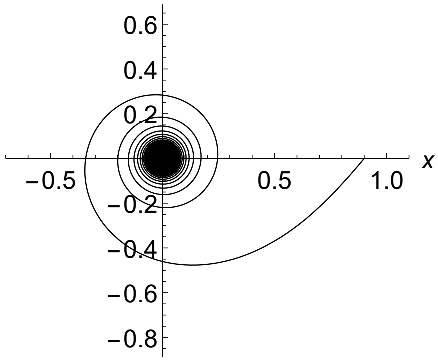

Figure 1.4: $a_0 = 0$, $\lambda = 0.9$; nonrectifiable.

The second main result in this paper is as follows.

Theorem 1.7. *Let $(x(t), y(t))$ be any nontrivial solution of (1.1). Suppose that (1.6), (1.7) and (1.8) hold. Then the zero solution of (1.1) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple, and (i) and (ii) below hold:*

(i) if

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{\sqrt{\left[\alpha_2(s) + \beta_2(s)(\rho_1(s;c))^{-1}\right]^2 + (\max\{|\gamma_1(s)|, |\gamma_2(s)|\})^2}}{(\rho_1(s;c))^{\frac{1}{2\lambda}}} ds < \infty \tag{1.15}$$

for each $c > 0$, then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{\sqrt{\left[\alpha_1(s) + \beta_1(s)(\rho_2(s;c))^{-1}\right]^2 + (\max\{\gamma_1(s), -\gamma_2(s), 0\})^2}}{(\rho_2(s;c))^{\frac{1}{2\lambda}}} ds = \infty \tag{1.16}$$

for each $c > 0$, then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

If $p(t) \equiv q(t) \equiv 0$, then system (1.1) reduces to the two-dimensional linear differential system

$$\begin{aligned} x' &= -e(t)x + f(t)y, \\ y' &= -g(t)x - h(t)y. \end{aligned} \tag{1.17}$$

Note that $\beta_1(t) \equiv \beta_2(t) \equiv 0$. For this linear system, using Theorem 1.7, we obtain the following corollary.

Corollary 1.8. *Let $(x(t), y(t))$ be any nontrivial solution of (1.17). Suppose that*

$$\alpha_1(t) > 0 \quad \text{for } t \geq t_0,$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \alpha_1(s) ds = \infty.$$

Then the zero solution of (1.17) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple, and (i) and (ii) below hold:

(i) if

$$\lim_{t\to\infty} \int_{t_0}^{t} \sqrt{\alpha_2^2(s) + (\max\{|\gamma_1(s)|, |\gamma_2(s)|\})^2} \exp\left(-\int_{t_0}^{s} \alpha_1(\tau) d\tau\right) ds < \infty,$$

then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if

$$\lim_{t\to\infty} \int_{t_0}^{t} \sqrt{\alpha_1^2(s) + (\max\{\gamma_1(s), -\gamma_2(s), 0\})^2} \exp\left(-\int_{t_0}^{s} \alpha_2(\tau) d\tau\right) ds = \infty,$$

then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

In particular, if $e(t) \equiv h(t)$, $f(t) \equiv g(t)$ then we have the two-dimensional linear differential system

$$\begin{aligned} x' &= -e(t)x + f(t)y, \\ y' &= -f(t)x - e(t)y. \end{aligned} \tag{1.18}$$

In this case, we know that $\alpha_1(t) \equiv \alpha_2(t) \equiv e(t)$, $\beta_1(t) \equiv \beta_2(t) \equiv 0$ and $\gamma_1(t) \equiv \gamma_2(t) \equiv -f(t)$. We can establish the following result by Corollary 1.8.

Corollary 1.9. *Let $(x(t), y(t))$ be any nontrivial solution of (1.18). Suppose that*

$$e(t) > 0 \quad \text{for } t \geq t_0, \tag{1.19}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} e(s) ds = \infty. \tag{1.20}$$

Then the zero solution of (1.18) is attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple, and the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable if and only if

$$\lim_{t\to\infty} \int_{t_0}^{t} \sqrt{e^2(s) + f^2(s)} \exp\left(-\int_{t_0}^{s} e(\tau) d\tau\right) ds < \infty. \tag{1.21}$$

Remark 1.10. It is well known that the local attractivity and the global attractivity are equivalent in the linear case (see [1, 20–22]). Hence, the attractivity of (1.18) means the global attractivity.

Consider the two-dimensional nonautonomous linear system

$$\begin{aligned} x' &= -\frac{1}{t}x + t^{\sigma}y, \\ y' &= -t^{\sigma}x - \frac{1}{t}y, \end{aligned} \tag{1.22}$$

where $\sigma \in \mathbf{R}$ and $t \geq 1$. Then assumptions (1.19) and (1.20) are easily satisfied. By Corollary 1.9, the zero solution of (1.22) is attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple. Moreover, we can see that the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable if and only if $\sigma < 0$ (The conditions of Corollary 1.9 will be confirmed in Section 5).

Remark 1.11. Our result on the rectifiability of orbits (or trajectories) of (1.22) is the same as one that the special case of the result given by Naito, Pašić and Tanaka [12, Example 5.2]. Note here that they dealt with half-linear systems. On the other hand, as related research, the rectifiability results of the authors [13, 14] can be mentioned, but note that this study has no inclusion relation with them. Moreover, we can find many results on the rectifiability and the fractal analysis of the systems and equations. For example, the reader is referred to [4–7, 9, 11, 15–19].

In the next section, we will discuss the rectifiability for more general systems under the assumption that the zero solution is globally attractive, and the orbit $\Gamma_{(t_0,x,y)}$ is simple. In Section 3, the simplicity and the global attractivity for (1.1) are considered. In Section 4, we prove Theorems 1.3 and 1.7. In Section 5, some examples and numerical simulations are presented.

2 Rectifiability

In this section, we consider the two-dimensional nonautonomous differential system

$$\begin{aligned} x' &= F_1(t, x, y), \\ y' &= F_2(t, x, y), \end{aligned} \tag{2.1}$$

where F_1 and F_2 are continuously differentiable with respect to (x, y), and satisfying

$$(F_1(t, 0, 0), F_2(t, 0, 0)) \equiv (0, 0).$$

For every solution $(x(t), y(t))$ of (2.1), we introduce the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$. Then we obtain

$$\begin{aligned} r' &= G_1(t, r, \theta), \\ r\theta' &= G_2(t, r, \theta), \end{aligned} \tag{2.2}$$

where G_1 and G_2 are defined by

$$G_1(t, r, \theta) = \cos\theta F_1(t, r\cos\theta, r\sin\theta) + \sin\theta F_2(t, r\cos\theta, r\sin\theta) \tag{2.3}$$

and

$$G_2(t, r, \theta) = \cos\theta F_2(t, r\cos\theta, r\sin\theta) - \sin\theta F_1(t, r\cos\theta, r\sin\theta). \tag{2.4}$$

The obtained result is as follows.

Theorem 2.1. *Let G_1 and G_2 be the functions given by (2.3) and (2.4), respectively. Let $(x(t), y(t))$ be any nontrivial solution of (2.1) on $[t_0, \infty)$. Suppose that the zero solution of (2.1) is globally attractive, and the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple. Then, (i) and (ii) below hold:*

(i) if there exist an $\bar{r} > 0$ and a continuous function $h : (0, \bar{r}) \to (0, \infty)$ such that

$$\|(G_1(t,r,\theta), G_2(t,r,\theta))\| \leq -h(r)G_1(t,r,\theta), \quad (t,r,\theta) \in [t_0,\infty) \times (0,\bar{r}) \times \mathbf{R}, \tag{2.5}$$

and

$$\lim_{r\to+0} \int_r^{\bar{r}} h(\eta)d\eta < \infty, \tag{2.6}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if there exist an $\bar{r} > 0$ and a continuous function $h : (0, \bar{r}) \to (0, \infty)$ such that

$$\|(G_1(t,r,\theta), G_2(t,r,\theta))\| \geq -h(r)G_1(t,r,\theta), \quad (t,r,\theta) \in [t_0,\infty) \times (0,\bar{r}) \times \mathbf{R}, \tag{2.7}$$

and

$$\lim_{r\to+0} \int_r^{\bar{r}} h(\eta)d\eta = \infty, \tag{2.8}$$

then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

Proof. Let $(x(t), y(t))$ be any nontrivial solution of (2.1). Define the functions r and θ by

$$x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t)$$

for $t \geq t_0$, where

$$r(t) = \|(x(t), y(t))\|.$$

Then $(r(t), \theta(t))$ is a solution to (2.2). Since the existence and uniqueness of solutions of (2.1) are guaranteed for the initial-value problem, the zero solution $(x(t), y(t)) \equiv (0, 0)$ is unique. Thus, $r(t) > 0$ for $t \geq t_0$. This together with the global attractivity of (2.1) implies that $\lim_{t\to\infty} r(t) = 0$, and there exists a $T > 0$ such that

$$r(t) \in (0, \bar{r}) \tag{2.9}$$

for $t \geq t_0 + T$.

Now, we consider case (i). Using (2.5) and (2.9), we have

$$\begin{aligned}
\|(x'(t), y'(t))\| &= \|(F_1(t,x(t),y(t)), F_2(t,x(t),y(t)))\| \\
&= \sqrt{(\cos\theta F_1 + \sin\theta F_2)^2 + (\cos\theta F_2 - \sin\theta F_1)^2} \\
&= \|(G_1(t,r(t),\theta(t)), G_2(t,r(t),\theta(t)))\| \\
&\leq -h(r(t))G_1(t,r(t),\theta(t)) = -h(r(t))r'(t)
\end{aligned}$$

for $t \geq t_0 + T$. Since $h(r)$ is a positive continuous function on $(0, \bar{r})$, and (2.9) holds, we see that

$$\begin{aligned}
\int_{t_0+T}^{t} \|(x'(s), y'(s))\|ds &\leq -\int_{t_0+T}^{t} h(r(s))r'(s)ds = \int_{r(t)}^{r(t_0+T)} h(\eta)d\eta \\
&\leq \int_{r(t)}^{\bar{r}} h(\eta)d\eta
\end{aligned}$$

for $t \geq t_0 + T$. Therefore, we have

$$\begin{aligned}\int_{t_0}^{t} \|(x'(s), y'(s))\| ds &= \int_{t_0}^{t_0+T} \|(x'(s), y'(s))\| ds + \int_{t_0+T}^{t} \|(x'(s), y'(s))\| ds \\ &\leq \int_{t_0}^{t_0+T} \|(x'(s), y'(s))\| ds + \int_{r(t)}^{\bar{r}} h(\eta) d\eta\end{aligned}$$

for $t \geq t_0 + T$. Using (2.6), (2.9) with $\lim_{t\to\infty} r(t) = 0$, we conclude that

$$\lim_{t\to\infty} \int_{t_0}^{t} \|(x'(s), y'(s))\| ds < \infty.$$

Hence, the simple orbit $\Gamma_{(t_0,x,y)}$ is rectifiable.

Next, we consider case (ii). From (2.7) and (2.9), we have

$$\|(x'(t), y'(t))\| \geq -h(r(t)) G_1(t, r(t), \theta(t)) = -h(r(t)) r'(t)$$

for $t \geq t_0 + T$. Since $h(r)$ is a positive continuous function on $(0, \bar{r})$, and (2.9) holds, we see that

$$\begin{aligned}\int_{t_0}^{t} \|(x'(s), y'(s))\| ds &\geq \int_{t_0+T}^{t} \|(x'(s), y'(s))\| ds \\ &\geq - \int_{t_0+T}^{t} h(r(s)) r'(s) ds = \int_{r(t)}^{r(t_0+T)} h(\eta) d\eta \\ &= \int_{r(t)}^{\bar{r}} h(\eta) d\eta - \int_{r(t_0+T)}^{\bar{r}} h(\eta) d\eta\end{aligned}$$

for $t \geq t_0 + T$. From (2.8), (2.9) with $\lim_{t\to\infty} r(t) = 0$, we get

$$\lim_{t\to\infty} \int_{t_0}^{t} \|(x'(s), y'(s))\| ds = \infty.$$

Consequently, the simple orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable. This completes the proof. □

For our main system (1.1), we find that

$$\begin{aligned}F_1(t, x, y) &= -e(t)x + f(t)y - p(t)x \left(x^2 + y^2\right)^{\lambda}, \\ F_2(t, x, y) &= -g(t)x - h(t)y - q(t)y \left(x^2 + y^2\right)^{\lambda},\end{aligned}$$

and

$$\begin{aligned}G_1(t, r, \theta) = &- \left(e(t)\cos^2\theta + h(t)\sin^2\theta\right) r + (f(t) - g(t)) r \sin\theta\cos\theta \\ &- \left(p(t)\cos^2\theta + q(t)\sin^2\theta\right) r^{2\lambda+1}, \\ G_2(t, r, \theta) = &- \left(g(t)\cos^2\theta + f(t)\sin^2\theta\right) r + (e(t) - h(t)) r \sin\theta\cos\theta \\ &+ (p(t) - q(t)) r^{2\lambda+1} \sin\theta\cos\theta.\end{aligned} \tag{2.10}$$

3 Simplicity and global attractivity

In this section, we deal with the simplicity and the global attractivity for our main system (1.1). First, we give two lemmas.

Lemma 3.1. *Let G_1 be the function given in (2.10). Then*

$$G_1(t,r,\theta) \le -\left(\alpha_1(t) + \beta_1(t)r^{2\lambda}\right) r$$

holds for $t \ge t_0$ and $r \in [0,\infty)$, where α_1 and β_1 are given in (1.3).

Proof. By (2.10), we get

$$\begin{aligned} G_1(t,r,\theta) &\le -\min\{e(t),h(t)\}r + \frac{|f(t)-g(t)|}{2}r - \min\{p(t),q(t)\}r^{2\lambda+1} \\ &= -\left(\alpha_1(t) + \beta_1(t)r^{2\lambda}\right) r \end{aligned}$$

for $t \ge t_0$ and $r \in [0,\infty)$. □

Lemma 3.2. *Suppose that (1.6) and (1.7) hold. Then*

$$\left(\alpha_1(t) + \beta_1(t)r^{2\lambda}\right) r > 0$$

holds for $t \ge t_0$ and $r \in (0,\infty)$, where α_1 and β_1 are given in (1.3).

Proof. By way of contradiction, we suppose that there exists a $t_1 \ge t_0$ such that

$$\left(\alpha_1(t_1) + \beta_1(t_1)r^{2\lambda}\right) r \le 0.$$

From (1.6) and $r \in (0,\infty)$, we have

$$\alpha_1(t_1) + \beta_1(t_1)r^{2\lambda} = 0.$$

This together with (1.6) says that $\alpha_1(t_1) = \beta_1(t_1) = 0$. However, this contradicts assumption (1.7). □

We now consider the simplicity of the nontrivial solutions to (1.1). The obtained result is as follows.

Lemma 3.3. *Let $(x(t),y(t))$ be a nontrivial solution of (1.1). Suppose that (1.6) and (1.7) hold. Then the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t),y(t))$ is simple.*

Proof. Let $(x(t),y(t))$ be a nontrivial solution of (1.1). Assume to the contrary that there exist $t_1,\ t_2 \in [t_0,\infty)$ such that $t_1 < t_2$ with $(x(t_1),y(t_1)) = (x(t_2),y(t_2))$. Let $(r(t),\theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t),y(t))$. Then $r(t_1) = r(t_2)$ holds. Since $(x(t),y(t))$ is a nontrivial solution and the zero solution is unique, we know that $r(t) > 0$ for all $t \ge t_0$. From Lemmas 3.1 and 3.2, we see that $r'(t) < 0$ for $t \ge t_0$. Integrating this inequality from t_1 to t_2, we obtain

$$r(t_2) - r(t_1) = \int_{t_1}^{t_2} r'(t)dt < 0.$$

This is a contradiction. Consequently, $\Gamma_{(t_0,x,y)}$ is a simple orbit. □

We will give an important inequality.

Lemma 3.4. *Let $(x(t), y(t))$ be a nontrivial solution of (1.1) with the initial condition $(x(t_0), y(t_0)) = (x_0, y_0)$. Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Suppose that $\beta_1(t) \geq 0$ holds for $t \geq t_0$. Then $(x(t), y(t))$ exists on $[t_0, \infty)$ and is the unique solution of (1.1) with $(x(t_0), y(t_0)) = (x_0, y_0)$, and the inequality*

$$0 < r(t) \leq \exp\left(-\int_{t_0}^{t} \alpha_1(s)ds\right)\left(r^{-2\lambda}(t_0) + 2\lambda \int_{t_0}^{t} \beta_1(s) \exp\left(-2\lambda \int_{t_0}^{s} \alpha_1(\tau)d\tau\right) ds\right)^{-\frac{1}{2\lambda}} \tag{3.1}$$

holds for $t \geq t_0$, where α_1 and β_1 are given in (1.3).

Proof. Let $(x(t), y(t))$ be a nontrivial solution of (1.1) with $(x(t_0), y(t_0)) = (x_0, y_0)$. Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Let $I \subset [t_0, \infty)$ be the maximal interval of the existence of $(x(t), y(t))$. Then $r(t) > 0$ holds for $t \in I$, from the uniqueness of the zero solution. Using Lemma 3.1, we have

$$r'(t) \leq -\left(\alpha_1(t) + \beta_1(t) r^{2\lambda}(t)\right) r(t)$$

for $t \in I$. Set $z(t) := r^{-2\lambda}(t)$. Then, it follows from the above inequality and $r(t) > 0$ that

$$z'(t) = -2\lambda r^{-2\lambda-1}(t) r'(t) \geq 2\lambda r^{-2\lambda}(t)\left(\alpha_1(t) + \beta_1(t) r^{2\lambda}(t)\right) = 2\lambda\alpha_1(t)z(t) + 2\lambda\beta_1(t)$$

for $t \in I$. Hence

$$\left(\exp\left(-2\lambda \int_{t_0}^{t} \alpha_1(s)ds\right) z(t)\right)' \geq 2\lambda\beta_1(t) \exp\left(-2\lambda \int_{t_0}^{t} \alpha_1(s)ds\right)$$

for $t \in I$. Integrating this inequality from t_0 to t, we get

$$\exp\left(-2\lambda \int_{t_0}^{t} \alpha_1(s)ds\right) z(t) \geq z(t_0) + 2\lambda \int_{t_0}^{t} \beta_1(s) \exp\left(-2\lambda \int_{t_0}^{s} \alpha_1(\tau)d\tau\right) ds,$$

and so that

$$r^{-2\lambda}(t) = z(t) \geq \exp\left(2\lambda \int_{t_0}^{t} \alpha_1(s)ds\right)\left(r^{-2\lambda}(t_0) + 2\lambda \int_{t_0}^{t} \beta_1(s) \exp\left(-2\lambda \int_{t_0}^{s} \alpha_1(\tau)d\tau\right) ds\right)$$

for $t \in I$. Therefore, if $\beta_1(t) \geq 0$ for $t \geq t_0$, then we obtain (3.1) for $t \in I$.

Using the above inequality and $\beta_1(t) \geq 0$ for $t \geq t_0$, we have

$$r^{-2\lambda}(t) \geq \exp\left(2\lambda \int_{t_0}^{t} \alpha_1(s)ds\right) r^{-2\lambda}(t_0),$$

and thus,

$$0 < \|(x(t), y(t))\| \leq \|(x_0, y_0)\| \exp\left(-\int_{t_0}^{t} \alpha_1(s)ds\right) \quad \text{for } t \in I. \tag{3.2}$$

This inequality means that $I = [t_0, \infty)$, that is, any nontrivial solution of (1.1) exists on $[t_0, \infty)$ by a standard argument of a general theory on ordinary differential equations. Consequently, the initial value problem (1.1) with $(x(t_0), y(t_0)) = (x_0, y_0)$ has a unique solution on $[t_0, \infty)$. □

Next, we consider the global attractivity for (1.1). Assuming a stronger condition, we can get stronger stability. The zero solution is said to be *globally exponentially stable* if there exists a $k > 0$ and, for any $\eta > 0$, there exists a $\delta(\eta) > 0$ such that $t_1 \in \mathbf{R}$ with $t_1 \geq t_0$ and $\|(x_0, y_0)\| < \eta$ imply

$$\|(x(t; t_1, x_0, y_0), y(t; t_1, x_0, y_0))\| \leq \delta(\eta)\|(x_0, y_0)\| e^{-k(t-t_1)}$$

for all $t \geq t_1$. The following lemma is established.

Lemma 3.5. *Suppose that* (1.6) *and* (1.8) *hold, where* α_1 *and* β_1 *are given in* (1.3). *Then the zero solution of* (1.1) *is globally attractive. In particular, if there exists an* $\underline{a} > 0$ *such that*

$$\alpha_1(s) \geq \underline{a} \quad \textit{for} \;\; t \geq t_0, \tag{3.3}$$

then the zero solution of (1.1) *is globally exponentially stable.*

Proof. Let t_1 satisfy $t_1 \geq t_0$. Let $(x(t), y(t))$ be any nontrivial solution of (1.1) with $(x(t_1), y(t_1)) = (x_0, y_0)$. Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Using Lemma 3.4, we have inequality (3.1) for $t \geq t_1$.

Now we consider the case $\lim_{t\to\infty} \int_{t_0}^{t} \alpha_1(s)ds < \infty$. This together with (1.8) yields

$$\lim_{t\to\infty} \int_{t_0}^{t} \beta_1(s)ds = \infty.$$

Let $L := \lim_{t\to\infty} \int_{t_0}^{t} \alpha_1(s)ds \geq 0$. Using this and (3.1), we obtain

$$0 < \|(x(t), y(t))\| = r(t) \leq \frac{1}{\left(r^{-2\lambda}(t_1) + 2\lambda e^{-2\lambda L} \int_{t_1}^{t} \beta_1(s)ds\right)^{\frac{1}{2\lambda}}} < \frac{1}{\left(2\lambda e^{-2\lambda L} \int_{t_1}^{t} \beta_1(s)ds\right)^{\frac{1}{2\lambda}}}$$

for $t \geq t_1$. Hence, any nontrivial solution of (1.1) tends to $(0,0)$ as $t \to \infty$. That is, the zero solution of (1.1) is globally attractive.

Next we consider the case $\lim_{t\to\infty} \int_{t_0}^{t} \alpha_1(s)ds = \infty$. Then, by assumption (1.6), we obtain inequality (3.2). Therefore, the zero solution of (1.1) is globally attractive. Moreover, if we suppose condition (3.3), then inequality (3.2) implies global exponential stability. This completes the proof. □

Remark 3.6. If $\alpha_1(t) \equiv 0$ then, it does not imply the (global) exponential stability for (1.1). For example, we consider the case $\lambda = 1$, $e(t) = h(t) = 0$ and $f(t) = g(t) = 1$ and $p(t) = q(t) = 1$ for $t \geq t_0$. That is, $\alpha_1(t) = \alpha_2(t) = 0$, $\beta_1(t) = \beta_2(t) = 1$ and $\gamma_1(t) = \gamma_2(t) = -1$ for $t \geq t_0$. From (2.2) and (2.10), we have

$$r' = -r^3.$$

Solving this equation, we get

$$r(t) = \frac{1}{\sqrt{2(t - t_0) + r^{-2}(t_0)}}$$

for $t \geq t_0$. Thus, the zero solution is not exponentially stable. Although not described here in detail, we can see that the zero solution of this system is uniformly asymptotically stable. It is well known that the exponential stability implies the uniform asymptotic stability; the uniform asymptotic stability implies the asymptotic stability (the zero solution is attractive and stable). If (1.1) is a periodic or autonomous system, then the asymptotic stability and the uniform asymptotic stability are equivalent. For example, see [2,3,8,21,22]. Moreover, if (1.1) is a linear system, the uniform asymptotic stability and the exponential stability are equivalent. For example, the reader is referred to [3,21,22] and the references cited therein. In general, our main equations are nonautonomous and nonlinear, so their stabilities are often different.

4 Proofs of the main theorems

Before proving the main theorems, we give three lemmas.

Lemma 4.1. *Let G_1 be the function given in (2.10). Then*

$$G_1(t,r,\theta) \geq -\left(\alpha_2(t) + \beta_2(t) r^{2\lambda}\right) r \tag{4.1}$$

holds for $t \geq t_0$ and $r \in [0,\infty)$, where α_2 and β_2 are given in (1.3).

Proof. By (2.10), we get

$$\begin{aligned} G_1(t,r,\theta) &\geq -\max\{e(t),h(t)\} r - \frac{|f(t)-g(t)|}{2} r - \max\{p(t),q(t)\} r^{2\lambda+1} \\ &= -\left(\alpha_2(t) + \beta_2(t) r^{2\lambda}\right) r \end{aligned}$$

for $t \geq t_0$ and $r \in [0,\infty)$. □

Lemma 4.2. *Let $(x(t),y(t))$ be any nontrivial solution of (1.1). Let $(r(t),\theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t),y(t))$. Suppose that $\beta_2(t) \geq 0$ holds for $t \geq t_0$. Then the inequality*

$$r(t) \geq \exp\left(-\int_{t_0}^{t} \alpha_2(s) ds\right) \left(r^{-2\lambda}(t_0) + 2\lambda \int_{t_0}^{t} \beta_2(s) \exp\left(-2\lambda \int_{t_0}^{s} \alpha_2(\tau) d\tau\right) ds\right)^{-\frac{1}{2\lambda}} \tag{4.2}$$

holds for $t \geq t_0$, where α_2 and β_2 are given in (1.3).

Proof. Let $(x(t),y(t))$ be any nontrivial solution of (1.1). Let $(r(t),\theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t),y(t))$. Using Lemma 4.1, we have

$$r'(t) \geq -\left(\alpha_2(t) + \beta_2(t) r^{2\lambda}(t)\right) r(t)$$

for $t \geq t_0$. Set $z(t) := r^{-2\lambda}(t)$. Then, it follows from the above inequality and $r(t) > 0$ that

$$z'(t) \leq 2\lambda \alpha_2(t) z(t) + 2\lambda \beta_2(t)$$

for $t \geq t_0$. Hence

$$\left(\exp\left(-2\lambda \int_{t_0}^{t} \alpha_2(s) ds\right) z(t)\right)' \leq 2\lambda \beta_2(t) \exp\left(-2\lambda \int_{t_0}^{t} \alpha_2(s) ds\right)$$

for $t \geq t_0$. Integrating this inequality from t_0 to t, we get

$$r^{-2\lambda}(t) \leq \exp\left(2\lambda \int_{t_0}^{t} \alpha_2(s) ds\right) \left(r^{-2\lambda}(t_0) + 2\lambda \int_{t_0}^{t} \beta_2(s) \exp\left(-2\lambda \int_{t_0}^{s} \alpha_2(\tau) d\tau\right) ds\right)$$

for $t \geq t_0$. Therefore, if $\beta_2(t) \geq 0$ for $t \geq t_0$, then we obtain the inequality in Lemma 4.2. □

Lemma 4.3. *Let G_2 be the function given in (2.10). Then*

$$\gamma_1(t) r \leq G_2(t,r,\theta) \leq \gamma_2(t) r \tag{4.3}$$

holds for $t \geq t_0$ and $r \in [0,1)$, where γ_1 and γ_2 are given by (1.4).

Proof. By (2.10) and $r \in [0,1)$, we obtain

$$G_2(t,r,\theta) \geq -\max\{f(t),g(t)\}r - \frac{|e(t)-h(t)|}{2}r - \frac{|p(t)-q(t)|}{2}r^{2\lambda+1}$$
$$\geq \gamma_1(t)r$$

and

$$G_2(t,r,\theta) \leq -\min\{f(t),g(t)\}r + \frac{|e(t)-h(t)|}{2}r + \frac{|p(t)-q(t)|}{2}r^{2\lambda+1}$$
$$\leq \gamma_2(t)r.$$

Thus, (4.3) holds. □

Now, we will prove the main theorems.

Proof of Theorem 1.3. From Lemma 3.5, the zero solution of (1.1) is globally attractive. Let $(x(t),y(t))$ be any nontrivial solution of (1.1). By Lemma 3.3, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t),y(t))$ is simple. Let $x = r\cos\theta$, $y = r\sin\theta$. Then we have (2.3) and (2.4). By Lemmas 3.1, 3.2, 4.1 and 4.3, the inequalities

$$0 < \left(\alpha_1(t) + \beta_1(t)r^{2\lambda}\right) r \leq |G_1(t,r,\theta)| = -G_1(t,r,\theta) \leq \left(\alpha_2(t) + \beta_2(t)r^{2\lambda}\right) r, \tag{4.4}$$

and

$$\max\{\gamma_1(t), -\gamma_2(t), 0\}r \leq |G_2(t,r,\theta)| \leq \max\{|\gamma_1(t)|, |\gamma_2(t)|\}r \tag{4.5}$$

hold for $t \geq t_0$ and $r \in (0,1)$. Therefore, we obtain

$$\frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t) + \beta_2(t)r^{2\lambda}} \leq \left|\frac{G_2(t,r,\theta)}{G_1(t,r,\theta)}\right| \leq \frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t) + \beta_1(t)r^{2\lambda}} \tag{4.6}$$

for $t \geq t_0$ and $r \in (0,1)$.

First, we consider case (i). Suppose that $\alpha_1(t) > 0$ for $t \geq t_0$, and (1.9), that is, there exists a $\mu > 0$ and a $t_1 \geq t_0$ such that

$$\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)} \leq \mu$$

holds for $t \geq t_1$. By (1.6), $\beta_1(t) \geq 0$ for $t \geq t_0$. This together with the above inequality implies

$$\sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t) + \beta_1(t)r^{2\lambda}}\right)^2} \leq \sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)}\right)^2} \leq \sqrt{1+\mu^2}$$

for $t \geq t_1$. Moreover, we can choose an $M_1 \geq \sqrt{1+\mu^2}$ such that

$$\sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t) + \beta_1(t)r^{2\lambda}}\right)^2} \leq M_1$$

for $t_0 \leq t \leq t_1$. Using these inequalities and (4.6), we have

$$\|(G_1(t,r,\theta), G_2(t,r,\theta))\| \leq -\sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t) + \beta_1(t)r^{2\lambda}}\right)^2}\, G_1(t,r,\theta) \leq -M_1 G_1(t,r,\theta)$$

for $t \geq t_0$ and $r \in (0,1)$, so that we get (2.5) with $\bar{r} = 1$ and $h(r) = M_1$. By

$$\lim_{r \to +0} \int_r^1 h(\eta) d\eta = M_1,$$

we have (2.6). Consequently, the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable.

Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Before proving cases (ii) and (iii), we will discuss some properties of $r(t)$. By the global attractivity for (1.1), there exits a $t_1 \geq t_0$ such that

$$0 < r(t) \leq 1 \quad \text{for} \quad t \geq t_1.$$

From Lemmas 3.4 and 4.2, we have

$$\rho_1(t; c_0) \leq r^{-2\lambda}(t) \leq \rho_2(t; c_0) \quad \text{for some} \quad c_0 > 0, \tag{4.7}$$

and for $t \geq t_0$, where ρ_1 and ρ_2 are given by (1.5). This together with (4.6) implies that

$$\frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t)\rho_2(t; c_0) + \beta_2(t)} r^{-2\lambda}(t) \leq \left| \frac{G_2(t, r(t), \theta(t))}{G_1(t, r(t), \theta(t))} \right| \leq \frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t; c_0) + \beta_1(t)} r^{-2\lambda}(t) \tag{4.8}$$

for $t \geq t_1$.

Now, we consider case (ii). Suppose that $0 < \lambda < 1/2$ and (1.10) hold, that is, there exists a $\mu > 0$ and a $t_2 \geq t_1$ such that

$$\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t; c) + \beta_1(t)} \leq \mu$$

holds for $t \geq t_2$. By (4.8), we have

$$\begin{aligned}
\sqrt{1 + \left| \frac{G_2(t, r(t), \theta(t))}{G_1(t, r(t), \theta(t))} \right|^2} &\leq \sqrt{r^{4\lambda}(t) + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t; c_0) + \beta_1(t)} \right)^2} \, r^{-2\lambda}(t) \\
&\leq \sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t; c_0) + \beta_1(t)} \right)^2} \, r^{-2\lambda}(t) \\
&\leq \sqrt{1 + \mu^2 r^{-2\lambda}(t)}
\end{aligned}$$

for $t \geq t_2$. Moreover, we can choose an $M_2 \geq \sqrt{1 + \mu^2}$ such that

$$\sqrt{1 + \left(\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)\rho_1(t; c_0) + \beta_1(t)} \right)^2} \leq M_2$$

for $t_0 \leq t \leq t_2$. Therefore, we see that

$$\begin{aligned}
\|(x'(t), y'(t))\| &= \|(F_1(t, x(t), y(t)), F_2(t, x(t), y(t)))\| = \|(G_1(t, r(t), \theta(t)), G_2(t, r(t), \theta(t)))\| \\
&= \sqrt{1 + \left| \frac{G_2(t, r(t), \theta(t))}{G_1(t, r(t), \theta(t))} \right|^2} \, |G_1(t, r(t), \theta(t))| \\
&\leq M_2 r^{-2\lambda}(t) |G_1(t, r(t), \theta(t))| = -M_2 r^{-2\lambda}(t) r'(t)
\end{aligned}$$

holds for $t \geq t_0$. Integrating this inequality, we obtain

$$\int_{t_0}^t \|(x'(s), y'(s))\| ds \leq M_2 \int_{r(t)}^{r(t_0)} \eta^{-2\lambda} d\eta = \frac{M_2}{1 - 2\lambda} \left(r^{1-2\lambda}(t_0) - r^{1-2\lambda}(t) \right) < \frac{M_2 r^{1-2\lambda}(t_0)}{1 - 2\lambda}$$

for $t \geq t_0$. Hence, we conclude that the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable.

Finally, we consider case (iii). Suppose that $\lambda \geq 1/2$ and (1.11) hold, that is, there exists a $\nu > 0$ and a $t_2 \geq t_1$ such that

$$\frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t)\rho_2(t;c) + \beta_2(t)} \geq \nu$$

holds for $t \geq t_2$. By (4.8), we have

$$\sqrt{1 + \left|\frac{G_2(t,r(t),\theta(t))}{G_1(t,r(t),\theta(t))}\right|^2} > \left|\frac{G_2(t,r(t),\theta(t))}{G_1(t,r(t),\theta(t))}\right| \geq \frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t)\rho_2(t;c_0) + \beta_2(t)} r^{-2\lambda}(t) \geq \nu r^{-2\lambda}(t)$$

for $t \geq t_2$. From this, we see that

$$\begin{aligned} \|(x'(t), y'(t))\| &= \sqrt{1 + \left|\frac{G_2(t,r(t),\theta(t))}{G_1(t,r(t),\theta(t))}\right|^2} |G_1(t,r(t),\theta(t))| \\ &> \nu r^{-2\lambda}(t) |G_1(t,r(t),\theta(t))| = -\nu r^{-2\lambda}(t) r'(t) \end{aligned} \tag{4.9}$$

for $t \geq t_2$. Now, we consider the case $\lambda = 1/2$. Integrating (4.9), we obtain

$$\int_{t_0}^{t} \|(x'(s), y'(s))\| ds \geq -\nu \int_{r(t_2)}^{r(t)} \eta^{-1} d\eta = -\nu \log \frac{r(t)}{r(t_2)}$$

for $t \geq t_2$. Since the zero solution of (1.1) is globally attractive, we conclude that the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable. On the other hand, we consider the case $\lambda > 1/2$. Integrating (4.9), we obtain

$$\int_{t_0}^{t} \|(x'(s), y'(s))\| ds \geq -\nu \int_{r(t_2)}^{r(t)} \eta^{-2\lambda} d\eta = \frac{\nu}{2\lambda - 1} \left(\frac{1}{r^{2\lambda-1}(t)} - \frac{1}{r^{2\lambda-1}(t_2)} \right)$$

for $t \geq t_2$. Consequently, $\Gamma_{(t_0,x,y)}$ is nonrectifiable. This completes the proof of Theorem 1.3. □

Proof of Theorem 1.7. Let $(x(t), y(t))$ be any nontrivial solution of (1.1). From Lemmas 3.3 and 3.5, the zero solution of (1.1) is globally attractive, and the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple. Let $(r(t), \theta(t))$ be the solution of (2.2) with (2.10) corresponding to $(x(t), y(t))$. Then the global attractivity for (1.1) implies that there exits a $t_1 \geq t_0$ such that

$$0 < r(t) < 1 \quad \text{for} \quad t \geq t_1.$$

From Lemmas 3.4 and 4.2, we have (4.7) for $t \geq t_0$. Using Lemmas 3.1, 3.2, 4.1 and 4.3, we get inequalities (4.4) and (4.5) for $t \geq t_0$ and $r \in (0,1)$. Therefore,

$$\|(G_1(t,r,\theta), G_2(t,r,\theta))\| \leq \sqrt{(\alpha_2(t) + \beta_2(t) r^{2\lambda})^2 + (\max\{|\gamma_1(t)|, |\gamma_2(t)|\})^2 r} \tag{4.10}$$

and

$$\|(G_1(t,r,\theta), G_2(t,r,\theta))\| \geq \sqrt{(\alpha_1(t) + \beta_1(t) r^{2\lambda})^2 + (\max\{\gamma_1(t), -\gamma_2(t), 0\})^2 r} \tag{4.11}$$

for $t \geq t_0$ and $r \in (0,1)$.

First we consider case (i). By (4.7), (4.10) and the fact

$$\|(x'(t), y'(t))\| = \|(F_1(t,x(t),y(t)), F_2(t,x(t),y(t)))\| = \|(G_1(t,r(t),\theta(t)), G_2(t,r(t),\theta(t)))\|,$$

we obtain

$$\|(x'(t),y'(t))\| \le \frac{\sqrt{[\alpha_2(t)+\beta_2(t)(\rho_1(t;c_0))^{-1}]^2+(\max\{|\gamma_1(t)|,|\gamma_2(t)|\})^2}}{(\rho_1(t;c_0))^{\frac{1}{2\lambda}}}$$

for $t \ge t_1$. Hence from (1.15) it follows that $\Gamma_{(t_0,x,y)}$ is rectifiable.

Next we consider case (iii). By (4.7) and (4.11), we obtain

$$\|(x'(t),y'(t))\| \ge \frac{\sqrt{[\alpha_1(t)+\beta_1(t)(\rho_2(t;c_0))^{-1}]^2+(\max\{\gamma_1(t),-\gamma_2(t),0\})^2}}{(\rho_2(t;c_0))^{\frac{1}{2\lambda}}}$$

for $t \ge t_1$. Integrating this inequality and using (1.16), we conclude that $\Gamma_{(t_0,x,y)}$ is nonrectifiable. This completes the proof of Theorem 1.7. □

Using Theorems 1.3 and 1.7, and Lemma 3.5, we can establish the following result.

Theorem 4.4. *Let $(x(t),y(t))$ be any nontrivial solution of (1.1). Suppose that (1.6) and (3.3) hold. Then the zero solution of (1.1) is globally exponentially stable, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t),y(t))$ is simple, and (i), (ii) and (iii) below hold:*

(i) if (1.9) holds, then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(ii) if (1.15) holds, then the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable;

(iii) if (1.16) holds, then the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable.

Corollary 1.9 and Lemma 3.5 imply the following.

Corollary 4.5. *Let $(x(t),y(t))$ be any nontrivial solution of (1.18). Suppose that there exists an $\underline{e} > 0$ such that*

$$e(t) \ge \underline{e} \quad \text{for} \quad t \ge t_0. \tag{4.12}$$

Then the zero solution of (1.18) is exponentially stable, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t),y(t))$ is simple, and the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable if and only if (1.21) holds.

5 Examples and numerical simulations

In this section we will present some examples and numerical simulations.

Example 5.1. Let $\lambda = 0.5$. Consider the two-dimensional nonautonomous differential system (1.1) with

$$e(t) = h(t) = \frac{1}{t}, \quad f(t) = g(t) = \frac{10\cos t}{t} \quad \text{and} \quad p(t) = q(t) = t. \tag{5.1}$$

Then

$$\alpha_1(t) = \alpha_2(t) = e(t) = \frac{1}{t}, \quad \beta_1(t) = \beta_2(t) = p(t) = t \quad \text{and} \quad \gamma_1(t) = \gamma_2(t) = -f(t) = -\frac{10\cos t}{t}.$$

Hence, assumptions (1.6), (1.7) and (1.8) are easily satisfied. Moreover,

$$\alpha_1(t) = \frac{1}{t} > 0 \quad \text{for} \quad t \ge 1,$$

and
$$\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\alpha_1(t)} = 10|\cos t| \le 10 \quad \text{for } \ t \ge 1.$$

By Theorem 1.3 (i), we conclude that the zero solution of (1.1) with (5.1) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ is simple and rectifiable. Fig. 5.1 shows the orbit $\Gamma_{(1,x,y)}$ corresponding to the nontrivial solution $(x(t), y(t))$ of (1.1) with (5.1) and $(x(1), y(1)) = (0.9, 0)$.

Example 5.2. Let $\lambda = 0.1$. Consider the two-dimensional nonautonomous differential system (1.1) with

$$e(t) = h(t) = 0, \quad f(t) = g(t) = \frac{1}{2} + \frac{\cos t}{t} \quad \text{and} \quad p(t) = q(t) = 0.1. \tag{5.2}$$

Then

$$\alpha_1(t) = \alpha_2(t) = 0, \quad \beta_1(t) = \beta_2(t) = 0.1 \quad \text{and} \quad \gamma_1(t) = \gamma_2(t) = -\frac{1}{2} - \frac{\cos t}{t}.$$

Hence, assumptions (1.6), (1.7) and (1.8) are easily satisfied. Moreover,

$$\frac{\max\{|\gamma_1(t)|, |\gamma_2(t)|\}}{\beta_1(t)} = 10\left(\frac{1}{2} + \frac{\cos t}{t}\right) \le 15 \quad \text{for } \ t \ge 1.$$

By Corollary 1.4 (ii), we conclude that the zero solution of (1.1) with (5.2) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ is simple and rectifiable. Fig. 5.2 shows the orbit $\Gamma_{(1,x,y)}$ corresponding to the nontrivial solution $(x(t), y(t))$ of (1.1) with (5.2) and $(x(1), y(1)) = (0.9, 0)$.

Example 5.3. Let $\lambda = 0.5$. Consider the two-dimensional nonautonomous differential system (1.1) with (5.2). Then

$$\frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\beta_2(t)} \ge \frac{-\gamma_2(t)}{\beta_2(t)} = 10\left(\frac{1}{2} + \frac{\cos t}{t}\right) > \frac{5}{2} \quad \text{for } \ t \ge 4.$$

By Corollary 1.4 (iii), the zero solution of (1.1) with (5.2) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ is simple and nonrectifiable. Fig. 5.3 shows the orbit $\Gamma_{(1,x,y)}$ corresponding to the nontrivial solution $(x(t), y(t))$ of (1.1) with (5.2) and $(x(1), y(1)) = (0.9, 0)$.

Example 5.4. Let $\lambda = 0.5$. Consider the two-dimensional nonautonomous differential system (1.1) with

$$e(t) = h(t) = \frac{1}{t}, \quad f(t) = g(t) = 2 + \cos t \quad \text{and} \quad p(t) = q(t) = \frac{1}{t^2}. \tag{5.3}$$

Then

$$\alpha_1(t) = \alpha_2(t) = \frac{1}{t}, \quad \beta_1(t) = \beta_2(t) = \frac{1}{t^2} \quad \text{and} \quad \gamma_1(t) = \gamma_2(t) = -2 - \cos t.$$

Hence, assumptions (1.6), (1.7) and (1.8) are easily satisfied. Since

$$\exp\left(2\lambda \int_{t_0}^{t} \alpha_2(s)ds\right) = \exp\left(\log \frac{t}{t_0}\right) = \frac{t}{t_0}$$

for $t \ge t_0$, we have

$$\rho_2(t; c) = \frac{t}{t_0}\left(c + t_0 \int_{t_0}^{t} s^{-3} ds\right) = t\left(\frac{c}{t_0} + \frac{1}{2t_0^2} - \frac{1}{2t^2}\right),$$

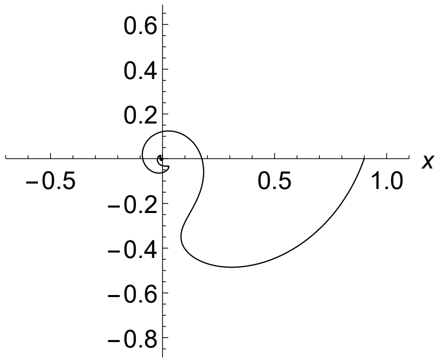

Figure 5.1: Example 5.1; Theorem 1.3 (i); rectifiable.

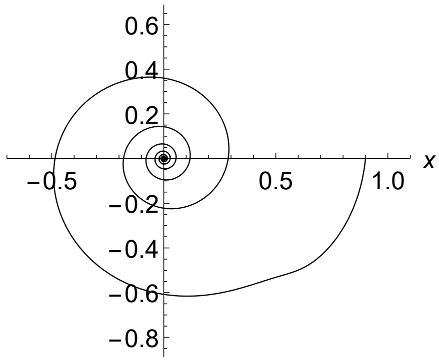

Figure 5.2: Example 5.2; Corollary 1.4 (ii); rectifiable.

and hence

$$\frac{\max\{\gamma_1(t), -\gamma_2(t), 0\}}{\alpha_2(t)\rho_2(t;c) + \beta_2(t)} \geq \frac{2+\cos t}{\frac{c}{t_0} + \frac{1}{2t_0^2} + \frac{1}{2t^2}} \geq \frac{1}{\frac{c}{t_0} + \frac{1}{t_0^2}}$$

for $t \geq t_0$. Hence (1.11) is satisfied. By Theorem 1.3 (iii), the zero solution of (1.1) with (5.3) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ is simple and nonrectifiable. Fig. 5.4 shows the orbit $\Gamma_{(1,x,y)}$ corresponding to the nontrivial solution $(x(t), y(t))$ of (1.1) with (5.3) and $(x(1), y(1)) = (0.9, 0)$.

Example 5.5. Consider the two-dimensional nonautonomous linear system (1.18) with

$$e(t) = 1 \quad \text{and} \quad f(t) = e^t. \tag{5.4}$$

Then assumption (4.12) is easily satisfied. It is clear that

$$\int_{t_0}^{t} \sqrt{e^2(s) + f^2(s)} \exp\left(-\int_{t_0}^{s} e(\tau)d\tau\right) ds \geq \int_{t_0}^{t} e^s e^{-s+t_0} ds = e^{t_0}(t - t_0)$$

for all $t \geq t_0$. Hence, by Corollary 4.5 we conclude that the zero solution of (1.18) with (5.4) is exponentially stable, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple and nonrectifiable. Fig. 5.5 shows the orbit $\Gamma_{(1,x,y)}$ corresponding to the nontrivial solution $(x(t), y(t))$ of (1.18) with (5.4) and $(x(1), y(1)) = (0.9, 0)$.

Example 5.6. Consider the two-dimensional nonautonomous linear system (1.22), where $\sigma \in \mathbf{R}$. Then assumptions (1.19) and (1.20) are easily satisfied. By Corollary 1.9 we conclude that the zero solution of (1.22) is globally attractive, the orbit $\Gamma_{(t_0,x,y)}$ corresponding to $(x(t), y(t))$ is simple. Moreover, the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable if and only if

$$\lim_{t\to\infty} \int_{t_0}^{t} \sqrt{e^2(s) + f^2(s)} \exp\left(-\int_{t_0}^{s} e(\tau)d\tau\right) ds < \infty.$$

Let

$$\omega(t) := \sqrt{e^2(t) + f^2(t)} \exp\left(-\int_{t_0}^{t} e(s)ds\right)$$

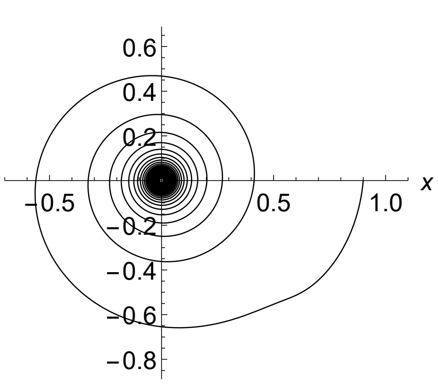

Figure 5.3: Example 5.3; Corollary 1.4 (iii); nonrectifiable.

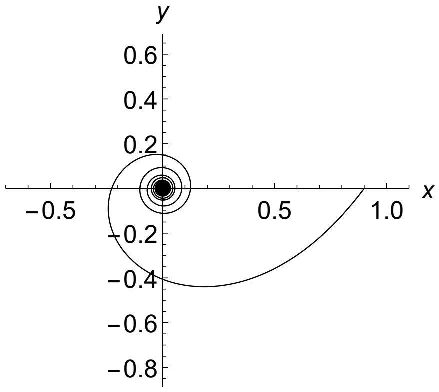

Figure 5.4: Example 5.4; Theorem 1.3 (iii); nonrectifiable.

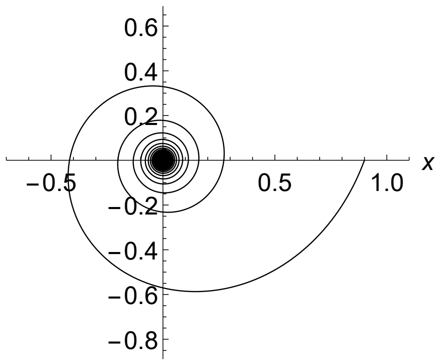

Figure 5.5: Example 5.5; Corollary 4.5; exponentially stable; nonrectifiable.

for all $t \geq 1$. Then we have

$$\omega(t) = t^{-1}\sqrt{t^{-2} + t^{2\sigma}} \tag{5.5}$$

holds for all $t \geq 1$. We will consider the three cases (i) $\sigma \leq -1$, (ii) $-1 < \sigma < 0$ and (iii) $\sigma \geq 0$.

Case (i). Using (5.5), we get

$$\int_1^t \omega(s)ds = \int_1^t s^{-2}\sqrt{1 + s^{2(\sigma+1)}}ds \leq \sqrt{2}\int_1^t s^{-2}ds = -\sqrt{2}(t^{-1} - 1) < \sqrt{2}$$

for all $t \geq 1$. By Theorem 1.9 we see that the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable.

Case (ii). From (5.5), we have

$$\int_1^t \omega(s)ds = \int_1^t s^{\sigma-1}\sqrt{s^{-2(\sigma+1)} + 1}ds \leq \sqrt{2}\int_1^t s^{\sigma-1}ds = \frac{\sqrt{2}}{\sigma}(t^\sigma - 1) < \frac{\sqrt{2}}{-\sigma}$$

for all $t \geq 1$. By Corollary 1.9 we see that the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable.

Case (iii). Using (5.5), we get

$$\int_1^t \omega(s)ds \geq \int_1^t s^{\sigma-1}ds \geq \int_1^t s^{-1}ds = \log t$$

for all $t \geq t_0$. By Corollary 1.9 we see that the orbit $\Gamma_{(t_0,x,y)}$ is nonrectifiable. Consequently, we can conclude that the orbit $\Gamma_{(t_0,x,y)}$ is rectifiable if and only if $\sigma < 0$.

Acknowledgements

Masakazu Onitsuka was supported by JSPS KAKENHI Grant Number 20K03668. Satoshi Tanaka was supported by JSPS KAKENHI Grant Number 19K03595. The authors would like to thank the referee for his/her careful reading and comments.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **19**, 1–3; https://doi.org/10.14232/ejqtde.2021.1.19 www.math.u-szeged.hu/ejqtde/

Corrigendum to “Topological entropy for impulsive differential equations” [*Electron. J. Qual. Theory Differ. Equ.* 2020, No. 68, 1–15]

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Received 2 March 2021, appeared 25 March 2021
Communicated by Gennaro Infante

Abstract. The aim of this corrigendum is two-fold: (i) to indicate the incorrect parts in two propositions of our recent paper with the same title, (ii) to state the correct statements.

Keywords: topological entropy, impulsive differential equations, entropy of composition, Ivanov’s inequality.

2020 Mathematics Subject Classification: Primary 34B37, 34C28, 37B40; Secondary 34C40, 37D45.

1 Incorrect propositions, their consequences and corrections

The vector impulsive differential equation under our consideration in [1] takes the form

$$\begin{cases} x' = F(t,x), \ t \neq t_j := j\omega, \ \text{for some given } \omega > 0, \\ x(t_j^+) = I(x(t_j^-)), \ j \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where $F\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the Carathéodory mapping such that $F(t,x) \equiv F(t+\omega, x)$, equation $x' = F(t,x)$ satisfies a uniqueness condition and a global existence of all its solutions on $(-\infty,\infty)$. Let, furthermore, $I\colon \mathbb{R}^n \to \mathbb{R}^n$ be a compact continuous impulsive mapping such that $K_0 := \overline{I(\mathbb{R}^n)}$ and $I(K_0) = K_0$.

Unfortunately, there is a gap in the second part of the proof of the following proposition.

Proposition 1.1 (cf. [1, Proposition 3.1]). *Let $T_\omega\colon \mathbb{R}^n \to \mathbb{R}^n$ be the associated Poincaré translation operator along the trajectories of $x' = F(t,x)$, such that $K_1 := T_\omega(K_0)$ and $K_0 \subset K_1$. Then the equality*

$$h\left(I\big|_{K_1} \circ T_\omega\big|_{K_0} \right) = h\left(I\big|_{K_0} \right) \tag{1.2}$$

holds for the topological entropies h of the maps $I\big|_{K_1} \circ T_\omega\big|_{K_0}\colon K_0 \to K_0$ and $I\big|_{K_0}\colon K_0 \to K_0$.

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Since the equality (1.2) was used in the proof of the first main theorem (see [1, Theorem 3.5]), this theorem can be corrected in the simplest way, when assuming (1.2) or, more generally the inequality

$$h\left(T_\omega|_{K_0} \circ I|_{K_1}\right) \geq h\left(I|_{K_0}\right), \tag{1.3}$$

explicitly. Then the following correction has rather a character of a proposition.

Theorem 1.2. *The vector impulsive differential equation* (1.1) *exhibits under* (1.3) *chaos in the sense of a positive topological entropy of the composition* $I|_{K_1} \circ T_\omega|_{K_0}$, *i.e.* $h(I|_{K_1} \circ T_\omega|_{K_0}) > 0$, *provided* $I(K_0) = K_0$ *and* $K_0 \subset K_1$, *where* $K_0 := \overline{I(\mathbb{R}^n)}$ *and* $K_1 := T_\omega(K_0)$, *jointly with* $h(I|_{K_0}) > 0$.

Despite this gap, all the related illustrative examples (see [1, Examples 3.7–3.9]) can be shown to be correct, when verifying (1.3), by means of e.g. a slightly generalized version of [2, Proposition 3.2].

The same type of a gap is in the proposition for the problem (1.1) considered, under the natural additional assumptions

$$F(t, \dots, x_j, \dots) \equiv F(t, \dots, x_{j+1}, \dots), \quad j = 1, \dots, n, \tag{1.4}$$

and

$$I(\dots, x_j, \dots) \equiv I(\dots, x_{j+1}, \dots) \pmod 1, \quad j = 1, \dots, n, \tag{1.5}$$

on the torus $\mathbb{R}^n/\mathbb{Z}^n$ (see [1, Proposition 4.1]). Quite analogously, the second main theorem (see [1, Theorem 4.3]) can be corrected by the additional technical assumption

$$h\left((\tau \circ T_\omega) \circ (\tau \circ I)\right) \geq h(\tau \circ I), \tag{1.6}$$

where $\tau\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ denotes the natural projection.

Since on tori, we have to our disposal the Ivanov inequality for the lower estimate of topological entropy in terms of the asymptotic Nielsen numbers (see [4] and cf. [1, Proposition 2.7]), the third main theorem in [1, Theorem 4.6] remains valid, even without verifying (1.6), in the following way.

Theorem 1.3. *Consider, under the above assumptions and* (1.4), (1.5), *the vector impulsive differential equation* (1.1) *on* $\mathbb{R}^n/\mathbb{Z}^n$. *Assume that the impulsive mapping* $(\tau \circ I)\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ *is homotopic to a continuous map* $f\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ *such that* $N^\infty(f) > 1$, *i.e.*

$$\limsup_{m\to\infty} |\lambda(f^m)|^{\frac{1}{m}} > 1,$$

where $\lambda(f^m)$ *stands for the Lefschetz number of the m-th iterate of* f.

Then

$$\begin{aligned} h\left((\tau \circ I) \circ (\tau \circ T_\omega)\right) &\geq \limsup_{m\to\infty} \frac{1}{m} \log N\Big(\left((\tau \circ I) \circ (\tau \circ T_\omega)\right)^m\Big) \\ &= \limsup_{m\to\infty} \frac{1}{m} \log N\left((\tau \circ I)^m\right) = \limsup_{m\to\infty} \frac{1}{m} \log N\left(f^m\right) > 0 \end{aligned}$$

holds, where $N(f^m)$ *denotes the Nielsen number of the m-th iterate of* f, *and subsequently equation* (1.1) *exhibits on* $\mathbb{R}^n/\mathbb{Z}^n$ *chaos in the sense of a positive topological entropy of the composition* $(\tau \circ I) \circ (\tau \circ T_\omega)$.

That is also why that all the related illustrative examples (see [1, Examples 4.5, 4.7, 4.9]) remain on this basis correct.

2 Concluding remarks

To verify the inequalities (1.3) and (1.6) is not an easy task (see e.g. [3]). We will try to affirm them at least in some particular cases elsewhere. In $\mathbb{R}$, the most promising way seems to be via the statements along the lines of [2, Proposition 3.2].

Acknowledgements

The author was supported by the Grant IGA_PrF_2020_015 "Mathematical Models" of the Internal Grant Agency of Palacký University in Olomouc.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 20, 1–17; https://doi.org/10.14232/ejqtde.2021.1.20 www.math.u-szeged.hu/ejqtde/

A sequence of positive solutions for sixth-order ordinary nonlinear differential problems

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Received 27 August 2020, appeared 25 March 2021
Communicated by Tibor Krisztin

In honor of Jeff Webb, great master of higher order ordinary differential equations, on the occasion of his seventy-fifth birthday. To Jeff with infinite admiration.

Abstract. Infinitely many solutions for a nonlinear sixth-order differential equation are obtained. The variational methods are adopted and an oscillating behaviour on the nonlinear term is required, avoiding any symmetry assumption.

Keywords: sixth-order equations, critical points, infinitely many solutions.

2020 Mathematics Subject Classification: 34B15, 34B18, 35B38.

1 Introduction

Equations of the following type

$$\frac{\partial u}{\partial t} = -\frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} - B\frac{\partial^2 u}{\partial x^2} + Cu - \lambda h(t, x, u(t, x)) \tag{1.1}$$

arise when an interface between two phases is examined because they help to reveal a more detailed structure of the interface and a description of the behaviour of phase fronts in materials that are undergoing a transition between the liquid and solid state [1,7,13]. Here we are concerned in periodic stationary solutions of (1.1). More precisely, we will give some multiple results for the following problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda f(x, u), & x \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{P_λ}$$

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where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, A, B and C are given real constants, while $\lambda > 0$.

In [13], starting from the interest for the stationary solutions of a class of fourth-order equations, the so-called extended Fisher–Kolmogorov equation, a variational approach is proposed for obtaining existence and non existence of stationary periodic solutions, observing that the same arguments apply also to sixth-order equations. In [6,16], taking advantage from a minimization theorem as well as Clark's theorem, the existence and the multiplicity of periodic solutions is investigated for a problem similar to (P_1), provided that A, B and C satisfy some suitable relations and the nonlinear term is a polynomial with a kind of symmetry. Again the variational methods have been exploited in [9] where two Brezis–Nirenberg linking theorems represent the main tool for assuring the existence of at least two or three periodic solutions for a sixth-order equation with super-quadratic nonlinearities, namely

$$\lim_{t\to+\infty} \frac{F(x,t)}{t^2} = +\infty, \qquad \lim_{t\to 0} \frac{F(x,t)}{t^2} = 0$$

uniformly with respect to x, where $F(x,t) = \int_0^t f(x,s)\, ds$ for every $x \in [0,1]$. We also cite [11, 21], where under suitable assumptions, in particular on the coefficients A, B, C, the existence of one or two positive solutions for problem (P_λ) is established by applying the theory of fixed point index in cones. Further nice results on higher-order differential equations are contained in [8,17–20], where non-local conditions have also been considered.

In this note we look at the existence of infinitely many solutions to problem (P_λ). In particular, under different assumptions on the parameters A, B and C and requiring a suitable oscillation on $f(x,\cdot)$ at infinity (see assumption ii) of Theorem 3.2), an unbounded sequence of classical solutions of (P_λ) is assured provided that λ belongs to a well determined interval. We explicitly stress that no symmetry conditions on the reaction term are involved. The variational structure of the problem is exploited and the solutions are obtained as local minima of the energy functional related to (P_λ). For this reason a crucial tool is a local minimum theorem proved in [2], see Theorem 2.8.

A further investigation is devoted to constant sign solutions of (P_λ). Whenever f is nonnegative the classical solutions are assured to be positive provided that suitable conditions on the coefficients are assumed (see Remark 3.3) owing to a strong maximum principle for sixth-order differential equations pointed out in Remark 3.4.

As example, here is a consequence of our main results.

Theorem 1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\liminf_{t\to+\infty} \frac{G(t)}{t^2} = 0, \qquad \limsup_{t\to+\infty} \frac{G(t)}{t^2} = +\infty,$$

where $G(t) = \int_0^t g(s)\, ds$ for every $t \in \mathbb{R}$, and fix $D \geq 0$.

Then, the problem

$$\begin{cases} -u^{(vi)} + 3Du^{(iv)} - 3D^2u'' + D^3u = g(u), & x \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{P}$$

admits an unbounded sequence of positive classical solutions.

Finally, when the oscillating behaviour is required at zero, instead that at infinity, a sequence of classical solutions that strong converges at zero is obtained (see Theorem 3.11).

In Section 2 we recall some useful preliminaries and detail the variational set pointing out the general strategy for obtaining classical solutions. The main results as well as their consequences and examples are contained in Section 3.

2 Basic notations and auxiliary results

Throughout the paper X denote the following Sobolev subspace of $H^3(0,1) \cap H^1_0(0,1)$

$$X = \{u \in H^3(0,1) \cap H^1_0(0,1) : \; u''(0) = u''(1) = 0\}$$

considered with the norm

$$\|u\| = \left(\|u'''\|_2^2 + \|u''\|_2^2 + \|u'\|_2^2 + \|u\|_2^2\right)^{1/2}, \qquad \forall\, u \in X, \tag{2.1}$$

where $\|\cdot\|_2$ denotes the usual norm in $L^2(0,1)$. It is well known that $\|\cdot\|$ is induced by the inner product

$$\langle u, v\rangle = \int_0^1 \left(u'''(x)v'''(x) + u''(x)v''(x) + u'(x)v'(x) + u(x)v(x)\right) dx, \qquad \forall\, u, v \in X.$$

Now, arguing as in [13], we point out some useful Poincaré type inequalities.

Proposition 2.1. *For every $u \in X$, if $k = 1/\pi^2$, one has*

$$\|u^{(i)}\|_2^2 \le k^{j-i}\|u^{(j)}\|_2^2, \qquad i = 0,1,2,\; j = 1,2,3 \text{ with } i < j. \tag{2.2}$$

Proof. Let us consider all the possible situations.

$j = 1$. In this case only $i = 0$ occurs and (2.2) reduces to the well known Poincaré inequality.

$j = 2$. The case $i = 1$ can be obtained observing that $\int_0^1 (u')^2 = -\int_0^1 uu''$. Hence, putting together the Hölder and the Poincaré inequalities one has

$$\|u'\|_2^2 \le \|u\|_2\|u''\|_2 \le k^{1/2}\|u'\|_2\|u''\|_2$$

from which directly follows (2.2).

For $i = 0$ condition (2.2) is derived putting together the Poincaré inequality with the case $i = 1$.

$j = 3$. The case $i = 2$ is directly the Poincaré inequality applied to $u'' \in H^1_0(0,1)$.

For $i = 1$, arguing as above one has

$$\|u'\|_2^2 \le \|u\|_2\|u''\|_2 \le k^{1/2}\|u'\|_2 k^{1/2}\|u'''\|_2 = k\|u'\|_2\|u'''\|_2,$$

where (2.2) for $i = 2$ has been also exploited. Hence, (2.2) is verified for $i = 1$.

Finally, for $i = 0$ the conclusion is achieved putting together the Poincaré inequality and using the case $i = 1$, indeed

$$\|u\|_2^2 \le k\|u'\|_2^2 \le k^3\|u'''\|_2^2. \qquad \square$$

Remark 2.2. The constants in (2.2) are the best ones as one can verify considering the function $\sin \pi x$ that realizes the equalities. Moreover, it is worth noting as follows. Indeed, we recall that, in general, one has

$$\|v\|_2^2 \leq 4k\|v'\|_2^2 \tag{2.3}$$

for all $v \in H^1([0,1])$ for which there is $c \in [0,1]$ such that $v(c) = 0$, and the equality for appropriate functions v also holds (see for instance [10, page 182]). So, if we apply the classical Poincaré inequality (2.3) to $v = u'$, then we obtain

$$\|u'\|_2^2 \leq 4k\|u''\|_2^2,$$

that, as shows (2.2), does not realize the best constant, on the contrary of (2.3). Clearly, this is due because in our case we have a greater regularity of u' (since $u \in X$).

We will introduce a convenient norm, equivalent to $\|\cdot\|$, that still makes X a Hilbert space. For this reason, for $A, B, C \in \mathbb{R}$ let us define the function $N : X \to \mathbb{R}$ by putting

$$N(u) = \|u'''\|_2^2 + A\|u''\|_2^2 + B\|u'\|_2^2 + C\|u\|_2^2$$

for every $u \in X$.

Now consider the following set of conditions according to the signs of the constants A, B and C:

$(H)_1$ $A \geq 0$, $B \geq 0$, $C \geq 0$;

$(H)_2$ $A \geq 0$, $B \geq 0$, $C < 0$ *and* $-Ak - Bk^2 - Ck^3 < 1$;

$(H)_3$ $A \geq 0$, $B < 0$, $C \geq 0$ *and* $-Ak - Bk^2 < 1$;

$(H)_4$ $A \geq 0$, $B < 0$, $C < 0$ *and* $-Ak - Bk^2 - Ck^3 < 1$;

$(H)_5$ $A < 0$, $B \geq 0$, $C \geq 0$ *and* $-Ak < 1$;

$(H)_6$ $A < 0$, $B \geq 0$, $C < 0$ *and* $\max\{-Ak,\ -Ak - Bk^2 - Ck^3\} < 1$;

$(H)_7$ $A < 0$, $B < 0$, $C \geq 0$ *and* $-Ak - Bk^2 < 1$;

$(H)_8$ $A < 0$, $B < 0$, $C < 0$ *and* $-Ak - Bk^2 - Ck^3 < 1$.

Moreover, fix $A, B, C \in \mathbb{R}$ and consider the following condition:

(H) $\max\{-Ak,\ -Ak - Bk^2,\ -Ak - Bk^2 - Ck^3\} < 1$.

We have the following result.

Proposition 2.3. *Condition (H) holds if and only if one of conditions $(H)_1$–$(H)_8$ holds.*

Proof. Assume (H). Clearly, according to the signs of the constants A, B, C, one of conditions $(H)_1$–$(H)_8$ is immediately verified. On the contrary, assuming one of conditions $(H)_1$–$(H)_8$, then a direct computation shows that (H) is verified. As an example, assume at first $(H)_5$ and next $(H)_8$. In the first of such cases, since $B \geq 0$ and $C \geq 0$, one has $-Ak - Bk^2 \leq -Ak$ and $-Ak - Bk^2 - Ck^3 \leq -Ak$, for which $\max\{-Ak,\ -Ak - Bk^2,\ -Ak - Bk^2 - Ck^3\} \leq Ak < 1$, that is, (H) holds. In the other case, we have that the sum of three positive addends is less than 1, that is, $0 < -Ak - Bk^2 - Ck^3 < 1$. If, arguing by a contradiction, either $-Ak \geq 1$ or $-Ak - Bk^2 \geq 1$, then $-Ak - Bk^2 - Ck^3 \geq 1$ and this is absurd. So, $-Ak < 1$, $-Ak - Bk^2 < 1$ and $-Ak - Bk^2 - Ck^3 < 1$, for which (H) is satisfied. $\square$

Proposition 2.4. *Assume (H). Then, there exits $m > 0$ such that*

$$N(u) \geq m\|u\|^2, \qquad \forall\, u \in X. \tag{2.4}$$

Proof. Fix $u \in X$ and distinguish the different cases, taking Proposition 2.3 into account.

Assume $(H)_1$.

Then, in view of (2.2) one has

$$N(u) \geq \|u'''\|_2^2 \geq \frac{1}{4}\left(\|u'''\|_2^2 + \frac{1}{k}\|u''\|_2^2 + \frac{1}{k^2}\|u'\|_2^2 + \frac{1}{k^3}\|u\|_2^2\right) \geq \frac{1}{4}\|u\|^2 \tag{2.5}$$

and (2.4) holds with $m = \frac{1}{4}$.

Assume $(H)_2$.

Then, in view of (2.2)

$$N(u) \geq \|u'''\|_2^2 + A\|u''\|_2^2 + (B + Ck)\,\|u'\|_2^2.$$

Hence, if $B + Ck \geq 0$ we can argue as in (2.5) and (2.4) holds with $m = \frac{1}{4}$. Otherwise, again from (2.2)

$$N(u) \geq \|u'''\|_2^2 + (A + Bk + Ck^2)\|u''\|_2^2.$$

So, if $A + Bk + Ck^2 \geq 0$ we can argue as in (2.5) and conclude that (2.4) holds with $m = \frac{1}{4}$. Conversely, always from (2.2) one obtains

$$N(u) \geq (1 + Ak + Bk^2 + Ck^3)\|u'''\|_2^2$$

and assumption $(H)_2$, combined with the same above arguments, leads to (2.4) with $m = \frac{1+Ak+Bk^2+Ck^3}{4}$. Summarizing, (2.4) holds with $m = \min\left\{\frac{1}{4}, \frac{1+Ak+Bk^2+Ck^3}{4}\right\}$.

Assume $(H)_3$.

Then, in view of (2.2) one has

$$N(u) \geq \|u'''\|_2^2 + (A + Bk)\|u''\|_2^2.$$

If $A + Bk \geq 0$, following the reasoning as in (2.5) we conclude that (2.4) holds with $m = \frac{1}{4}$. Otherwise, (2.2) implies

$$N(u) \geq (1 + Ak + Bk^2)\|u'''\|_2^2.$$

and assumption $(H)_3$ implies that (2.4) holds with $m = \frac{1+Ak+Bk^2}{4}$.

Summarizing, (2.4) holds with $m = \min\left\{\frac{1}{4}, \frac{1+Ak+Bk^2}{4}\right\}$.

Assume $(H)_4$.

Then, from (2.2) one has

$$N(u) \geq \|u'''\|_2^2 + (A + Bk + Ck^2)\|u''\|_2^2.$$

If $A + Bk + Ck^2 \geq 0$ we conclude choosing $m = \frac{1}{4}$. Otherwise, with the same technique,

$$N(u) \geq (1 + Ak + Bk^2 + Ck^3)\|u'''\|_2^2$$

and we can complete also this case, pointing out that $m = \min\left\{\frac{1}{4}, \frac{1+Ak+Bk^2+Ck^3}{4}\right\}$.

Assume $(H)_5$.

Then, from (2.2) one has

$$N(u) \geq (1 + Ak)\|u'''\|_2^2$$

and (2.4) holds with $m = \frac{1+Ak}{4}$.

Assume $(H)_6$.

Then, from (2.2) one has

$$N(u) \geq (1 + Ak)\|u'''\| + (B + Ck)\|u'\|_2.$$

If $B + Ck \geq 0$ then assumption $(H)_6$ implies (2.4) with $m = \frac{1+Ak}{4}$. Otherwise,

$$N(u) \geq (1 + Ak + Bk^2 + Ck^3)\|u'''\|_2^2$$

we can conclude again but with $m = \frac{1+Ak+Bk^2+Ck^3}{4}$. Summarizing, in this case one has $m = \min\left\{\frac{1+Ak}{4}, \frac{1+Ak+Bk^2+Ck^3}{4}\right\}$.

Assume $(H)_7$.

Then, from (2.2) one has

$$N(u) \geq (1 + Ak + Bk^2)\|u'''\|_2^2$$

and (2.4) holds with $m = \frac{1+Ak+Bk^2}{4}$.

Assume $(H)_8$.

Then, from (2.2) one has

$$N(u) \geq (1 + Ak + Bk^2 + Ck^3)\|u'''\|_2^2$$

and (2.4) holds with $m = \frac{1+Ak+Bk^2+Ck^3}{4}$. □

From the above considerations one can derive the following

Proposition 2.5. *Assume that* (H) *holds and put*

$$\|u\|_X = \sqrt{N(u)}, \qquad \forall\, u \in X. \tag{2.6}$$

Then, $\|\cdot\|_X$ *is a norm equivalent to the usual one defined in* (2.1) *and* $(X, \|\cdot\|_X)$ *is a Hilbert space.*

Proof. The definition of N and Proposition 2.4 assure that $\|\cdot\|_X$ is the norm induced by the inner product

$$\langle\cdot,\cdot\rangle_X = \int_0^1 (u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x))\, dx, \qquad \forall\, u, v \in X.$$

It is simple to observe that there exists $M > 0$ such that

$$\|u\|_X^2 \leq M\|u\|^2 \tag{2.7}$$

for every $u \in X$. Hence, the equivalence is an immediate consequence of (2.4) and (2.7) and the proof is complete. □

Clearly $(X, \|\cdot\|_X) \hookrightarrow (C^0(0,1), \|\cdot\|_\infty)$ and the embedding is compact. For a qualitative estimate of the constant of this embedding it is useful to introduce the following number

$$\delta = \begin{cases} 1, & \text{if } (H)_1 \text{ holds,} \\ \min\{1, 1 + Ak + Bk^2 + Ck^3\}, & \text{if } (H)_2 \text{ or } (H)_4 \text{ holds,} \\ \min\{1, 1 + Ak + Bk^2\}, & \text{if } (H)_3 \text{ holds,} \\ 1 + Ak, & \text{if } (H)_5 \text{ holds,} \\ \min\{1 + Ak, 1 + Ak + Bk^2\}, & \text{if } (H)_6 \text{ holds,} \\ 1 + Ak + Bk^2, & \text{if } (H)_7 \text{ holds,} \\ 1 + Ak + Bk^2 + Ck^3, & \text{if } (H)_8 \text{ holds.} \end{cases} \tag{2.8}$$

We explicitly observe that the proof of Proposition 2.4 shows in addition that

$$\|u\|_X^2 \geq \delta \|u'''\|_2^2 \tag{2.9}$$

for every $u \in X$, and $\delta = 4m$, where m is the number assured from the same Proposition 2.4.

Proposition 2.6. *Assume that (H) holds. One has*

$$\|u\|_\infty \leq \frac{k}{2\sqrt{\delta}} \|u\|_X \tag{2.10}$$

for every $u \in X$, where δ is given in (2.8)*.*

Proof. It is well known that $H_0^1(0,1) \hookrightarrow C^0(0,1)$ and $\|u\|_\infty \leq \frac{1}{2}\|u'\|_2$, thus, taking in mind (2.2),

$$\|u\|_\infty \leq \frac{k}{2} \|u'''\|_2. \tag{2.11}$$

Moreover, from (2.9) one has

$$\|u'''\|_2 \leq \frac{1}{\sqrt{\delta}} \|u\|_X$$

and (2.10) holds, in view of (2.11). □

In order to clarify the variational structure of problem (P_λ), we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by putting

$$\Phi(u) = \frac{1}{2}\|u\|_X^2, \qquad \Psi(u) = \int_0^1 F(x, u(x))\, dx, \qquad \forall\, u \in X, \tag{2.12}$$

where $F(x,t) = \int_0^t f(x,s)\, ds$ for every $(x,t) \in [0,1] \times \mathbb{R}$.

With standard arguments one can verify that Φ and Ψ are continuously Gâteaux differentiable, being in particular

$$\Phi'(u)(v) = \int_0^1 \left(u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x)\right)\, dx$$

and

$$\Psi'(u)(v) = \int_0^1 f(x, u(x))v(x)\, dx$$

for every $u, v \in X$.

Recall that a weak solution of problem (P_λ) is any $u \in X$ such that

$$\int_0^1 \left(u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x)\right)\, dx$$
$$= \lambda \int_0^1 f(x, u(x))v(x)\, dx, \qquad \forall v \in X. \quad (2.13)$$

Hence, the weak solutions of (P_λ) are exactly the critical points of the functional $\Phi - \lambda\Psi$.

Proposition 2.7. *Every weak solution of* (P_λ) *is also a classical solution.*

Proof. Let $u \in X$ be a weak solution of (P_λ). Then, since

$$A \int_0^1 u''(x)v''(x)\, dx = -A \int_0^1 u'(x)v'''(x)\, dx$$

and

$$B \int_0^1 u'(x)v'(x)\, dx = -B \int_0^1 u''(x)v(x)\, dx,$$

one can observe that

$$\int_0^1 \left(u'''(x) - Au'(x)\right) v'''(x)\, dx = \int_0^1 \left(Bu''(x) - Cu(x) + \lambda f(x, u(x))\right) v(x)\, dx$$

for every $v \in X$. Hence, $u''' - Au' \in H^3(0, 1)$ and

$$\left(u''' - Au'\right)''' = -Bu'' + Cu - \lambda f(x, u). \quad (2.14)$$

The continuity of f and the embedding $X \hookrightarrow C^2(0, 1)$ imply that $u''' - Au' \in C^3(0, 1)$. Thus, since

$$u''' = u''' - Au' + Au' \quad (2.15)$$

it is clear that $u \in C^4(0, 1)$, namely $u' \in C^3(0, 1)$ and (2.15) leads to $u \in C^6(0, 1)$. From (2.14) one obtains

$$-u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda f(x, u). \quad (2.16)$$

At this point, integrating by parts (2.13) and exploiting (2.16) one has

$$\left[-u^{(iv)}(x)v'(x)\right]_0^1 = 0$$

for every $v \in X$, thus $u^{(iv)}(0) = u^{(iv)}(1) = 0$ and the proof is complete. □

The main tool in our approach is the following critical point theorem (see [2, Theorem 7.4])

Theorem 2.8. *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below. Put*

$$\gamma = \liminf_{r\to+\infty} \varphi(r), \qquad \chi = \liminf_{r\to(\inf_X \Phi)^+} \varphi(r),$$

where

$$\varphi(r) = \inf_{\Phi(v)<r} \frac{\sup_{\Phi(u)<r} \Psi(u) - \Psi(v)}{r - \Phi(v)} \qquad \left(r > \inf_X \Phi\right).$$

(a) *If $\gamma < +\infty$ and for each $\lambda \in \left]0, \frac{1}{\gamma}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$-condition for all $r \in \mathbb{R}$ then, for each $\lambda \in \left]0, \frac{1}{\gamma}\right[$, the following alternative holds:*
either

(a_1) *I_λ possesses a global minimum,*

or

(a_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$.*

(b) *If $\chi < +\infty$ and for each $\lambda \in \left]0, \frac{1}{\chi}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$-condition for some $r > \inf_X \Phi$ then, for each $\lambda \in \left]0, \frac{1}{\chi}\right[$, the following alternative holds:*
either

(b_1) *there is a global minimum of Φ which is a local minimum of I_λ,*

or

(b_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ such that $\lim_{n\to\infty} \Phi(u_n) = \inf_X \Phi$.*

For the sake of completeness, we recall that for $r \in \mathbb{R}$, $I_\lambda = \Phi - \lambda\Psi$ is said to satisfy the $(PS)^{[r]}$-condition if any sequence $\{u_n\}$ such that

(α_1) $\{I_\lambda(u_n)\}$ is bounded,

(α_2) $\|I'_\lambda(u_n)\|_{X^*} \to 0$ as $n \to \infty$,

(α_3) $\Phi(u_n) < r \quad \forall\, n \in \mathbb{N}$

has a convergent subsequence.

3 Main results

In this section we are going to present the announced multiplicity result. The following technical constant will be useful

$$\tau = 4\delta\pi^4 \left(96\left(\frac{12}{5}\right)^5 + 4A\left(\frac{12}{5}\right)^4 + B\frac{1248}{175} + C\frac{493}{756} \right)^{-1} \tag{3.1}$$

where A, B and C are the real numbers involved in problem (P_λ) and such that (H) holds, while δ has been introduced in (2.8).

Remark 3.1. We wish to stress a useful estimate for τ. If we consider the function

$$w(x) = \begin{cases} v(x), & \text{if } x \in [0, 5/12[, \\ 1, & \text{if } x \in [5/12, 7/12], \\ v(1-x), & \text{if } x \in]7/12, 1], \end{cases} \tag{3.2}$$

where $v(x) = \left(\frac{12}{5}\right)^4 x^4 - 2\left(\frac{12}{5}\right)^3 x^3 + \frac{24}{5}x$ for every $x \in [0, 5/12]$, a straightforward computation shows that $w \in X = H^3(0,1) \cap H^1_0(0,1)$ and, in particular

$$\|w\|^2_X = \frac{4\delta\pi^4}{\tau}.$$

Recalling that (H) holds, the positivity of τ follows from the positivity of δ as seen in the arguments presented in the previous section (see also Proposition 2.4). Moreover, from (2.10), since $\|w\|_\infty = 1$, one can even conclude that

$$0 < \tau \leq 1.$$

Here is the first main result.

Theorem 3.2. *Assume that*

i) $F(x,t) \geq 0$ *for every* $(x,t) \in ([0,5/12] \cup [7/12,1]) \times \mathbb{R}$,

ii) $\liminf_{t\to+\infty} \dfrac{\int_0^1 \max_{|s|\leq t} F(x,s)\,dx}{t^2} < \tau \limsup_{t\to+\infty} \dfrac{\int_{5/12}^{7/12} F(x,t)\,dx}{t^2}$.

Then, for every

$$\lambda \in \Lambda = \left]\frac{2\delta\pi^4}{\tau}\frac{1}{\limsup_{t\to+\infty}\frac{\int_{5/12}^{7/12} F(x,t)\,dx}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to+\infty}\frac{\int_0^1 \max_{|s|\leq t} F(x,s)\,dx}{t^2}}\right[$$

the problem (P_λ) *admits an unbounded sequence of classical solutions.*

Proof. We wish to apply Theorem 2.8, case (a), with $X = H^3(0,1) \cap H^1_0(0,1)$ endowed with the norm $\|\cdot\|_X$ defined in (2.6), Φ and Ψ as in (2.12).

In the previous section we have already pointed out that Φ, $\Psi \in C^1(X)$. It is simple to verify that Φ is bounded from below, coercive and its derivative is a homeomorphism. Moreover, the compactness of the embedding $X \hookrightarrow C^0(0,1)$ assures that Ψ' is a compact operator. Hence, we can conclude that, for every $\lambda > 0$ (indeed for every $\lambda \in \mathbb{R}$) the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for every $r \in \mathbb{R}$ (see [2, Remark 2.1]). Our aim is now to verify that $\gamma < +\infty$. Let us begin by observing that, in view of (2.10) one has

$$\{v \in X : \Phi(v) < r\} \subset \left\{v \in C^0(0,1) : \|v\|_\infty \leq \frac{k}{\sqrt{\delta}}\sqrt{\frac{r}{2}}\right\}$$

for all $r > 0$. Let $\{t_n\}$ be in $\mathbb{R}^+$ such that $t_n \to +\infty$ and

$$\lim_{n\to\infty} \frac{\int_0^1 \max_{|s|\leq t_n} F(x,s)\,dx}{t_n^2} = \liminf_{t\to+\infty} \frac{\int_0^1 \max_{|s|\leq t} F(x,s)\,dx}{t^2}.$$

Put $r_n = 2\delta\pi^4 t_n^2$ for every $n \in \mathbb{N}$. Hence, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{\Phi(v)<r_n} \frac{\sup_{\Phi(u)<r_n} \Psi(u) - \Psi(v)}{r - \Phi(v)} \\
&\leq \frac{\sup_{\Phi(u)<r_n} \Psi(u)}{r_n} \\
&\leq \frac{1}{2\delta\pi^4} \frac{\int_0^1 \max_{|s|\leq t_n} F(x,s)\,dx}{t_n^2}.
\end{aligned}$$

Passing to the lim inf in the previous inequality one obtains

$$\gamma \leq \liminf_{n\to\infty} \varphi(r_n) \leq \frac{1}{2\delta\pi^4} \liminf_{t\to+\infty} \frac{\int_0^1 \max_{|s|\leq t} F(x,s)\, dx}{t^2} < +\infty.$$

In particular, we have also verified that

$$\Lambda \subset \left]0, \frac{1}{\gamma}\right[.$$

Fix now $\lambda \in \Lambda$ and let us check that I_λ is unbounded from below. We can explicitly observe that

$$\frac{1}{2\delta\pi^4} \liminf_{t\to+\infty} \frac{\int_0^1 \max_{|s|\leq t} F(x,s)\, dx}{t^2} < \frac{1}{\lambda} < \frac{\tau}{2\delta\pi^4} \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\, dx}{t^2}.$$

Pick $\eta > 0$ such that

$$\frac{1}{\lambda} < \eta < \frac{\tau}{2\delta\pi^4} \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\, dx}{t^2}$$

and consider a sequence $\{d_n\}$ in $\mathbb{R}^+$ such that $d_n \to +\infty$ and

$$\frac{\int_{5/12}^{7/12} F(x,d_n)}{d_n^2} > \eta \frac{2\delta\pi^4}{\tau}$$

for every $n \in \mathbb{N}$. If, for every $n \in \mathbb{N}$ we define

$$w_n(x) = d_n w(x),$$

where w has been defined in (3.2), it is clear that $0 \leq w_n(x) \leq d_n$ for every $x \in [0,1]$, $w_n \in X$ and, in particular

$$\|w_n\|_X^2 = \frac{4\delta\pi^4}{\tau} d_n^2.$$

Thus, also in view of i),

$$\begin{aligned} I_\lambda(w_n) &= \Phi(w_n) - \lambda\Psi(w_n) \\ &= \frac{2\delta\pi^4}{\tau} d_n^2 - \lambda \int_0^1 F(x, w_n(x))\, dx \\ &< \frac{2\delta\pi^4}{\tau}(1-\lambda\eta) d_n^2. \end{aligned}$$

Namely, passing to the limit and taking in mind that $1 - \lambda\eta < 0$ one achieves that I_λ is unbounded from below.

We are now in the position to apply Theorem 2.8, case (a), and obtain a sequence $\{u_n\}$ in X of critical points (local minima) of I_λ such that $\|u_n\|_X \to +\infty$. Taking in mind that the critical points of I_λ are classical solutions of (P_λ), see Proposition 2.7, we have completed the proof. □

The following remark will be useful in order to obtain a sign condition on the solutions of (P_λ).

Remark 3.3. Recall that if $\mu \geq 0$ and $w \in H^2(0,T)$ is such that

$$\begin{cases} -w'' + \mu w \geq 0, & x \in [0,1], \\ w(0) = w(1) = 0, \end{cases}$$

then $w(x) \geq 0$ for every $x \in [0,1]$ (see also [4, Théorème VIII.17]).

Remark 3.4. We wish to point out that if $u \in C^6(0,1)$ is a nonnegative and nontrivial function such that

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu \geq 0, & x \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases}$$

and there exist three nonnegative numbers X, Y and Z such that

$$\begin{cases} X + Y + Z = A, \\ XY + XZ + YZ = B, \\ XYZ = C, \end{cases} \tag{3.3}$$

then $u(x) > 0$ for every $x \in (0,1)$. To justify this, we can observe that for the following linear differential operators

$$L_1(w) = -w'' + Xw, \qquad L_2(w) = -w'' + Yw, \qquad L_3(w) = -w'' + Zw$$

is possible to apply the strong maximum principle (see [14]). Hence, in particular, since in view of (3.3)

$$\begin{aligned} L_1(L_2(L_3(u))) &= -u^{(vi)} + (X+Y+Z)u^{(iv)} - (XY+XZ+YZ)u'' + XYZu \\ &= -u^{(vi)} + Au^{(iv)} - Bu'' + Cu, \end{aligned}$$

one has, using several times Remark 3.3,

$$L_1(L_2(L_3(u))) \geq 0 \;\Rightarrow\; L_2(L_3(u)) \geq 0 \;\Rightarrow\; L_3(u) \geq 0 \;\Rightarrow\; u \geq 0 \quad \text{in } [0,1].$$

Finally, from [14, Theorem 3] one can conclude that $u(x) > 0$ for every $x \in (0,1)$.

We refer to [12] for further considerations on maximum principle for high-order differential equations.

Example 3.5. If $u \in C^6(0,1)$ is such that

$$\begin{cases} -u^{(vi)} + 3u^{(iv)} - 3u'' + u \geq 0, & x \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases}$$

then $u > 0$ in $(0,1)$. It suffices to take $X = Y = Z = 1$ in (3.3), so that $A = B = 3$ and $C = 1$.

Example 3.6. If $C = 0$ and A, $B \geq 0$ are such that $A^2 - 4B \geq 0$ then every nonnegative and nontrivial $u \in C^6(0,1)$ such that

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' \geq 0, & x \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{iv}(0) = u^{iv}(1) = 0, \end{cases}$$

is positive in $(0,1)$. Indeed, we can recall Remark 3.4, in (3.3) consider $X = \frac{A+\sqrt{A^2-4B}}{2}$, $Y = \frac{A-\sqrt{A^2-4B}}{2}$, $Z = 0$ and conclude that $u > 0$ in $(0,1)$.

In the following we say that $A,\ B,\ C$ satisfy the (H_+) condition if

(H_+) there exist nonnegative numbers $X,\ Y$ and Z such that (3.3) holds.

The existence of constant sign solutions can be pointed out, provided

$$f(x,t) \geq 0, \qquad \forall (x,t) \in [0,1] \times [0,+\infty[. \tag{3.4}$$

In particular the following result holds.

Theorem 3.7. *Assume that assumption* (3.4) *and* (H_+) *hold and*

ii') $\displaystyle \liminf_{t\to+\infty} \frac{\int_0^1 F(x,t)\,dx}{t^2} < \tau \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\,dx}{t^2}.$

Then, for every

$$\lambda \in \tilde{\Lambda} = \left] \frac{2\delta\pi^4}{\tau} \frac{1}{\limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\,dx}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to+\infty} \frac{\int_0^1 F(x,t)\,dx}{t^2}} \right[$$

the problem (P_λ) *admits an unbounded sequence of positive classical solutions.*

Proof. Put

$$f^+(x,t) = \begin{cases} f(x,t), & \text{if } (x,t) \in [0,1] \times [0,+\infty[\,, \\ \\ f(x,0), & \text{if } (x,t) \in [0,1] \times]-\infty,0[\,, \end{cases}$$

and $F^+(x,t) = \int_0^t f^+(x,s)\,ds$. Clearly $f^+_{|[0,1]\times[0,+\infty[} = f_{|[0,1]\times[0,+\infty[}$ as well as $F^+_{|[0,1]\times[0,+\infty[} = F_{|[0,1]\times[0,+\infty[}$. Hence, in view of ii'),

$$\begin{aligned} \liminf_{t\to+\infty} \frac{\int_0^1 \max_{|s|\leq t} F^+(x,s)\,dx}{t^2} &= \liminf_{t\to+\infty} \frac{\int_0^1 \max_{0\leq s\leq t} F(x,s)\,dx}{t^2} \\ &= \liminf_{t\to+\infty} \frac{\int_0^1 F(x,t)\,dx}{t^2} \\ &< \tau \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\,dx}{t^2} \\ &= \tau \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F^+(x,t)\,dx}{t^2}. \end{aligned}$$

Thus, we can apply Theorem 3.2 to f_+ and F_+ and assure that for every $\lambda \in \tilde{\Lambda}$, the problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda f^+(x,u), & x \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{3.5}$$

admits an unbounded sequence of classical solutions.

We claim that

Every solution of (3.5) is a nonnegative solution of (P_λ).

Indeed, if u solves (3.5), since $f^+(x,t) \geq 0$ for every $(x,t) \in [0,1] \times \mathbb{R}$, we can recall Remark 3.4 and deduce that u is positive. Hence, $f^+(x,u(x)) = f(x,u(x))$ for every $x \in [0,1]$ and u solves (P_λ). The claim is now verified and the proof is completed. □

We now present an autonomous version of the previous result.

Theorem 3.8. *Suppose (H_+) holds and assume that g is a nonnegative continuous function such that*

$$\liminf_{t\to+\infty} \frac{G(t)}{t^2} < \frac{\tau}{6} \limsup_{t\to+\infty} \frac{G(t)}{t^2}, \tag{3.6}$$

where $G(t) = \int_0^t g(s)\, ds$ for every $t \in \mathbb{R}$.

Then, for every $\lambda \in \left] \frac{12\delta\pi^4}{\tau} \frac{1}{\limsup_{t\to+\infty} \frac{G(t)}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to+\infty} \frac{G(t)}{t^2}} \right[$ the problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda g(u), & x \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{$\tilde{P}_\lambda$}$$

admits an unbounded sequence of classical positive solutions.

Proof. Apply Theorem 3.7 to $f(x,t) = g(t)$ for all $(x,t) \in [0,1] \times \mathbb{R}$ and observe that

$$F(x,t) = G(t), \qquad \int_{5/12}^{7/12} F(x,t)\, dx = \frac{1}{6} G(t).$$

□

Example 3.9. Fix A, B and C (as usual such that (H_+) holds) let τ be the number defined in (3.1), pick $\rho > \frac{6-\tau}{\tau}$ and consider the continuous function $g : \mathbb{R} \to \mathbb{R}$ defined by putting

$$g(t) = \begin{cases} 2t \left[1 + \rho \sin^2(\ln(\rho^2 + \ln^2 t)) + \sin(2\ln(\rho^2 + \ln^2 t)) \frac{\rho \ln t}{\rho^2 + \ln^2 t} \right], & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Then, for every $\lambda \in \left] \frac{12\delta\pi^4}{\tau(1+\rho)}, 2\delta\pi^4 \right[$ the problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda g(u), & x \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases}$$

admits an unbounded sequence of classical positive solutions.

Indeed, a direct computation shows that $0 < \tau \leq 1$ (see Remark 3.1). Hence, $\rho > 0$ and exploiting the boundedness of the function $t \to \sin(2\ln(\rho^2 + \ln^2 t)) \frac{\rho \ln t}{\rho^2 + \ln^2 t}$ one has

$$g(t) \geq 2t \left[\frac{1}{2} + \rho \sin^2(\ln(\rho^2 + \ln^2 t)) \right] > 0$$

for every $t > 0$. Moreover,

$$G(t) = \begin{cases} t^2(1 + \rho \sin^2(\ln(\rho^2 + \ln^2 t))), & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Hence, if we put $a_n = e^{\sqrt{e^{n\pi}-\rho^2}}$ and $b_n = e^{\sqrt{e^{(2n+1)\pi/2}-\rho^2}}$ for every $n \in \mathbb{N}$ with $n > (2/\pi)\ln\rho$, one has

$$\begin{aligned}\liminf_{t\to+\infty}\frac{G(t)}{t^2} &\le \lim_{n\to+\infty}\frac{G(a_n)}{a_n^2} = 1\\ &< \frac{\tau}{6}(1+\rho)\\ &= \frac{\tau}{6}\lim_{n\to+\infty}\frac{G(b_n)}{b_n^2}\\ &\le \frac{\tau}{6}\limsup_{t\to+\infty}\frac{G(t)}{t^2}.\end{aligned}$$

At this point we can apply Theorem 3.8 observing that

$$\left]\frac{12\delta\pi^4}{\tau(1+\rho)}, 2\delta\pi^4\right[\subseteq \left]\frac{12\delta\pi^4}{\tau}\frac{1}{\limsup_{t\to+\infty}\frac{G(t)}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to+\infty}\frac{G(t)}{t^2}}\right[.$$

We can directly derive the proof of Theorem 1.1 from Theorem 3.8.

Proof of Theorem 1.1. Apply Theorem 3.8, with $A = 3D$; $B = 3D^2$; $C = D^3$, and exploit Remark 3.4, by choosing $X = Y = Z = D$. □

Remark 3.10. Clearly, if (H) holds and (H_+) is not satisfied, the assumptions of Theorems 3.7 and 3.8 ensure the existence of infinitely many classical solutions.

We conclude the present note pointing out that, adapting the previous arguments, one can exploit case (b) of Theorem 2.8 in order to prove the existence of arbitrary small solutions of problem (P_λ).

Theorem 3.11. *Assume that*

j) there exists $r > 0$ *such that* $F(x,t) \ge 0$ *for every* $(x,t) \in ([0,5/12] \cup [7/12,1]) \times [0,r]$,

jj) $\displaystyle\liminf_{t\to0^+}\frac{\int_0^1 \max_{|s|\le t} F(x,s)\,dx}{t^2} < \tau\limsup_{t\to0^+}\frac{\int_{5/12}^{7/12} F(x,t)\,dx}{t^2}.$

Then, for every

$$\lambda \in \Gamma = \left]\frac{2\delta\pi^4}{\tau}\frac{1}{\limsup_{t\to0^+}\frac{\int_{5/12}^{7/12}F(x,t)\,dx}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to0+}\frac{\int_0^1\max_{|s|\le t}F(x,s)\,dx}{t^2}}\right[$$

the problem (P_λ) *admits a sequence of pairwise distinct nontrivial classical solutions, which strongly converges to* 0 *in* X.

Clearly, starting from Theorem 3.11 and arguing as above, further results dealing with the existence of arbitrary small (positive) classical solutions could be furnished.

Acknowledgements

The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The paper is partially supported by PRIN 2017 – Progetti di Ricerca di rilevante Interesse Nazionale, "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods" (2017AYM8XW).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **21**, 1–6; https://doi.org/10.14232/ejqtde.2021.1.21 www.math.u-szeged.hu/ejqtde/

λ-lemma for nonhyperbolic point in intersection

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Received 23 May 2020, appeared 26 March 2021
Communicated by Tibor Krisztin

Abstract. The well known λ-Lemma has been proved by J. Palis for a hyperbolic fixed point of a C^1-diffeomorphism. In this paper we show that the result is true for some cases of nonhyperbolic point.

Keywords: λ-lemma, nonhyperbolic point.

2020 Mathematics Subject Classification: 37D30, 37C29.

1 Introduction

The well known λ-Lemma [9] gives an important description of chaotic dynamics. A basic assumption of this theorem is hyperbolicity.

Theorem 1.1 (Palis)**.** *Let f be a C^1 diffeomorphism of $\mathbb{R}^n$ with a hyperbolic fixed point at 0 and m- and p-dimensional stable and unstable manifolds W^S and W^U $(m+p=n)$. Let D be a p-disk in W^U, and w be another p-disk in W^U meeting W^S at some point A* transversely*. Then $\bigcup_{n\geq 0} f^n(w)$ contains p-disks arbitrarily C^1-close to D.*

Generally, for C^1 diffeomorphism f of compact manifold M periodic point z is called hyperbolic if there exists a splitting $T_z(M) = E^s \oplus E^u$ with constants $k > 0$ and $0 < \lambda < 1$ such that

$$\|(Df^n)|_{E^s}\| \leq k\lambda^n \qquad (n > 0),$$
$$\|(Df^{-n})|_{E^u}\| \leq k\lambda^n \qquad (n > 0).$$

Here E^s and E^u are called stable and unstable subspaces of f, respectively. If z is nonhyperbolic this splitting can be written as $T_z(M) = E^s \oplus E^u \oplus E^c$, where E^s and E^u are the same as above and E^c is called the center subspace of f.

Some extensions of this lemma can be found in the [1–4, 11]. One question that arises is whether it is possible to put weaker conditions in this lemma instead of being hyperbolic. In this paper we append some new cases in which we have affirmative answer. We think these cases can be used in extending the connecting lemma of Hayashi[5]. Our result can help to generalize [7, 8] to some new cases.

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Definition 1.2. We say that a nonhyperbolic periodic point z satisfies the invariant conditions (IC) if there is a local chart (U, ϕ) at z such that in $\phi(U)$ one of the following is true:

I_1) $E^s \oplus E^u$ is invariant under f;

I_2) $E^c \oplus E^s$ is invariant under f;

I_3) $E^c \oplus E^u$ is invariant under f.

Notice that f in I_1, I_2 and I_3 is in fact $\tilde{f} = \phi f \phi^{-1}$.
Let us give an example of a system which satisfies in IC.

Example 1.3. Let $\mathbb{R}^3_\infty$ be the compactification* of $\mathbb{R}^3$. As known this is a C^∞ manifold with two charts, one at the origin and the other at ∞. We define the diffeomorphism

$$f(x, y, z) = \left(2x, \frac{y}{2}, z\right)$$

It is easy to see that the axes are the three invariant manifolds of the origin and the whole of coordinate surface are invariant under f. But, origin is not hyperbolic.

2 Preliminaries

Let $A = Df(0)$ and Let p be a nonhyperbolic fixed point of f satisfying IC, i.e. f satisfies either I_1, I_2 or I_3.

First assume I_1 is true. Since f is locally invariant on $E^s \oplus E^u$, if $W^s_{loc}(0)$ and $W^u_{loc}(0)$ are the graphs of ϕ^s and ϕ^u respectively, then locally we can write

$$\phi^s : B^s \to E^u \quad \text{and} \quad \phi^u : B^u \to E^s.$$

Here ϕ^s and ϕ^u are C^r, $D\phi^s(0) = 0$, $D\phi^u(0) = 0$, $\phi^s(0) = 0$ and $\phi^u(0) = 0$. Consider the map

$$\phi : B^s \oplus B^u \oplus E^c \to E^s \oplus E^u \oplus E^c$$

$$(x_s, x_u, x_c) \mapsto (x_s - \phi^u(x_u), x_u - \phi^s(x_s), x_c).$$

It is clear that ϕ is C^r and $D\phi(0)$ is a the identity and ϕ is diffeomorphism when restricted to some neighborhood of 0. Let $\tilde{f} = \phi f \phi^{-1}$ then $\tilde{f}$ is a diffeomorphism on a neighborhood of 0 and $\tilde{f}(0) = 0$, $D\tilde{f}(0) = A$ and E^s, E^u are local stable and unstable manifold of $\tilde{f}$. It is clear that $E^s \oplus E^u$ is still invariant. This shows that in this case we can always assume that local stable and unstable manifolds of f are discs in E^s and E^u, respectively.

Let $B^s \subseteq E^s$ and $B^u \subseteq E^u$ be such that $B^s \subseteq W^s_{loc}(0)$ and $B^u \subseteq W^u_{loc}(0)$. Let B^c be the intersection of local chart containing z, with E^c.

Now we can rewrite the proof of λ-lemma in [10] as the following lemmas.

Lemma 2.1. *Let z be a nonhyperbolic fixed point of f which satisfies I_1. Let $V = B^s \times B^u \times B^c$, and let D be a disc transversal to B^s at q with $\dim(D) = \dim(E^u)$. If D_n is the connected component of $f^n(D) \cap V$ to which $f^n(q)$ belongs, then for any given small positive ϵ we can find n such that D_n is ϵ-C^1 close to B^u.*

*We have to suppose compactification because in the definition of a hyperbolic fixed point that was mentioned above we need a compact manifold.

The proof is very similar to the proof of λ-lemma in [10]. Notice that the existence of E^c does not change the the main flow of the original proof, since $E^s \oplus E^u$ is invariant under f.

Let I_2 be true. We get the C^r map $\phi^u : B^u \to E^s \oplus E^c$ that its graph is $W^s_{loc}(0)$. Thus, $D\phi^u(0) = 0$ and $\phi^u(0) = 0$. Assume that $\phi^u(x_u) = (\phi^{us}(x_u), \phi^{uc}(x_u))$. Consider the map

$$\phi : B^s \oplus E^u \oplus E^c \to E^s \oplus E^u \oplus E^c,$$

$$(x_s, x_u, x_c) \mapsto (x_s - \phi^{us}(x_u), x_u, x_c - \phi^{uc}(x_c)),$$

where ϕ is C^r and $D\phi(0)$ is identity. Thus, ϕ is a diffeomorphism defined on a neighborhood of 0. Let $\tilde{f} = \phi f \phi^{-1}$, then $\tilde{f}$ is a diffeomorphism of a neighborhood of 0 with $\tilde{f}(0) = 0$, and, $D\tilde{f}(0) = A$. Moreover, E^u and $E^c \oplus E^s$ are invariant under $\tilde{f}$. This implies that for every f which satisfies I_2 for a nonhyperbolic fixed point, we can find a local chart such that E^u and $E^s \oplus E^c$ are invariant with respect to f.

Lemma 2.2. *Let z be a nonhyperbolic fixed point of f which satisfies I_2 and D be a transversal disc to $E^c \oplus E^s$ at $q \in E^s$ and $D^u \subseteq E^u$ a disc containing 0, then for an arbitrary small positive ϵ, there exists n such that a section of $f^n(D)$ is ϵ-C^1 close to D^u.*

Proof. Let $A = Df(0)$ and A^{cs} and A^u be respectively restriction of A to subspaces $E^{cs} = E^c \oplus E^s$ and E^u, thus f on a neighborhood V of origin becomes:

$$f(x_{cs}, x_u) = (A^{cs}x_{cs} + \phi_{cs}(x_{cs}, x_u), A^u x_u + \phi_u(x_{cs}, x_u)),$$

whence

$$(Df)_0 = (A^{cs}, A^u), \qquad x_{cs} \in B^{cs} = V \cap E^{cs}, \qquad x_u \in B^u = V \cap E^u,$$
$$\|A^{cs}\| \le 1, \qquad \|A^u\| \ge a > 1,$$

$$\left.\frac{\partial \phi_{cs}}{\partial x_u}\right|_{B^u} = \left.\frac{\partial \phi_u}{\partial x_{cs}}\right|_{B^{cs}} = 0.$$

From above and continuity of partial differential we can find $0 < k < 1$ such that $k < \frac{a-1}{8}$ and for $V' \subset V$,

$$\max_{V'} = \left\| \frac{\partial \phi_i}{\partial x_j} \right\| \le k, \qquad i, j = cs, u.$$

Let $q \in V'$, $B^u \subset V'$ take arbitrary unit vector v_0 in $(TD)_q$. Because $V = B^{cs} \times B^u$ then $v_0 = (v_0^{cs}, v_0^u)$. If λ_0 is the slope of v_0 then $\lambda_0 = \frac{\|v_0^{cs}\|}{\|v_0^u\|}$. In this fraction $\|v_0^u\| \ne 0$ because D is transversal disc to B^{cs}.

$$\begin{aligned} q_1 &= f(q), & v_1 &= Df_q(v_0) \\ q_2 &= f(q_1), & v_2 &= Df_{q_1}(v_1) \\ &\vdots & &\vdots \\ q_n &= f(q_{n-1}), & v_n &= Df_{q_{n-1}}(v_{n-1}). \end{aligned} \tag{2.1}$$

for $q \in \partial B^{cs}$

$$Df_q(v_0) = \begin{pmatrix} A^{cs} + \frac{\partial \phi_{cs}}{\partial x_{cs}}(q) & \frac{\partial \phi_{cs}}{\partial x_u}(q) \\ 0 & A^u + \frac{\partial \phi_u}{\partial x_u}(q) \end{pmatrix} \begin{pmatrix} v_0^{cs} \\ v_0^u \end{pmatrix}$$
$$= \begin{pmatrix} A^{cs}v_0^{cs} + \frac{\partial \phi_{cs}}{\partial x_{cs}}(q)v_0^{cs} + \frac{\partial \phi_{cs}}{\partial x_u}(q)v_0^u \\ A^u v_0^u + \frac{\partial \phi_u}{\partial x_u}(q)v_0^u \end{pmatrix}.$$

Thus
$$\lambda_1 = \frac{\|v_1^{cs}\|}{\|v_1^u\|} = \frac{\left\|A^{cs}v_0^{cs} + \frac{\partial\phi_{cs}}{\partial x_{cs}}(q)v_0^{cs} + \frac{\partial\phi_{cs}}{\partial x_u}(q)v_0^u\right\|}{\left\|A^u v_0^u + \frac{\partial\phi_u}{\partial x_u}(q)v_0^u\right\|}.$$
The numerator of above fraction is less than
$$\|A^{cs}v_0^{cs}\| + \left\|\frac{\partial\phi_{cs}}{\partial x_{cs}}(q)v_0^{cs}\right\| + \left\|\frac{\partial\phi_{cs}}{\partial x_u}(q)v_0^u\right\| \leq (1+k)\|v_0^{cs}\| + k\|v_0^s\|$$
and its denominator is greater than
$$\|A^u v_0^u\| - \left\|\frac{\partial\phi_u}{\partial x_u}(q)v_0^u\right\| \geq (a-k)\|v_0^u\|,$$
then
$$\begin{aligned}
\lambda_1 &\leq \frac{(1+k)\lambda_0 + k}{a-k} \leq \frac{1+k}{a-k}\lambda_0 + \frac{k}{a-k},\\
\lambda_2 &= \frac{\|v_2^{cs}\|}{\|v_2^u\|} \leq \frac{(1+k)\lambda_1 + k}{a-k} \leq \left(\frac{1+k}{a-k}\right)^2\lambda_0 + \frac{k}{1+k}\sum_{i=1}^{2}\left(\frac{1+k}{a-k}\right)^i\\
&\vdots\\
\lambda_n &= \frac{\|v_n^{cs}\|}{\|v_n^u\|} \leq \left(\frac{1+k}{a-k}\right)^n\lambda_0 + \frac{k}{1+k}\sum_{i=1}^{n}\left(\frac{1+k}{a-k}\right)^i \leq \left(\frac{1+k}{a-k}\right)^n\lambda_0 + \frac{a-k}{a-1-2k}.
\end{aligned}$$
Because $\left(\frac{1+k}{a-k}\right)^n\lambda_0 \to 0$, then there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have $\lambda_n < \frac{a-k}{a-1-2k}$.

Consider the number k_1 such that $0 < k_1 < \min(\epsilon, k)$. Because $\frac{\partial\phi_{cs}}{\partial x_u}\big|_{B^u} = 0$ and B^u is compact, there exists $\delta < \epsilon$ such that $V_1 = \delta B^{cs} \times B^u \subset V$ so
$$\max_{V_1}\left\|\frac{\partial\phi_{cs}}{\partial x_u}\right\| \leq k_1.$$
Let δB^{cs} be a ball with radius δ times radius of B^{cs} . We can assume that v_0 is a vector in $(TD)_q$ that has maximal slope, so for $n \geq n_0$ the slope of all unit vectors in $(TD_n)_{q_n}$ is less than $\frac{a-k}{a-1-2k}$. For a properly chosen n_0 we have $q_{n_0} \in V_1$. From the continuity of the tangent space D_{n_0}, we can find a disk $\tilde{D}$ embedded in D_{n_0} with center q_{n_0} such that for all $p \in \tilde{D}$ the slope of all unit vectors in $(T\tilde{D})_p$ is less than $\frac{2(a-k)}{a-1-2k}$.

Let $v \in (T\tilde{D})_p$ be a unit vector. If $v = (v^{cs}, v^u)$ and its slope is $\lambda_{n_0} = \frac{\|v^{cs}\|}{\|v^u\|}$ then
$$Df_p = \begin{pmatrix} A^{cs}v^{cs} + \frac{\partial\phi_{cs}}{\partial x_{cs}}(p)v^{cs} + \frac{\partial\phi_{cs}}{\partial x_u}(p)v^u \\ \frac{\partial\phi_u}{\partial x_{cs}}(p)v^{cs} + A^u v^u + \frac{\partial\phi_u}{\partial x_u}(p)v^u \end{pmatrix}.$$
Thus
$$\lambda_{n_0+1} = \frac{\left\|A^{cs}v^{cs} + \frac{\partial\phi_{cs}}{\partial x_{cs}}(p)v^{cs} + \frac{\partial\phi_{cs}}{\partial x_u}(p)v^u\right\|}{\left\|\frac{\partial\phi_u}{\partial x_{cs}}(p)v^{cs} + A^u v^u + \frac{\partial\phi_u}{\partial x_u}(p)v^u\right\|}.$$
The numerator of above fraction is less that $(1+k)\|v^{cs}\| + k_1\|v^u\|$ and its denominator is greater than
$$\|A^u v^u\| - \left\|\frac{\partial\phi_u}{\partial x_u}(p)v^u\right\| - \left\|\frac{\partial\phi_u}{\partial x_{cs}}(p)v^{cs}\right\| \geq (a-k)\|v^u\| - k\|v^{cs}\|.$$

Thus

$$\lambda_{n_0+1} \leq \frac{(1+k)\lambda_{n_0}+k_1}{a-k-k\lambda_{n_0}} \leq \frac{(1+k)\lambda_{n_0}+k_1}{a-k-k\frac{2(a-k)}{a-1-2k}}$$
$$\leq \frac{(1+k)\lambda_{n_0}+k_1}{\frac{(a-k)(a-1-4k)}{a-1-2k}}.$$

Let $b = \frac{(a-k)(a-1-4k)}{a-1-2k}$. It is easy to see that $k+1 < b$. Therefore we have

$$\lambda_{n+n_0} \leq \left(\frac{1+k}{b}\right)^n \lambda_{n_0} + k_1 \frac{b}{(b-1-k)(k+1)}.$$

Then there exits $\tilde{n}$ such that for $n \geq \tilde{n}$

$$\lambda_{n+n_0} \leq \epsilon \left(1 + \frac{b}{(b-1-k)(k+1)}\right).$$

This shows that for $n \geq \tilde{n}$ the slope of nonzero tangent vectors to $f^n(\tilde{D}) \cap V_1$ is less than given ϵ.

Now we show that the length of any tangent vector to $f^n(\tilde{D}) \cap V_1$ is growing as n is increasing. We denote the image of (v_n^{cs}, v_n^u) under Df as $(v_{n+1}^{cs}, v_{n+1}^u)$, thus

$$\frac{\sqrt{\|v_{n+1}^{cs}\|^2 + \|v_{n+1}^u\|^2}}{\sqrt{\|v_n^{cs}\|^2 + \|v_n^u\|^2}} = \frac{\|v_{n+1}^u\|}{\|v_n^u\|}\sqrt{\frac{1+\lambda_{n+1}^2}{1+\lambda_n^2}}.$$

But

$$\frac{\|v_{n+1}^u\|}{\|v_n^u\|} \geq a - k - \lambda_n.$$

As n is growing, λ_n and λ_{n+1} become small enough; then the length of the tangent vectors to $f^n(\tilde{D}) \cap V_1$ are increasing with ratio $a - k > 1$. This fact and tendency to zero of the slope of the tangent vectors imply that for $n > \tilde{n}$ the $f^n(\tilde{D}) \cap V_1$ are approaching in C^1 topology to B^u. □

Finally suppose that condition I_3 is true, we replace f by f^{-1}, then condition I_2 is true for f^{-1} and using the above lemma we have:

Lemma 2.3. *Let z be a nonhyperbolic fixed point of f that satisfies I_3 and D be a disc transversal to $E^u \oplus E^c$ at $q \in E^u$ and $D^s \subseteq E^s$ a disc containing 0, then for an arbitrary small positive ϵ there exists n that $f^{-n}(d)$ is ϵ-C^1 close to D^s.*

As a consequence of the above lemmas, the following proposition can be obtained. We first need the definition of *forwardly related* from [6].

For any C^1 diffeomorphism f of compact manifold M and $p \in M$ the forward orbit of p is

$$\mathcal{O}_f^+ = \{x \in M : \exists n \in \mathbb{Z} \ s.t. \ f^n(p) = x\}.$$

Definition 2.4. A point $p \in M$ is called forwardly related to $q \in M$ if $q \notin \mathcal{O}_f^+(p)$ and there exists a sequence diffeomorphisms $\{f_n\}$ such that $f_n \to f$ and a sequence of strings $\gamma_n = \{f_n^k(p_n) : k = 0, \ldots, s_n\}$ such that $p_n \to p$ and $f_n^{s_n}(p_n) \to q$.

Proposition 2.5. *Let z be a nonhyperbolic fixed point satisfying IC, let $p \in W_{loc}^s(z)$, and, $q \in W_{loc}^u(z)$. Then p is forwardly related to q.*

All the above results are true for periodic point p. It is sufficient to replace f by f^n where n is the period of p.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **22**, 1–14; https://doi.org/10.14232/ejqtde.2021.1.22 www.math.u-szeged.hu/ejqtde/

On solvability of focal boundary value problems for higher order functional differential equations with integral restrictions

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Received 8 October 2020, appeared 30 March 2021
Communicated by Leonid Berezansky

Abstract. Sharp conditions are obtained for the unique solvability of focal boundary value problems for higher-order functional differential equations under integral restrictions on functional operators. In terms of the norm of the functional operator, unimprovable conditions for the unique solvability of the boundary value problem are established in the explicit form. If these conditions are not fulfilled, then there exists a positive bounded operator with a given norm such that the focal boundary value problem with this operator is not uniquely solvable. In the symmetric case, some estimates of the best constants in the solvability conditions are given. Comparison with existing results is also performed.

Keywords: functional differential equations, focal boundary value problem, unique solvability.

2020 Mathematics Subject Classification: 34K06, 34K10.

1 Introduction

We consider here boundary value problems

$$\begin{cases} (-1)^{(n-k)} x^{(n)}(t) + (Tx)(t) = f(t), \quad t \in [0,1], \\ x^{(i)}(0) = 0, \quad i = 0, \dots, k-1, \\ x^{(j)}(1) = 0, \quad j = k, \dots, n-1, \end{cases} \tag{1.1}$$

where $n \in \{2, 3, \dots\}$, $k \in \{1, 2, \dots, n-1\}$, $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is a linear bounded operator, $\mathbf{C}[0,1]$ and $\mathbf{L}[0,1]$ are the space of real continuous and integrable functions (respectively) with the standard norms, $f \in \mathbf{L}[0,1]$. A real absolutely continuous function with absolutely continuous derivatives up to $(n-1)$-th order which satisfies the boundary conditions from (1.1) and satisfies the functional differential equation from (1.1) almost everywhere on $[0,1]$ is called a solution to problem (1.1).

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The boundary value problems with such kind of boundary conditions are called focal ones. The solvability of such problems for linear and non-linear functional differential equations occupies a special place in many studies of physical, chemical, and biological processes (see, for example, [1,2,7,14,31,37] end references there).

The focal problem for the ordinary differential equation

$$\begin{cases} (-1)^{(n-k)}x^{(n)}(t) = f(t), & t \in [0,1], \\ x^{(i)}(0) = 0, & i = 0, \dots, k-1, \\ x^{(j)}(1) = 0, & j = k, \dots, n-1, \end{cases}$$

has a unique solution $x(t) = \int_0^1 G(t,s)f(s)\,ds$, $t \in [0,1]$, where Green's function $G(t,s)$ is defined by the equality [20]

$$G(t,s) = \frac{1}{(n-k-1)!}\frac{1}{(k-1)!}\int_0^{\min(t,s)} (s-\tau)^{n-k-1}(t-\tau)^{k-1}\,d\tau, \quad t,s \in [0,1]. \tag{1.2}$$

Note, that the function $G(t,s)$ is an oscillating kernel by the Kalafaty–Gantmacher–Krein Theorem [17] (see also [18,19,22,34]), therefore, in particular, the inequality

$$\begin{vmatrix} G(\tau_1,s_1) & G(\tau_1,s_2) \\ G(\tau_2,s_1) & G(\tau_2,s_2) \end{vmatrix} > 0 \tag{1.3}$$

holds for all $0 < \tau_1 < \tau_2 \le 1$, $0 < s_1 < s_2 \le 1$. Problem (1.1) enjoys the Fredholm property [8, Ch. 2]. Thus, if the homogeneous problem has only a trivial solution, then problem (1.1) has a unique solution for all $f \in \mathbf{L}[0,1]$.

Obviously, boundary value problem (1.1) is equivalent to the equation

$$x(t) = -\int_0^1 G(t,s)(Tx)(s)\,ds + \int_0^1 G(t,s)f(s)\,ds, \quad t \in [0,1]. \tag{1.4}$$

Applying some fixed point theorems, for example, the classical methods for estimating the norm of the operator $G : \mathbf{C}[0,1] \to \mathbf{C}[0,1]$ defined by the equality

$$(Gx)(t) = -\int_0^1 G(t,s)(Tx)(s)\,ds, \quad t \in [0,1],$$

one can obtain various unique solvability conditions for problem (1.1).

Conditions for the solvability of focal boundary value problems for higher-order differential equations were obtained in the works by R. Agarwal [1,4], R. Agarwal and I. Kiguradze [3], and others [5,6,15,20,21,23,28,29,31,32,35,36,38]. As for those conditions as applied to the linear higher-order functional differential equations, among the results related to the norm of the operator T, the author does not know of any that would significantly improve the following.

Denote

$$\widetilde{\mathcal{T}}_{n,k} \equiv (n-1)(n-k-1)!(k-1)!$$

Proposition 1.1. *Problem* (1.1) *is uniquely solvable if*

$$\|T\|_{\mathbf{C}\to\mathbf{L}} \le \widetilde{\mathcal{T}}_{n,k}. \tag{1.5}$$

Proof. We have

$$G(1,1) = \frac{1}{\widetilde{\mathcal{T}}_{n,k}} > G(t,s) \geq 0$$

for all $(t,s) \in [0,1] \times [0,1]$, $(t,s) \neq (1,1)$. Therefore, if the condition of the statement is fulfilled, then for any non-zero solution x to equation (1.4) for $f \equiv 0$ the following inequalities hold:

$$\begin{aligned} |x(t)| = \left| \int_0^1 G(t,s)(Tx)(s)\,ds \right| &< G(1,1) \int_0^1 |(Tx)(s)|\,ds \\ &\leq G(1,1) \| T \|_{\mathbf{C}\to\mathbf{L}} \| x \|_{\mathbf{C}} \leq \| x \|_{\mathbf{C}} \quad \text{for all} \quad t \in [0,1]. \end{aligned}$$

Since the continuous function $|x(t)|$ has a maximal value at a corresponding point $t^\star \in [0,1]$, the inequality $|x(t^\star)| < \|x\|_{\mathbf{C}}$ is impossible. It follows that the homogeneous boundary value problem has only the trivial solution. Therefore, the Fredholm boundary value problem (1.1) is uniquely solvable. □

Examples show that the constant in the right-hand side of inequality (1.5) is unimpovable. Let us define a linear bounded operator $T_\theta : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$, $\theta \in (0,1)$, by the equality

$$(T_\theta x)(t) = \begin{cases} 0, & t \in [0,\theta], \\ -\frac{x(1)}{\int_\theta^1 G(1,s)\,ds}, & t \in (\theta,1]. \end{cases}$$

Homogeneous problem (1.1) for $T = T_\theta$ and $f \equiv 0$ has a non-trivial solution

$$x(t) = \int_\theta^1 G(t,s)\,ds, \quad t \in [0,1].$$

Therefore, this problem isn't uniquely solvable. Since

$$\lim_{\theta\to 1-} \| T_\theta \|_{\mathbf{C}\to\mathbf{L}} = \lim_{\theta\to 1-} \frac{1-\theta}{\int_\theta^1 G(1,s)\,ds} = \widetilde{\mathcal{T}}_{n,k},$$

for every $\varepsilon > 0$ there exists a linear bounded operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with $\| T \|_{\mathbf{C}\to\mathbf{L}} = \widetilde{\mathcal{T}}_{n,k} + \varepsilon$ such that problem (1.1) isn't uniquely solvable.

However, it was shown in [24–26] that for certain monotone functional operators and for some boundary value problems, the solvability conditions based on contraction mapping principle can be essentially weakened.

An operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is called positive if it maps non-negative functions from $\mathbf{C}[0,1]$ to almost everywhere non-negative functions from $\mathbf{L}[0,1]$. The norm of such an operator is defined by the equality $\| T \|_{\mathbf{C}\to\mathbf{L}} = \int_0^1 (T\mathbf{1})(t)\,dt$, where $\mathbf{1}(t) = 1$, $t \in [0,1]$, is the unit function. For $p \in \mathbf{L}[0,1]$ and a measurable function $h : [0,1] \to [0,1]$, the operator

$$(Tx)(t) = p(t)x(h(t)), \quad t \in [0,1],$$

is positive if the function $p \in \mathbf{L}[0,1]$ is non-negative. Its norm equals $\| T \|_{\mathbf{C}\to\mathbf{L}} = \int_0^1 p(t)\,dt$.

This work is devoted to weakening the solvability conditions (1.5) for problem (1.1) with positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$. We obtain a necessary and sufficient condition for the focal boundary value problem (1.1) to be uniquely solvable for all positive operators T with a given norm.

For some other boundary value problems, similar unimprovable conditions are obtained by R. Hakl, A. Lomtatidze, S. Mukhigulashvili, B. Půža, J. Šremr, and others [10,16,24–27,30].

2 Main results

Theorem 2.1. *Let a non-negative number $\mathcal{T}$ be given. Problem (1.1) is uniquely solvable for all positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with norm $\mathcal{T}$ if and only if*

$$\mathcal{T} \leq \min_{0<t<1,\ 0<s<1} \frac{G(t,1)+G(1,s)+2\sqrt{G(t,s)G(1,1)}}{G(t,s)G(1,1)-G(t,1)G(1,s)} \equiv \mathcal{T}_{n,k}.$$

Taking into account (1.3), the constants $\mathcal{T}_{n,k}$ are well-defined. Green's function $G(t,s)$ has explicit representation (1.2), therefore, the best constant $\mathcal{T}_{n,k}$ from the solvability conditions can be easily calculated approximately. Note, since Green's functions of corresponding problems are symmetric, we have

$$\mathcal{T}_{n,k} = \mathcal{T}_{n,n-k}.$$

In some cases, the constants are calculated exactly. In particular, $\mathcal{T}_{2,1}$, $\mathcal{T}_{4,2}$, $\mathcal{T}_{6,3}$ are obtained in Example 3.3, and the constant $\mathcal{T}_{3,1}$ is obtained in Example 3.9. For even n in Theorem 3.2, the constants $\mathcal{T}_{n,n/2}$ are represented using one-dimensional minimization. In Corollaries 3.5, 3.6, asymptotically unimprovable estimates for $\mathcal{T}_{n,n/2}$ are obtained.

The proof of Theorem 2.1 is based on the following assertion [11, Theorem 2.28, p. 106] (see also a similar proof in [12]).

Proposition 2.2 ([11,12])**.** *Let $\mathcal{T}$ be a non-negative number. Problem (1.1) is uniquely solvable for all positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with norm $\mathcal{T}$ if and only if for all numbers c, d, τ_1, τ_2, $\mathcal{T}_1$, $\mathcal{T}_2$ satisfying the conditions*

$$c,d \in [0,1], \quad 0 \leq \tau_1 \leq \tau_2 \leq 1, \quad \mathcal{T}_1 \geq 0, \quad \mathcal{T}_2 \geq 0, \quad \mathcal{T}_1 + \mathcal{T}_2 \leq \mathcal{T}, \tag{2.1}$$

the inequality

$$\begin{aligned} \Delta &\equiv \Delta(\tau_1,\tau_2,c,d,\mathcal{T}_1,\mathcal{T}_2) \\ &\equiv 1+\mathcal{T}_1 G(\tau_1,c)+\mathcal{T}_2 G(\tau_2,d)+\mathcal{T}_1\mathcal{T}_2(G(\tau_1,c)G(\tau_2,d)-G(\tau_2,c)G(\tau_1,d)) \geq 0 \end{aligned} \tag{2.2}$$

holds.

Proof of Theorem 2.1. We will use Proposition 2.2. Let

$$R \equiv G(\tau_1,c)G(\tau_2,d)-G(\tau_2,c)G(\tau_1,d).$$

If $R \geq 0$, then $\Delta = 1+\mathcal{T}_1 G(\tau_1,c)+\mathcal{T}_2 G(\tau_2,d)+\mathcal{T}_1\mathcal{T}_2 R > 0$.

Let further $R < 0$ and $0 < \tau_1 < \tau_2 < 1$. From (1.3) and $R < 0$ it follows that

$$0 < d < c \leq 1. \tag{2.3}$$

For fixed points τ_1, τ_2, c, d, and $\mathcal{T}_1$, Δ takes its minimum at $\mathcal{T}_2 = \mathcal{T} - \mathcal{T}_1$ or at $\mathcal{T}_2 = 0$. In the latter case, $\Delta = 1+\mathcal{T}_1 G(\tau_1,c) \geq 1$.

Thus, the inequality (2.2) should be verified only at $\mathcal{T}_2 = \mathcal{T} - \mathcal{T}_1$ for all $\mathcal{T}_1 \in [0,\mathcal{T}]$. In this case, we have

$$\begin{aligned} \Delta &\equiv \Delta(\tau_1,\tau_2,c,d,\mathcal{T}_1) \\ &\equiv 1+\mathcal{T}_1 G(\tau_1,c)+(1-\mathcal{T}_1)G(\tau_2,d)+\mathcal{T}_1(1-\mathcal{T}_1)R \\ &= -\mathcal{T}_1^2 R+\mathcal{T}_1(G(\tau_1,c)-G(\tau_2,d)+\mathcal{T}R)+1+\mathcal{T}G(\tau_2,d). \end{aligned}$$

Let us find the minimum of this value in the variable $\mathcal{T}_1$ at fixed values of other variables.

Denote $B \equiv G(\tau_1, c) - G(\tau_2, d)$.

If $|B/R| > \mathcal{T}$, then the value Δ takes its minimum on $\mathcal{T}_1 \in [0, \mathcal{T}]$ at $\mathcal{T}_1 = 0$ or $\mathcal{T}_1 = \mathcal{T}$. In the first case, we have $\Delta = 1 + \mathcal{T} G(\tau_2, d) \geq 1$, in the second one, $\Delta = 1 + \mathcal{T} G(\tau_1, c) \geq 1$.

If $|B/R| \leq \mathcal{T}$, then the minimum of Δ occurs at

$$\mathcal{T}_1 = \frac{G(\tau_1, c) - G(\tau_2, d) + \mathcal{T} R}{2R} \equiv \frac{\mathcal{T} + B/R}{2}.$$

This minimum value is equal to

$$\Delta_{\min} = \frac{R}{4}\mathcal{T}^2 + \mathcal{T}\left(\frac{B}{2} + G(\tau_2, d)\right) + 1 + \frac{B^2}{4R},$$

therefore, $\Delta_{\min} \geq 0$ if and only if the following inequalities hold:

$$Q(\tau_1, \tau_2, c, d) \leq \mathcal{T} \leq S(\tau_1, \tau_2, c, d),$$

where

$$Q(\tau_1, \tau_2, c, d) \equiv \frac{G(\tau_1, c) + G(\tau_2, d) - 2\sqrt{G(\tau_1, d) G(\tau_2, c)}}{|R|},$$

$$S(\tau_1, \tau_2, c, d) \equiv \frac{G(\tau_1, c) + G(\tau_2, d) + 2\sqrt{G(\tau_1, d) G(\tau_2, c)}}{|R|}.$$

From the inequality (1.3) for $s_1 = d$ and $s_2 = c$ it follows that

$$\frac{G(\tau_1, c) + G(\tau_2, d) - 2\sqrt{G(\tau_1, d) G(\tau_2, c)}}{|R|} \leq \frac{|G(\tau_1, c) - G(\tau_2, d)|}{|R|} \leq \frac{|B|}{|R|} \leq \mathcal{T}.$$

Therefore, inequality (2.2) is satisfied for all parameters satisfying the conditions (2.1) if and only if

$$\mathcal{T} \leq \min_{\substack{0 \leq \tau_1 \leq \tau_2 \leq 1 \\ c, d \in [0,1],\ R < 0}} S(\tau_1, \tau_2, d, c) \equiv \widetilde{\mathcal{T}}.$$

Since (2.3), we have

$$\widetilde{\mathcal{T}} = \min_{\substack{0 < \tau_1 < \tau_2 \leq 1 \\ 0 < d < c \leq 1}} S(\tau_1, \tau_2, d, c).$$

Our aim is to simplify the expression for evaluating $\widetilde{\mathcal{T}}$.

For $0 \leq \tau_1 \leq \tau_2 \leq 1$, $0 < d < c \leq 1$, we prove that

$$S'_{\tau_2}(\tau_1, \tau_2, d, c) = \frac{1}{R^2}\left(\frac{G'_{\tau_2}(\tau_2, d)}{G(\tau_2, d)} A - \frac{G'_{\tau_2}(\tau_2, c)}{G(\tau_2, c)} B\right) \leq 0, \tag{2.4}$$

where

$$A = G(\tau_1, c)^2 G(\tau_2, d) + G(\tau_1, d) G(\tau_2, d) G(\tau_2, c) + 2G(\tau_1, c) G(\tau_2, d)\sqrt{G(\tau_1, d) G(\tau_2, c)},$$

$$\begin{aligned} B = {} & G(\tau_1, c) G(\tau_1, d) G(\tau_2, c) + G(\tau_1, d) G(\tau_2, d) G(\tau_2, c) \\ & + \left(G(\tau_1, c) G(\tau_2, d) + G(\tau_1, d) G(\tau_2, c)\right)\sqrt{G(\tau_1, d) G(\tau_2, c)}. \end{aligned}$$

Since the function $G(t,s)$ is an oscillating kernel, we easily see that $B \geq A \geq 0$. Indeed, we have

$$B - A = (G(\tau_1,c) + \sqrt{G(\tau_1,d)G(\tau_2,c)})(G(\tau_1,d)G(\tau_2,c) - G(\tau_1,c)G(\tau_2,d)) \geq 0.$$

Let us prove that for each $t \in (0,1]$ the function $\frac{G_t'(t,s)}{G(t,s)}$ does not decrease in the second argument for $s \in (0,1]$. It suffices to show that for all $0 < t_1 < t_2 \leq 1$, $0 < s_1 < s_2 \leq 1$, the inequality

$$\frac{G(t_2,s_2) - G(t_1,s_2)}{G(t_1,s_2)} \geq \frac{G(t_2,s_1) - G(t_1,s_1)}{G(t_1,s_1)}$$

holds. This inequality is a direct consequence of the inequality (1.3). It follows that inequality $B \geq A$ implies inequality (2.4).

Similarly, it is verified that for $0 \leq \tau_1 \leq \tau_2 \leq 1$, $0 < d < c \leq 1$, the inequality

$$S_c'(\tau_1,\tau_2,d,c) \leq 0 \tag{2.5}$$

holds. From (2.4) and (2.5) it follows that in (2.5) the value $\widetilde{\mathcal{T}}$ has the minimum point at $\tau_2 = 1$ and $c = 1$. This implies the assertion of the theorem. □

3 Consequences

For calculating the constants $\mathcal{T}_{n,n/2}$, we need the following lemma, a technical proof of which was carried out in the paper [13].

Lemma 3.1. *Let $n = 2k$. Then the function*

$$M(t,s) = \sqrt{G(t,s)G(1,1)} - \sqrt{G(t,1)G(s,1)}, \quad t,s \in [0,1],$$

has its maximum value at $t = s$.

Let us show that for even n to calculate the constants $\mathcal{T}_{n,n/2}$, it is sufficient to solve an one-dimensional optimization problem.

Theorem 3.2. *Let a non-negative number $\mathcal{T}$ and $n = 2k$ be given. Problem (1.1) is uniquely solvable for all positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with the norm $\mathcal{T}$ if and only if*

$$\mathcal{T} \leq \frac{2\left((n/2-1)!\right)^2}{\max\limits_{0<t<1}\left(\frac{t^{(n-1)/2}}{n-1} - \int_0^t (t-\tau)^{n/2-1}(1-\tau)^{n/2-1}\,d\tau\right)} \equiv \mathcal{T}_{n,n/2}. \tag{3.1}$$

Proof. Let us use the Theorem 2.1. We have

$$\frac{G(t,1) + G(1,s) + 2\sqrt{G(t,s)G(1,1)}}{G(t,s)G(1,1) - G(t,1)G(1,s)}$$
$$= \frac{(\sqrt{G(t,1)} - \sqrt{G(1,s)})^2}{G(t,s)G(1,1) - G(t,1)G(1,s)} + \frac{2}{\sqrt{G(t,s)G(1,1)} - \sqrt{G(t,1)G(1,s)}}.$$

It follows that if $t_0 = s_0$ and the point $(t,s) = (t_0,s_0)$ is the minimum point of the function

$$\frac{2}{\sqrt{G(t,s)G(1,1)} - \sqrt{G(t,1)G(1,s)}},$$

then the minimum of the value expressing the exact estimate of the norm of the operator T under the conditions of the Theorem 2.1 will be taken at this point.

Lemma 3.1 implies that for $n = 2k$ the minimum under the conditions of Theorem 2.1 is taken namely at $s = t$. Calculating $G(t,t)$ and $G(t,1)$ using representation (1.2), we obtain the assertion of the theorem. □

Example 3.3. Under the conditions of the Theorem 3.2 for $n = 2$, $n = 4$, and $n = 6$ the values $\mathcal{T}_{n,n/2}$ are calculated exactly. We have

$$\mathcal{T}_{2,1} = 8$$

(the maximum in the representation of $\mathcal{T}_{2,1}$ (3.1) occurs at $t_2 = 1/4$);

$$\mathcal{T}_{4,2} = 66 + 30\sqrt{5},$$

(the maximum in the representation of $\mathcal{T}_{4,2}$ (3.1) occurs at $t_4 = \frac{3-\sqrt{5}}{2}$);

$$\mathcal{T}_{6,3} = 120\frac{2t_6^3 - 10t_6^2 + 20t_6 + 12\sqrt{t_6}}{t_6^3(1-t_6)(t_6^4 - 9t_6^3 + 36t_6^2 - 64t_6^1 + 36)} \approx 2610,$$

where the point of the maximum t_6 in representation (3.1) of $\mathcal{T}_{6,3}$ is defined by the equalities

$$t_6 = ((C - 1 - \sqrt{27 - C^2 + 22/C})/4)^2 \approx 0.49,$$

$$C = \sqrt{2(124 + 4\sqrt{97})^{1/3} + 9 + 48(124 + 4\sqrt{97})^{-1/3}}.$$

Remark 3.4. Apparently only the constant

$$\mathcal{T}_{2,1} = 8 \tag{3.2}$$

was previously known. In particular, equality (3.2) follows from the results of the work [33] on the solvability of two-dimensional systems functional differential equations. The solvability conditions associated with the rest of the found constants $\mathcal{T}_{n,k}$ are new.

For even $n \geq 8$, we obtain sufficient conditions for solvability (lower bounds for the constants $\mathcal{T}_{n,n/2}$).

Corollary 3.5. *Let $n = 2k \geq 8$ and a linear operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be positive. If*

$$\|T\|_{\mathbf{C}\to\mathbf{L}} \leq \frac{(n^2-9)(n^2-1)\left((n/2-1)!\right)^2}{3 + (n-2)\left(\frac{n-7}{n-3}\right)^{\frac{n+1}{2}}},$$

then problem (1.1) is uniquely solvable.

Corollary 3.6. *Let $n = 2k \geq 8$ and a linear operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be positive. If*

$$\|T\|_{\mathbf{C}\to\mathbf{L}} \leq e^2(n-3)^3\left((n/2-1)!\right)^2, \tag{3.3}$$

then problem (1.1) is uniquely solvable.

Remark 3.7. In (3.3), the constant e^2 and the exponent 3 are sharp.

Proof of Corollary 3.5. Let us introduce the notation

$$y_n(t) \equiv \frac{t^{(n-1)/2}}{n-1} - \int_0^t (t-\tau)^{n/2-1}(1-\tau)^{n/2-1}\,d\tau,$$

$$Y_n \equiv \max_{0<t<1} y_n(t), \quad \mathcal{T}_n \equiv \mathcal{T}_{n,n/2}.$$

By Theorem 3.2, it is obvious that

$$\mathcal{T}_n \equiv \frac{2\left((n/2-1)!\right)^2}{Y_n}.$$

We obtain the estimate $\widehat{Y}_n \geq Y_n$. Then

$$\mathcal{T}_n \geq \widehat{\mathcal{T}}_n \equiv \frac{2\left((n/2-1)!\right)^2}{\widehat{Y}_n},$$

therefore, the condition $\mathcal{T} \leq \widehat{\mathcal{T}}_n$ ensures the unique solvability of the problem (1.1) for each positive operator T with given norm $\mathcal{T}$.

It is convenient to present y_n' using the hypergeometric function ${}_2F_1$ [9, p. 69]:

$$\begin{aligned} y_n'(t) &= \frac{t^{(n-3)/2}}{2} - (n/2-1)\int_0^t (t-\tau)^{n/2-2}(1-\tau)^{n/2-1}\,d\tau \\ &= \frac{t^{(n-3)/2}}{2} - (n/2-1)t^{n/2-1}\int_0^1 (1-\theta)^{n/2-2}(1-t\theta)^{n/2-1}\,d\theta \\ &= \frac{t^{(n-3)/2}}{2} - t^{n/2-1}\,{}_2F_1(1-n/2,1;n/2;t) \equiv \frac{t^{(n-3)/2}}{2} z_n(t), \end{aligned} \tag{3.4}$$

where

$$z_n(t) \equiv 1 - 2\sqrt{t}\,{}_2F_1(1,1-n/2;n/2;t).$$

Further, for the hypergeometric function, the following properties will be used (it is obvious that in our case the hypergeometric function is a polynomial, moreover, we only need real parameters and a real argument). [9, p. 71–72] :

$$\frac{d^m}{dt^m}\,{}_2F_1(a,b;c;t) = \frac{(a)_m(b)_m}{(c)_m}\,{}_2F_1(a+m,b+m;c+m;t), \quad t\in[0,1],$$

$$(a)_m = a(a+1)\cdot\ldots\cdot(a+m-1), \quad m=1,2,3,\ldots,\ (a)_0=1,$$

$${}_2F_1(a,b;c;t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{\theta^{b-1}(1-\theta)^{c-b-1}}{(1-t\theta)^{c-b-1}}\,\theta \quad t\in[0,1],\ c>b>0. \tag{3.5}$$

Estimating $z_n(t)$, we obtain an approximation for $y_n'(t)$. Let

$$\widehat{z}_n(t) \equiv (t-1)\left(\frac{1}{2(n-3)} + \frac{t-1}{8}\right).$$

Lemma 3.8. *For every $n \geq 8$, the inequality*

$$z_n(t) \geq \widehat{z}_n(t), \quad t\in[0,1], \tag{3.6}$$

holds.

Proof. It suffices to show that

$$H_n(t) \equiv {}_2F_1(1, 1-n/2; n/2; t) \le \frac{1+(1-t)\left(\frac{1}{2(n-3)}+\frac{t-1}{8}\right)}{2\sqrt{t}} \equiv Z_n(t), \quad t \in (0,1]. \tag{3.7}$$

We have

$$Z_n(1) = H_n(1) = 1/2, \quad Z_n'(1) = H_n'(1) = -\frac{n-2}{4(n-3)}, \quad Z_n''(1) = H_n'(1) = \frac{n-2}{4(n-3)}.$$

To prove (3.7), it is now sufficient to prove that for all $t \in (0,1]$

$$H_n'''(t) = \frac{6\,(1-\frac{n}{2})_3}{(\frac{n}{2})_3}\, {}_2F_1(4, 4-n/2; n/2+3; t) \ge Z_n'''(t) = \frac{3(n(t^2+2t-35)+3t^2-10t+85)}{128(n-3)t^{7/2}}.$$

It remains to verify the chain of the inequalities

$$H_n'''(t) \ge w_0(t) \ge w_1(t) \ge w_2(t) \ge Z_n'''(t), \quad t \in (0,1], \tag{3.8}$$

where

$$w_0(t) \equiv H_n'''(0) + t(H_n'''(1) - H_n'''(0)),$$

$$H_n'''(0) = -6\frac{(n/2-3)(n/2-2)(n/2-1)}{(n/2)(n/2+1)(n/2+2)}, \quad H_n'''(1) = -6\frac{(n/2-1)^2(n/2-2)(n/2-3)}{(n-5)(n-4)(n-3)(n-2)},$$

$$w_1(t) \equiv \frac{45}{8}t - 6, \quad w_2(t) = \frac{3(t^2+2t-35)}{128t^{7/2}}.$$

To prove the first inequality in (3.8), we use the equality [9, p. 71]

$$H_n^{(m)}(t) = \frac{(1-n/2)_m(1)_m}{(n/2)_m}\, {}_2F_1(1-n/2+m, 1+m; n/2+m; t),$$

from which it follows that the sign of the function $H_n^{(m)}(t)$ coincides with $(-1)^m$, in particular, for $m=3$, $m=4$, $m=5$ (it is also taken into account that due to the integral representation (3.5) [9, p. 72] the function ${}_2F_1(1-n/2+m, 1+m; n/2+m; t)$ is non-negative. The rest inequalities can be verified directly. □

Define the function $\widehat{y}_n$ by the equality

$$\widehat{y}_n(t) \equiv -\frac{1}{2}\int_t^1 s^{\frac{n-3}{2}} \widehat{z}_n(s)\, ds, t \in (0,1].$$

It is clear that $\widehat{y}_n(1) = y_n(1) = 0$. From (3.4) and (3.6) it follows that

$$\widehat{y}_n(t) \ge y_n(t), \quad t \in [0,1].$$

Its maximum $\widehat{Y}_n \ge Y_n$ the function $\widehat{y}_n(t)$ takes at the point $t_n \in (0,1)$ defined by the equality

$$\widehat{y}_n'(t_n) = \frac{t_n^{\frac{n-3}{2}}}{2}\widehat{z}_n(t_n) = 0.$$

therefore, we get

$$t_n = \frac{n-7}{n-3},$$

$$\widehat{Y}_n = \int_{t_n}^1 \frac{s^{\frac{n-3}{2}}}{2}(1-s)\left(\frac{1}{2(n-3)}+\frac{s-1}{8}\right) ds = \frac{6+2(n-2)\left(\frac{n-7}{n-3}\right)^{n/2+1/2}}{(n^2-9)(n^2-1)}.$$

This implies the assertion of Corollary 3.5. □

Proof of Corollary 3.6. It is easy to see that

$$\lim_{n\to\infty}(n-3)^3\widehat{Y}_n = \frac{2}{e^2}.$$

Moreover, $(n-3)^3\widehat{Y}_n < \frac{2}{e^2}$ for all $n \geq 8$. Thus, the statement of Corollary 3.6 is also true. □

Example 3.9. Consider problem (1.1) for the third-order equation for $k = 1$

$$\begin{cases} x'''(t) + (Tx)(t) = f(t), \quad t \in [0,1], \\ x(0) = 0, \\ x'(1) = 0, \quad x''(1) = 0, \end{cases} \tag{3.9}$$

By Theorem 2.1 problems (3.9) is uniquely solvable for all positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with the norm $\mathcal{T}$ if and only if

$$\mathcal{T} \leq \min_{0<s\leq t<1} 2\frac{t^2 - s^2 + 2s + 2\sqrt{(2t-s)s}}{s(1-t)(2t-s-st)} = 6(3+2\sqrt{3}) \approx 38.8.$$

Note, the minimum occurs at $s = (3-\sqrt{3})/6$, $t = (3-\sqrt{3})/3$.

For each $\varepsilon > 0$, there is a positive operator with the norm $6(3+2\sqrt{3}) + \varepsilon$, for which problem (3.9) is not uniquely solvable.

Proposition 1.1 only allows us to claim that problems (3.9) is uniquely solvable if the norm of the operator T is less than or equal to two.

Example 3.10. It is clear that the constant $\mathcal{T}_{n,k}$ from the necessary and sufficient conditions of Theorem 2.1 is equal or greater than the constants $\widetilde{\mathcal{T}}_{n,k}$ from Preposition 1.1. With the help of approximate computation, we make the following table containing the integer parts of the quotients $\mathcal{T}_{n,k}/\widetilde{\mathcal{T}}_{n,k}$, which shows how the classical results are improved by Theorem 2.1:

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=2$	8				
$n=3$	19				
$n=4$	31	44			
$n=5$	42	75			
$n=6$	54	109	130		
$n=7$	66	145	190		
$n=8$	78	184	255	275	
$n=9$	90	226	326	366	
$n=10$	101	269	404	464	481

Every element of this table shows approximately how many times the conditions of Theorem 2.1 are weaker than in Proposition 1.1 for given n and k, and gives a sufficient solvability conditions for corresponding problem (1.1). Formulate, for example, one such sufficient condition.

Proposition 3.11. *For $n = 10$ and $k = 1$ problem (1.1) is uniquely solvable if $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is a linear positive operator and $\int_0^1 (T\mathbf{1})(t)\,dt \leq 101 \cdot 9!$. There exists a linear positive operator T with $\int_0^1 (T\mathbf{1})(t)\,dt \geq 102 \cdot 9!$ such that problem (1.1) isn't uniquely solvable.*

Acknowledgements

The author thanks the anonymous referee for her/his comments and valuable suggestions. This work was supported by the Russian Foundation for Basic Research (Project 18-01-00332) and was performed as part of the State Task of the Ministry of Science and Higher Education of the Russian Federation (project FSNM-2020-0028).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **23**, 1–31; https://doi.org/10.14232/ejqtde.2021.1.23 www.math.u-szeged.hu/ejqtde/

Periodic stationary solutions of the Nagumo lattice differential equation: existence regions and their number

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Received 3 September 2020, appeared 30 March 2021
Communicated by Sergei Trofimchuk

Abstract. The Nagumo lattice differential equation admits stationary solutions with arbitrary spatial period for sufficiently small diffusion rate. The continuation from the stationary solutions of the decoupled system (a system of isolated nodes) is used to determine their types; the solutions are labelled by words from a three-letter alphabet. Each stationary solution type can be assigned a parameter region in which the solution can be uniquely identified. Numerous symmetries present in the equation cause some of the regions to have identical or similar shape. With the help of combinatorial enumeration, we derive formulas determining the number of qualitatively different existence regions. We also discuss possible extensions to other systems with more general nonlinear terms and/or spatial structure.

Keywords: reaction-diffusion equation, lattice differential equation, graph differential equation, stationary solutions, enumeration, symmetry groups.

2020 Mathematics Subject Classification: 34A33, 39A12, 05A05, 34B45.

1 Introduction

In this paper, we consider the Nagumo lattice differential equation (LDE)

$$u_i'(t) = d\big(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\big) + f\big(u_i(t); a\big) \tag{1.1}$$

for $i \in \mathbb{Z}, t > 0$ with $d > 0$, where the nonlinear term f is given by

$$f(s; a) = s(1-s)(s-a), \tag{1.2}$$

with $a \in (0, 1)$. The LDE (1.1) is used as a prototype bistable equation arising from the modelling of a nerve impulse propagation in a myelinated axon [4]. The bistable equations have their use in modelling of active transmission lines [32, 33], cardiophysiology [3], neurophysiology [4], nonlinear optics [25], population dynamics [28] and other fields.

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Throughout this paper, we shall use correspondence of the LDE (1.1) and the Nagumo graph differential equation on a cycle (1.6). The graph and lattice reaction-diffusion differential equations are used in modelling of dynamical systems whose spatial structure is not continuous but can be described by individual vertices (possibly infinitely many) and their interaction via edges. The main difference is such that a lattice (the underlying structure of (1.1)) is infinite but there are strong assumptions on its regularity whereas graphs are usually (but not exclusively) finite and nothing is assumed about their structure in general. Such models arise in population dynamics [1], image processing [29], chemistry [27], epidemiology [26] and other fields. Alternative focus lies in the numerical analysis where the graph differential equations describe spatial discretizations of partial differential equations [17, 23]. Mathematically, the interaction between analytic and graph theoretic properties represent new and interesting challenges. The graph and lattice reaction-diffusion differential equations exhibit behaviour which cannot be observed in their partial differential equation counterparts such as a rich structure of stationary solutions [36], or other phenomena described in the forthcoming text such as pinning, multichromatic waves and other.

The LDE (1.1) is known to possess travelling wave solutions of the form

$$\begin{gathered} u_i(t) = \varphi(i - ct), \\ \lim_{s\to-\infty} \varphi(s) = 0, \quad \lim_{s\to+\infty} \varphi(s) = 1. \end{gathered} \tag{1.3}$$

As the authors in [24] and [38] had shown, there are nontrivial parameter (a, d)-regimes preventing the solutions of type (1.3) from travelling ($c = 0$) creating the so-called *pinning region*. This propagation failure phenomenon can be partially clarified by the existence of countably many stable stationary solutions (including the periodic ones) of (1.1) which inhabit mainly the pinning region, see Figure 1.1. This pinning phenomenon was observed in other lattice systems [10], experimentally in chemistry [27] and also hinted in systems of coupled oscillators [6]. It is worth mentioning that the equation (1.1) can be obtained via spatial discretization of the Nagumo partial differential equation

$$u_t(x, t) = du_{xx}(x, t) + f\big(u(x, t); a\big),$$

which possesses travelling wave solutions of type

$$\begin{gathered} u(x, t) = \varphi(x - ct), \\ \lim_{s\to-\infty} \varphi(s) = 0, \quad \lim_{s\to+\infty} \varphi(s) = 1; \end{gathered} \tag{1.4}$$

the waves are pinned if and only if $\int_0^1 f(s; a)\,\mathrm{d}s = 0$.

The waves of type (1.3) (whether the travelling or the pinned ones) can be perceived as solutions connecting two homogeneous stable states of the LDE (1.1); constant 0 and constant 1. This concept can be generalized to the solutions connecting the nonhomogeneous periodic steady states. Let $\mathrm{u}, \mathrm{v} \in \mathbb{R}^n$ be two vectors such that their periodic extensions are asymptotically stable stationary solutions of the LDE (1.1). The multichromatic wave is then a solution of a form

$$\begin{gathered} u_i(t) = \phi(i - ct), \\ \lim_{s\to-\infty} \phi(s) = \mathrm{u}, \quad \lim_{s\to+\infty} \phi(s) = \mathrm{v}, \end{gathered} \tag{1.5}$$

where

$$\phi = (\phi_1, \phi_2, \dots, \phi_n) : \mathbb{R} \to \mathbb{R}^n.$$

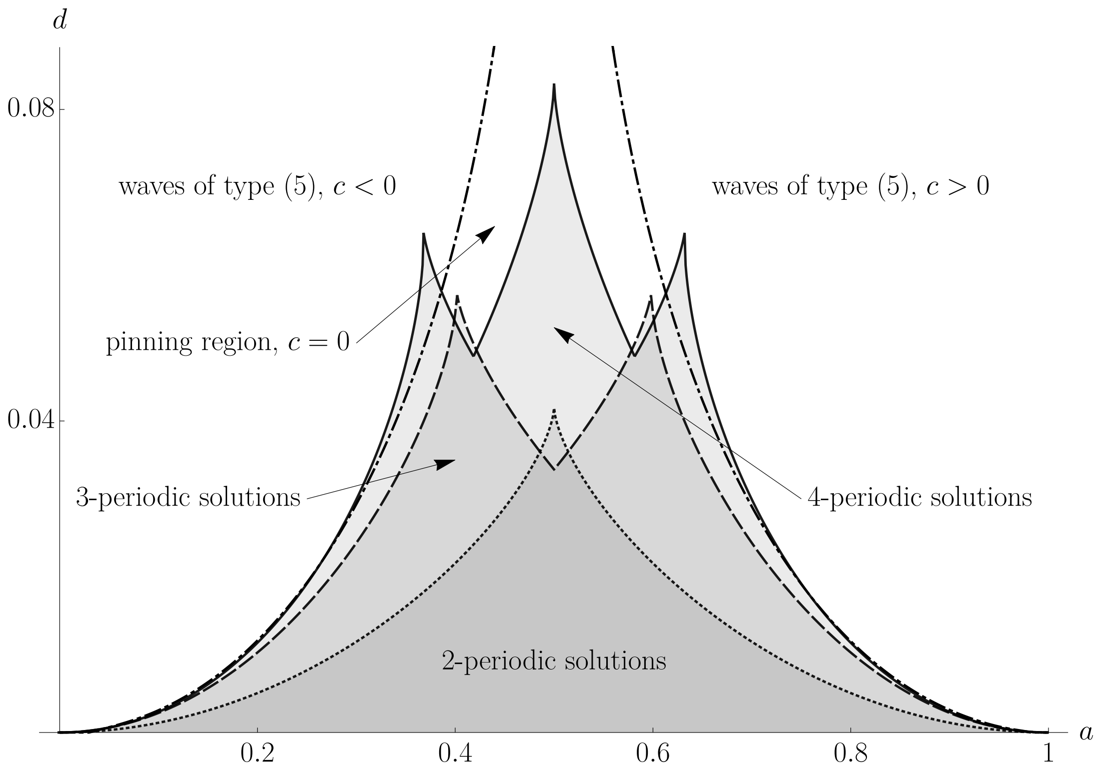


Figure 1.1: Numerically computed regions in the (a, d)-plane in which the waves of the type (1.3) travel (the regions above the two dot-dashed curves) and the pinning region (the region between the a-axis and the two dot-dashed curves). To better illustrate the significance and the presence of the stable heterogeneous n-periodic stationary solutions of the LDE (1.1) in the pinning region, we include the existence regions for the two-periodic stable stationary solutions (dotted edge), the three-periodic stable stationary solutions (dashed edge) and the four-periodic stable stationary solutions (solid edge).

The bichromatic waves connecting homogeneous and two-periodic solutions were examined in [19]. The tri- and quadrichromatic waves incorporating three- and four-periodic solutions were studied in detail in [20]. Stationary solutions with analogous construction idea, the *oscillatory plateaus* whose limits approach homogeneous steady states and there exists a middle section close to a periodic stationary solution, were analysed in [7].

Motivated by the importance of detailed understanding of the existence of the stationary solutions to the analysis of the advanced structures, the focal point of this paper is the examination of the (a, d)-regions in which particular periodic stationary solutions of the LDE (1.1) exist. Our aim is to derive counting formulas for inequivalent existence regions; the notion of equivalence is rigorously defined in the forthcoming section since it requires certain technical preliminaries. It is useful to have a detailed knowledge of the shape of the regions because of their connection to other phenomena. It has been shown in [19,20] that they are closely related to the travelling regions of the multichromatic waves. As simulations hint (see Figure 1.1), the regions corresponding to the stable stationary solutions seem to inhabit mainly the pinning region. Finally, we emphasize their obvious significance as the condition for emergence of spatial patterns in the LDE (1.1). To reach the goal, we employ the idea from [21] where we have shown a one-to-one correspondence of the LDE (1.1) n-periodic stationary solutions and stationary solutions of the Nagumo graph differential equation (GDE) on an n-vertex cycle

$$\begin{cases} \mathrm{u}_1'(t) = d\big(\mathrm{u}_n(t) - 2\mathrm{u}_1(t) + \mathrm{u}_2(t)\big) + f\big(\mathrm{u}_1(t); a\big), \\ \mathrm{u}_2'(t) = d\big(\mathrm{u}_1(t) - 2\mathrm{u}_2(t) + \mathrm{u}_3(t)\big) + f\big(\mathrm{u}_2(t); a\big), \\ \qquad \vdots \\ \mathrm{u}_i'(t) = d\big(\mathrm{u}_{i-1}(t) - 2\mathrm{u}_i(t) + \mathrm{u}_{i+1}(t)\big) + f\big(\mathrm{u}_i(t); a\big), \\ \qquad \vdots \\ \mathrm{u}_n'(t) = d\big(\mathrm{u}_{n-1}(t) - 2\mathrm{u}_n(t) + \mathrm{u}_1(t)\big) + f\big(\mathrm{u}_n(t); a\big), \end{cases} \tag{1.6}$$

and, subsequently, with vectors of length n having elements in the three letter alphabet $\mathcal{A}_3 = \{\mathfrak{0}, \mathfrak{a}, \mathfrak{1}\}$, also called the *words*. The words encode the origin of the bifurcation branches for $d = 0$ whose existence can be shown by using the implicit function theorem for $d > 0$ small enough. Moreover, the implicit function theorem also implies that the solutions preserve their stability and the asymptotically stable solutions can be thus identified with words created with the two letter alphabet $\mathcal{A}_2 = \{\mathfrak{0}, \mathfrak{1}\}$. The region in the (a, d)-space belonging to a solution labelled by a word w is denoted by $\Omega_{\mathrm{w}} \subset \mathcal{H} = [0, 1] \times \mathbb{R}^+$. Since the stationary problem for (1.6) is equivalent to the problem of searching for the roots of a 3^n-th order polynomial it is a convoluted task to derive some information about the regions. There are known lower estimates for their upper boundaries, [12], asymptotics near threshold points $a \approx 0$, $a \approx 1$ and numerical results, both [19, 20]. The computations and the numerical simulations can be cumbersome to carry out and thus the exploitation of the equation symmetries is beneficial. The idea is to observe, whether a symmetry present in the equation relates two regions Ω_{w} without any a-priori knowledge of their shapes. For example, the LDE (1.1) is invariant to an index shift and the GDE (1.6) is invariant to the rotation of indices. Consider $n = 3$, then given a parameter tuple $(a, d) \in \mathcal{H}$, if there exists a stationary solution u_1 of the GDE (1.6) emerging from $(0, 0, 1)$ for $d = 0$, then there surely exist solutions $\mathrm{u}_2, \mathrm{u}_3$ emerging from $(0, 1, 0)$, $(1, 0, 0)$, respectively. Moreover, $\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3$ have identical values which are just rotated by one element to the left. We can thus say that the regions of existence of the solutions emerging from $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$ are identical, i.e., $\Omega_{\mathfrak{001}} = \Omega_{\mathfrak{010}} = \Omega_{\mathfrak{100}}$.

We show how the symmetries of the LDE (1.1) and the GDE (1.6) correspond and how they propagate to the set of the labelling words $\mathcal{A}_3^n$. Namely, the index rotation $i \mapsto i + 1$, the reflection $i \mapsto n - i + 1$ create word subsets whose respective regions are identical. The value switch $\mathfrak{0} \leftrightarrow \mathfrak{1}$ relates solution types whose respective regions are axially symmetric to each other. To this end, we define groups acting on the set of the words $\mathcal{A}_3^n$ and compute the number of their orbits (the number of the word subsets which are pairwise unreachable by the action of the group) via Burnside's lemma, Theorem 2.6. We next restrict the computations to the words whose primitive period is equal to their length since the periodic extension of the GDE (1.6) stationary solution of a certain type (e.g., $\mathfrak{0a0a0a}$) is identical to a periodic extension of its subword with the length equal to the original word's primitive period ($\mathfrak{0a}$ here). The main tool is Möbius inversion formula in this case, Theorem 2.7. The division of the word set $\mathcal{A}_3^n$ into orbits with respect to the action of a group can be achieved with the cost proportional to the number of the words (3^n in this case), see [9]. Our results do not help with the generation of the representative words directly but enable us to easily determine their number. All results are also provided for asymptotically stable stationary solutions of the LDE (1.1) whose corresponding labelling set is $\mathcal{A}_2^n$.

The paper is organized as follows. In §2 we provide an overview of the properties of the periodic stationary solutions of the LDE (1.1) including the introduction of the labelling scheme and the statement of our main result, Theorem 2.9. We next include a list of relevant symmetries of the equation and their influence on the regions Ω_{w} and conclude with presentation of the used group-theoretical tools together with the commentary of the known results. Using the formal definitions from the preceding text, §3 is devoted to the derivation of lemmas needed for the proof of the main statement in §4. The final paragraphs elaborate on possible extensions to other models and we discuss open questions therein.

2 Preliminaries

2.1 Periodic stationary solutions and existence regions

Searching for a general stationary solution of the LDE (1.1) requires solving a countable system of nonlinear analytic equations. The restriction to periodic solutions simplifies the case to a finite-dimensional problem. Indeed, the problem is thus reduced to finding stationary solutions of the GDE (1.6).

Lemma 2.1 ([21, Lemma 1])**.** *Let $n \geq 3$. The vector $\mathrm{u} = (\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n)$ is a stationary solution of the GDE (1.6) if and only if its periodic extension u is an n-periodic stationary solution of the LDE (1.1). Moreover, u is an asymptotically stable solution of the GDE (1.6) if and only if u is an asymptotically stable solution of the LDE (1.1) with respect to the ℓ^∞-norm.*

If $\mathrm{u} = (\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n) \in \mathbb{R}^n$ is a vector then the periodic extension $(u_i)_{i\in\mathbb{Z}} \in \ell^\infty$ of u satisfies $u_i = \mathrm{u}_{1+\mathrm{mod}\,(i,n)}$ for all $i \in \mathbb{Z}$. In the further text, the function $\mathrm{mod}\,(a,b)$ denotes the remainder of the integer division of a/b for $a, b \in \mathbb{N}$.

Let us denote the function on the right-hand side of the GDE (1.6) by $h\colon \mathbb{R}^n \times (0,1) \times \mathbb{R}^+_0 \to \mathbb{R}^n$,

$$h(\mathrm{u}; a, d) = \begin{pmatrix} d\big(\mathrm{u}_n - 2\mathrm{u}_1 + \mathrm{u}_2\big) + f(\mathrm{u}_1; a) \\ \vdots \\ d\big(\mathrm{u}_{i-1} - 2\mathrm{u}_i + \mathrm{u}_{i+1}\big) + f(\mathrm{u}_i; a) \\ \vdots \\ d\big(\mathrm{u}_{n-1} - 2\mathrm{u}_n + \mathrm{u}_1\big) + f(\mathrm{u}_n; a) \end{pmatrix}. \tag{2.1}$$

The problem of finding a stationary solution of the GDE (1.6) can be now reformulated as

$$h(\mathrm{u}; a, d) = 0. \tag{2.2}$$

The problems of type (2.2), i.e., a diagonal nonlinear perturbation of a finite-dimensional linear operator, are being treated with a wide spectrum of methods ranging from variational techniques, topological approaches to monotone operator theory, see [37] and references therein. We derive some information about the system using the perturbation theory. Suppose $d = 0$, then the problem

$$h(\mathrm{u}; a, 0) = 0 \tag{2.3}$$

has precisely 3^n solutions $\mathrm{u} \in \mathbb{R}^n$ which are vectors of length n with the coordinates in the set $\{0, a, 1\}$; the system (2.3) contains n independent equations. There is also an easy way to determine the stability of the roots of (2.3). One can readily calculate that

$$f'(0; a) = -a, \qquad f'(a; a) = a(1-a), \qquad f'(1; a) = a - 1,$$

which gives $f'(s;a) < 0$ for either $s = 0$ or $s = 1$ and $f'(s;a) > 0$ for $s = a$. The derivative of the function h with respect to the first variable, $D_1 h(\mathrm{u}; a, 0)$, is a regular diagonal matrix at each solution of (2.3)

$$D_1 h(\mathrm{u}; a, 0) = \operatorname{diag}\big(f'(\mathrm{u}_1; a), f'(\mathrm{u}_2; a), \dots, f'(\mathrm{u}_n; a)\big).$$

If the solution vector contains the value a then it is an unstable stationary solution of the GDE (1.6) and it is stable otherwise for $d = 0$. Let some $a^* \in (0,1)$ be given. The implicit function theorem now ensures the existence of the solutions of the system (2.2) for $(a,d) \in \mathcal{U} \cap \mathcal{H}$, where $\mathcal{U}$ is some neighbourhood of the point $(a^*, 0)$. The parameter dependence is smooth and the sign of the Jacobian is preserved.

The discussion above justifies the introduction of the naming scheme for the roots of (2.2) where each solution is identified with the origin of its bifurcation branch at $d = 0$. It is important to realize that the parameter $a \in (0,1)$ is allowed to vary in our considerations. The identification must be made through the substitute alphabet $\mathcal{A}_3 = \{\mathfrak{o}, \mathfrak{a}, \mathfrak{1}\}$ and we define a function $\mathrm{w}_{|a} \colon \mathcal{A}_3^n \to \{0, a, 1\}^n$ for given $a \in [0,1]$ by

$$(\mathrm{w}_{|a})_i = \begin{cases} 0, & \mathrm{w}_i = \mathfrak{o}, \\ a, & \mathrm{w}_i = \mathfrak{a}, \\ 1, & \mathrm{w}_i = \mathfrak{1}. \end{cases}$$

Definition 2.2 ([20, Definition 2.1])**.** Consider a word $\mathrm{w} \in \mathcal{A}_3^n$ together with a triplet

$$(\mathrm{u}, a, d) \in [0,1]^n \times (0,1) \times \mathbb{R}_0^+.$$

Then we say that u is an equilibrium of the type w if there exists a C^1-smooth curve

$$[0,1] \ni t \mapsto \big(\mathrm{v}(t), \alpha(t), \delta(t)\big) \in [0,1]^n \times (0,1) \times \mathbb{R}_0^+$$

so that we have

$$\begin{aligned} (\mathrm{v}, \alpha, \delta)(0) &= (\mathrm{w}_{|a}, a, 0), \\ (\mathrm{v}, \alpha, \delta)(1) &= (\mathrm{u}, a, d), \end{aligned}$$

together with

$$h(\mathrm{v}(t); \alpha(t), \delta(t)) = 0, \qquad \det D_1 h(\mathrm{v}(t); \alpha(t), \delta(t)) \neq 0$$

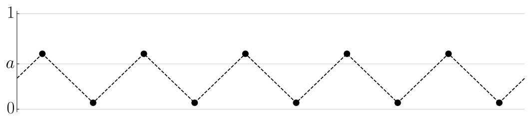
for all $0 \le t \le 1$.

We define an open pathwise connected set for each $\mathrm{w} \in \mathcal{A}_3^n$ by

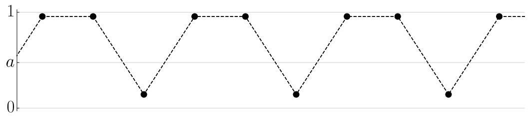
$$\Omega_{\mathrm{w}} = \big\{(a,d) \in \mathcal{H} \mid \text{the system (2.2) admits a solution of the type w}\big\}.$$

Under further considerations, it can be shown that any parameter-dependent solution $u_{\mathrm{w}}(a,d)$ of type w of the system (2.2) is uniquely defined in Ω_{w} and if $(a,d) \in \Omega_{\mathrm{w}_1} \cap \Omega_{\mathrm{w}_2} \neq \emptyset$ for any two given words $\mathrm{w}_1 \neq \mathrm{w}_2$ then $u_{\mathrm{w}_1}(a,d) \neq u_{\mathrm{w}_2}(a,d)$. We recommend the reader to consult [20, §2.1] for a full-length discussion. The notion of solution type can be now passed on to the periodic stationary solutions of the LDE (1.1) via the statement of Lemma 2.1, see Figure 2.1 for illustration.

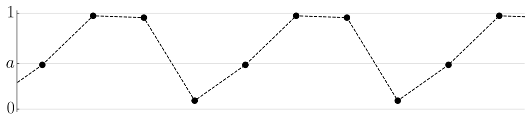
Remark 2.3. The definition of the naming scheme, Definition 2.2, ensures that a solution u_{w} of a given type $\mathrm{w} \in \mathcal{A}_3^n$ preserves its stability inside Ω_{w} since the determinant of the Jacobian matrix is not allowed to change its sign. Words from $\mathcal{A}_2^n = \{\mathfrak{o}, \mathfrak{1}\}^n$ thus represent asymptotically stable steady states.



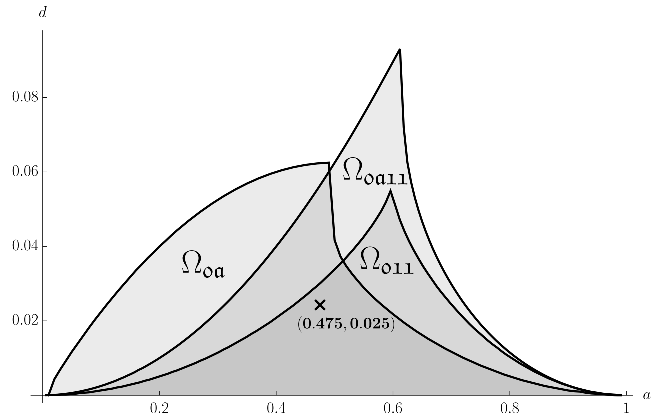
(a) A two-periodic stationary solution of type $\mathfrak{0a}$.



(b) A three-periodic stationary solution of type $\mathfrak{011}$.



(c) A four-periodic stationary solution of type $\mathfrak{0a11}$.



(d) Regions of existence Ω_{w} for solutions of type $\mathfrak{0a}$, $\mathfrak{011}$, $\mathfrak{0a11}$.

Figure 2.1: Examples of two-, three- and four-periodic stationary solutions of the LDE (1.1) and the regions of existence for solutions of their respective type. The parameters $(a,d) = (0.475, 0.025)$ are set to be identical in all three cases.

2.2 Symmetries of the periodic solutions

We start with a list of symmetries of the system (2.2) which are relevant to the similarities of the regions Ω_{w} and then discuss their general impact on the number of the regions. Note that the results apply to the periodic stationary solutions of the LDE (1.1) via Lemma 2.1.

2.2.1 Rotations

Let the rotation operator $r : \mathcal{A}_3^n \to \mathcal{A}_3^n$ be defined by

$$\big(r(\mathrm{w})\big)_i = \mathrm{w}_{1+\mathrm{mod}\,(i,n)} \tag{2.4}$$

for $i = 1, 2, \dots, n$ with obvious extension to vectors in $[0,1]^n$. A shift in indexing in (2.2) shows that $\mathrm{u} \in [0,1]^n$ is the system (2.2) solution of type $\mathrm{w} \in \mathcal{A}_3^n$ if and only if $r(\mathrm{u})$ is a solution of type $r(\mathrm{w})$. Note that this is true in general even if u cannot be assigned a type; the claim "u is a solution of the system (2.2)" is invariant with respect to the rotation r. As a direct consequence, we have

$$\Omega_{\mathrm{w}} = \Omega_{r(\mathrm{w})}$$

for all $\mathrm{w} \in \mathcal{A}_3^n$.

The transformation r generates a finite cyclic group of order n which we denote by

$$C_n = \big(\{r^0, r^1, \dots, r^{n-1}\}, \circ\big).$$

where the group operation $\circ$ is composition of the rotations $r^i \circ r^j = r^{\mathrm{mod}\,(i+j,n)}$. For the sake of consistency with the future notation, we denote the identity element e by r^0 and $r^1 = r$. Let us mention one fact which is implicitly used throughout the paper. If i and n are relatively coprime, then r^i is a generator of the group C_n. For example, let $n = 4$, then the repetitive composition of r^3 gives the sequence $r^3 \to r^2 \to r^1 \to r^0 \to r^3 \to \ldots$ which covers the whole element set of C_4. On the other hand, the composition of r^2 gives $r^2 \to r^0 \to r^2 \to \ldots$ which does not span the whole element set of C_4.

2.2.2 Reflections

Let the reflection operator $s : \mathcal{A}_3^n \to \mathcal{A}_3^n$ be defined by

$$\big(s(\mathrm{w})\big)_i = \mathrm{w}_{n-i+1} \tag{2.5}$$

together with its natural extension to vectors in $[0,1]^n$. Similar argumentation as in the previous paragraph shows that $\mathrm{u} \in [0,1]^n$ is a solution of type w of the system (2.2) if and only if $s(\mathrm{u})$ is a solution of type $s(\mathrm{w})$.

Adding the reflection s to the cyclic group C_n results in construction of the dihedral group D_n which is generated by the transformations r and s. Let us denote the composition of the rotation r^i and the reflection s by $sr^i = r^i \circ s$ (i.e., we first reflect and then rotate). For the sake of consistency, we also set $sr^0 = s$. This allows us to define the dihedral group

$$D_n = \big(\{r^0, r^1, \ldots, r^{n-1}, sr^0, sr^1, \ldots, sr^{n-1}\}, \circ\big)$$

and

$$\Omega_{\mathrm{w}} = \Omega_{g(\mathrm{w})}$$

holds for all $\mathrm{w} \in \mathcal{A}_3^n$ and $g \in D_n$.

2.2.3 Value permutation

The third symmetry exploits a specific property of the cubic nonlinearity

$$f(s;a) = -f(1-s;1-a)$$

with $s, a \in [0,1]$. We therefore have

$$h(\mathrm{u};a,d) = -h(\mathbf{1}-\mathrm{u};1-a,d) \tag{2.6}$$

for any $\mathrm{u} \in [0,1]^n$ and $a \in [0,1]$ where the subtraction $\mathbf{1}-\mathrm{u}$ is element-wise. Let us define the value permutation $\pi : \mathcal{A}_3^n \to \mathcal{A}_3^n$ by

$$\big(\pi(\mathrm{w})\big)_i = \begin{cases} \mathtt{1}, & \mathrm{w}_i = \mathtt{0}, \\ \mathtt{a}, & \mathrm{w}_i = \mathtt{a}, \\ \mathtt{0}, & \mathrm{w}_i = \mathtt{1}. \end{cases} \tag{2.7}$$

The equality (2.6) now shows that u is a solution of type w of the system (2.2) if and only if $\mathbf{1}-\mathrm{u}$ is a solution of type $\pi(\mathrm{w})$ with $a \mapsto 1-a$. As a direct consequence,

$$\Omega_{\mathrm{w}} = \mathcal{T}\big(\Omega_{\pi(\mathrm{w})}\big)$$

holds for all $\mathrm{w} \in \mathcal{A}_3^n$ where $\mathcal{T} \colon \mathcal{H} \to \mathcal{H}$ is

$$\mathcal{T}(a,d) = (1-a,d). \tag{2.8}$$

The transformation $\mathcal{T}$ is a vertical reflection with respect to the line $a = 1/2$.

The operation π generates the two element group

$$\Pi = \big(\{e, \pi\}, \circ\big),$$

where e is the identity element. The group Π can be also restricted to operate on the set of all words made with the two letter alphabet $\mathcal{A}_2$ by

$$\big(\pi(\mathrm{w})\big)_i = \begin{cases} 1, & \mathrm{w}_i = 0, \\ 0, & \mathrm{w}_i = 1. \end{cases}$$

To enlighten the notation, we denote the symbol permutation group by the letter Π regardless of the used alphabet.

In virtue of the previous notation, let us define $\pi r^i = r^i \circ \pi$ and $\pi s r^i = r^i \circ s \circ \pi$ and the group C_n^Π by

$$C_n^\Pi = \big(\{r^0, r^1, \dots, r^{n-1}, \pi r^0, \pi r^1, \dots, \pi r^{n-1}\}, \circ\big).$$

Note that the group C_n^Π contains elements from C_n and the elements from C_n composed with the symbol permutation π. Equivalently, we define the group D_n^Π by

$$D_n^\Pi = \left(\left\{\begin{array}{l} r^0, r^1, \dots, r^{n-1}, \pi r^0, \pi r^1, \dots, \pi r^{n-1}, \\ s r^0, s r^1, \dots, s r^{n-1}, \pi s r^0, \pi s r^1, \dots, \pi s r^{n-1} \end{array}\right\}, \circ\right).$$

Although our main aim is the examination of the action of the group D_n^Π it is convenient to study the group C_n^Π separately to be able to obtain partial results which are used in the proof of the main theorem. Let us also emphasize that the action of the groups C_n^Π and D_n^Π preserves stability of the corresponding solutions.

2.2.4 Primitive periods

Let us assume that a word w of length n has a primitive period of length $m < n$ (say, 1aa1aa), i.e., it consists of n/m-times repeated word w_m of length m (1aa in this case). Then surely

$$\Omega_{\mathrm{w}} = \Omega_{\mathrm{w}_m};$$

their respective regions are identical. It is not difficult to include this in the counting formulas alone but the interplay with the group operations $(C_n, D_n, C_n^\Pi, D_n^\Pi)$ is more intricate and is treated later via Möbius inversion formula, Theorem 2.7.

2.2.5 Other solution properties

It is clear that regions belonging to the constant solutions of type $00\dots0$, $\mathrm{aa}\dots\mathrm{a}$ and $11\dots1$ are trivial

$$\Omega_{00\dots0} = \Omega_{\mathrm{aa}\dots\mathrm{a}} = \Omega_{11\dots1} = \mathcal{H}.$$

Another notable similarity of regions can be illustrated on the words 01 and 0011. Argumentation in [20, Section 4] shows that the region Ω_{0011} has exactly the same shape as twice vertically stretched region Ω_{01}. Indeed, we can consider $u_1 = u_2$ and $u_3 = u_4$ for solution of type 0011 and the system (2.2) reduces to two equations with halved diffusion coefficient

d. We were however not able to generalize this observation to other types of solutions since, e.g., the natural candidate Ω_{000111} does not possess this property since $u_1 \neq u_2 \neq u_3$ holds in general.

Motivated by the previous paragraphs, we define the notion of similarity of the sets Ω_w.

Definition 2.4. Two regions $\Omega_{w_1}, \Omega_{w_2} \subset \mathcal{H}$ are called *qualitatively equivalent* if either

$$\Omega_{w_1} = \Omega_{w_2} \quad \text{or} \quad \Omega_{w_1} = \mathcal{T}(\Omega_{w_2}).$$

Two sets are called *qualitatively distinct* if they are not qualitatively equivalent.

2.3 Orbits and equivalence classes

Orbit of a word from $\mathcal{A}_3^n$ is a subset of $\mathcal{A}_3^n$ reachable by the action of some group G. As indicated in the previous section, we are interested in the number of different orbits since each orbit with respect to the group D_n^Π contains words whose respective regions are qualitatively equivalent. In fact, the orbits divide the sets of words $\mathcal{A}_2^n, \mathcal{A}_3^n$ into equivalence classes, i.e., two words w_1, w_2 belong to the same equivalence class (have the same orbit) if there exists a group operation $g \in G$ such that $w_1 = g(w_2)$. Burnside's lemma (Theorem 2.6) and Möbius inversion formula (Theorem 2.7) are the main tools for determining the number of the classes and the classes representing words with a given primitive period, respectively.

Example 2.5. There are 27 words of length $n = 3$ made with the alphabet $\mathcal{A}_3 = \{0, a, 1\}$:

$$\begin{aligned} W_{\mathcal{A}_3}(3) = \{&000, 00a, 001, 0a0, 0aa, 0a1, 010, 01a, 011,\\ &a00, a0a, a01, aa0, aaa, aa1, a10, a1a, a11,\\ &100, 10a, 101, 1a0, 1aa, 1a1, 110, 11a, 111\}. \end{aligned}$$

Taking into account the action of the group C_3, there are 11 equivalence classes

$$\begin{aligned} W_{\mathcal{A}_3}^{C_3}(3) = \{&\{000\}, \{aaa\}, \{111\},\\ &\{00a, 0a0, a00\}, \{001, 010, 100\}, \{0aa, aa0, a0a\},\\ &\{0a1, a10, 10a\}, \{01a, 1a0, a01\}, \{011, 110, 101\},\\ &\{aa1, a1a, 1aa\}, \{a11, 11a, 1a1\}\}, \end{aligned}$$

while the action of the group D_3 merges two of these classes

$$\begin{aligned} W_{\mathcal{A}_3}^{D_3}(3) = \{&\{000\}, \{aaa\}, \{111\},\\ &\{00a, 0a0, a00\}, \{001, 010, 100\}, \{0aa, aa0, a0a\},\\ &\{0a1, a10, 10a, 1a0, a01, 01a\}, \{011, 110, 101\},\\ &\{aa1, a1a, 1aa\}, \{a11, 11a, 1a1\}\}. \end{aligned}$$

The action of the groups C_3^Π and D_3^Π divides the set of the words into the same system of 6 equivalence classes

$$\begin{aligned} W_{\mathcal{A}_3}^{C_3^\Pi}(3) = W_{\mathcal{A}_3}^{D_3^\Pi}(3) = \{&\{000, 111\}, \{aaa\},\\ &\{00a, 0a0, a00, 11a, 1a1, a11\}, \{001, 010, 100,\\ &110, 101, 011\}, \{0aa, aa0, a0a, 1aa, aa1, a1a\},\\ &\{0a1, a10, 10a, 1a0, a01, 01a\}\}. \end{aligned}$$

See Figure 2.2 for a graphical illustration of the equivalence classes.

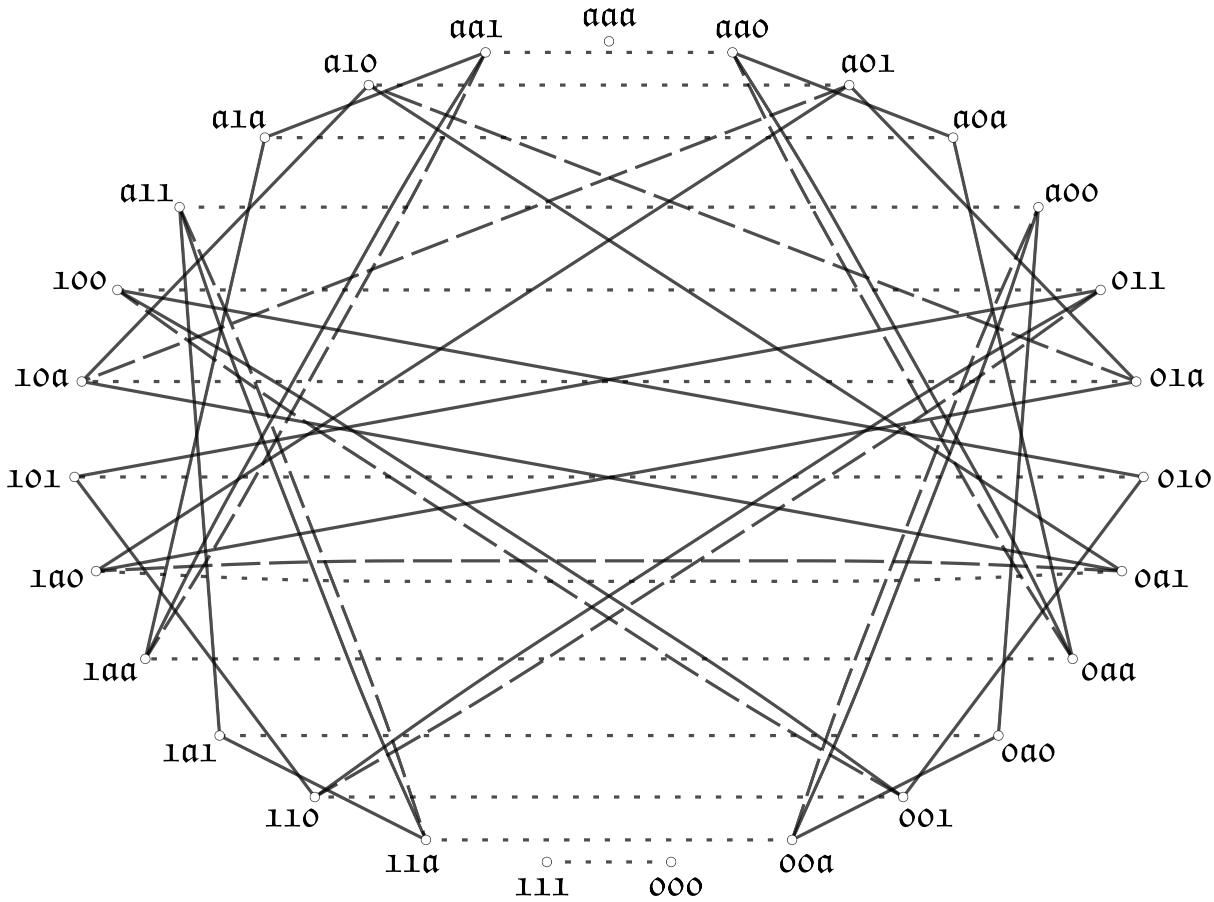


Figure 2.2: A diagram capturing the action of the groups C_3, D_3, C_3^Π and D_3^Π on the set of three-letter words made with the alphabet $\mathcal{A}_3$. The presence of a line connecting two words indicates the existence of an operation transforming the solutions onto each other. The rotation r is expressed by a solid line, the reflection s is expressed by a dashed line and the symbol permutation π is expressed by a dotted line. Every maximal connected subgraph with appropriate line types represents one equivalence class with respect to the action of a certain group, e.g., the action of C_3^Π is depicted by solid and dotted lines.

The crucial question is whether we can determine the number of equivalence classes in a systematic manner. A useful tool for this is Burnside's lemma [8].

Theorem 2.6 (Burnside's lemma)**.** *Let G be a finite group operating on a finite set S. Let $I(g)$ be the number of set elements such that the group operation $g \in G$ leaves them invariant. Then the number of distinct orbits O is given by the formula*

$$O = \frac{1}{|G|} \sum_{g \in G} I(g).$$

The power of Burnside's lemma lies in the fact that one counts fixed points of the group operations instead of the orbits themselves. This can be much simpler in many cases as can be seen in the forthcoming sections.

The number of the orbits induced by the action of the group C_n is usually called the number of the *necklaces* made with n beads in two (the alphabet $\mathcal{A}_2$) or three (the alphabet $\mathcal{A}_3$)

colors. The *bracelets* are induced by the action of the dihedral group D_n. Due to the lack of a common terminology, we call the classes induced by the action of the groups C_n^Π and D_n^Π the *permuted necklaces* and the *permuted bracelets*, respectively.

Burnside's lemma does not take into account the primitive period of the words. For example, the existence region of the solutions of type $0\alpha 1$ coincides with the region of $0\alpha 10\alpha 1$ and thus cannot be counted twice. The assumption of the primitive period of a given length together with the action of the cyclic group C_n create classes which are called the *Lyndon necklaces*. The *Lyndon bracelets* are a natural counterpart resulting from the action of the dihedral group D_n together with the assumption of a given primitive period length. The classes representing words with a given primitive period length without specification of the group are called the *Lyndon words*. We emphasize that the terminology is not fully unified in the literature but the one presented here suits our purpose best without rising any unnecessary confusion.

Theorem 2.7 (Möbius inversion formula)**.** *Let $f, g : \mathbb{N} \to \mathbb{R}$ be two arithmetic functions such that*

$$f(n) = \sum_{d|n} g(d),$$

holds for all $n \in \mathbb{N}$. Then the values of the latter function g can be expressed as

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d),$$

where μ is the Möbius function.

The Möbius function μ was first introduced in [30] as

$$\mu(n) = \begin{cases} (-1)^{P(n)}, & \text{each prime factor of } n \text{ is present at most once,} \\ 0, & \text{otherwise,} \end{cases}$$

where $P(n)$ number of the prime factors of n. Use of Möbius inversion formula is a straightforward one. Let us assume, that we know the number $f(n)$ of the equivalence classes induced by the action of one of the above defined groups (C_n, D_n, C_n^Π, D_n^Π) for each $n \in \mathbb{N}$ (note that the group actions preserve the length of the primitive period of each of the words). Then for each n, this number $f(n)$ is given as the sum of the number of equivalence classes representing the words with primitive period of length d dividing n.

In the further text, we extensively exploit two crucial properties of Möbius inversion formula. Firstly, the formula is linear in the sense, that

$$\sum_{i=1}^{m} \alpha_i f_i(n) = \sum_{d|n} g(d)$$

implies

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{i=1}^{m} \alpha_i f_i(d) = \sum_{i=1}^{m} \alpha_i \sum_{d|n} \mu\left(\frac{n}{d}\right) f_i(d),$$

and thus, each f_i can be treated separately. Secondly, we can freely exchange indices in the following manner

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

since $n = n/d \cdot d$.

Example 2.8. We complement Example 2.5 with the list of equivalence classes of the of the words with length $n = 4$ made with the alphabet $\mathcal{A}_2 = \{0, 1\}$. There exist words of length 4 with the primitive period 2 and thus the set of the equivalence classes and the set of the Lyndon words will differ not by only the trivial constant words $\{0000\}, \{1111\}$. There are 16 words of length 4

$$\begin{aligned} \mathrm{W}_{\mathcal{A}_2}(4) = \{&0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, \\ &1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}. \end{aligned}$$

We next include the equivalence classes induced by the action of the groups C_4, D_4, C_4^Π, D_4^Π. The words with the primitive period of length smaller than 4 are highlighted by a grey color

$$\begin{aligned} \mathrm{W}^{C_4}_{\mathcal{A}_2}(4) = \mathrm{W}^{D_4}_{\mathcal{A}_2}(4) &= \{\{0000\}, \{1111\}, \{0101, 1010\}, \{0011, 0110, 1100, 1001\}, \\ &\quad \{0001, 0010, 0100, 1000\}, \{0111, 1110, 1101, 1011\}\}, \\ \mathrm{W}^{C_4^\Pi}_{\mathcal{A}_2}(4) = \mathrm{W}^{D_4^\Pi}_{\mathcal{A}_2}(4) &= \{\{0000, 1111\}, \{0101, 1010\}, \{0011, 0110, 1100, 1001\}, \\ &\quad \{0001, 0010, 0100, 1000, 1110, 1101, 1011, 0111\}\}. \end{aligned}$$

This introduction allows us to state the main theorem of the paper which gives an upper estimate of qualitatively distinct regions belonging to words of length m which ranges from one up to some given value $n \in \mathbb{N}$. We must combine the action of the dihedral group D_m^Π with the assumption of the primitive period equal to the word length (Lyndon bracelets) for each $m \leq n$. It is however upper estimate only, since we cannot be sure whether there exist two qualitatively equivalent regions whose labelling words are not related via any of the above mentioned symmetries. Numerical simulations however indicate that the upper bound may be close to optimal, [20].

Since the expressions in the theorem may look confusing at the first sight we include a short preliminary commentary. The function $BL_k^\pi(m)$ denotes the number of the permuted Lyndon bracelets of length m and the formulas are defined by parts since they incorporate the number of the bracelets which cannot be written in a consistent form for even and odd m's. The functions $\#^{\leq}_{\mathcal{A}_k}(n)$ just add the numbers of Lyndon bracelets of length ranging from two to n including the one region $\Omega_0 = \Omega_a = \Omega_1 = \mathcal{H}$ identical for all homogeneous solutions.

Theorem 2.9. *Let $n \geq 2$ be given. There are at most*

$$\#^{\leq}_{\mathcal{A}_3}(n) = 1 + \sum_{m=2}^{n} BL^\pi_{\mathcal{A}_3}(m) \tag{2.9}$$

qualitatively distinct regions Ω_w, $\mathrm{w} \in \mathcal{A}_3^m$, out of which at most

$$\#^{\leq}_{\mathcal{A}_2}(n) = 1 + \sum_{m=2}^{n} BL^\pi_{\mathcal{A}_2}(m) \tag{2.10}$$

regions belong to the asymptotically stable stationary solutions, where

$$BL^{\pi}_{\mathcal{A}_3}(m) = \frac{1}{4m}\left[\sum_{d|m,\, d\, \mathrm{odd}} \mu(d)\, 3^{\frac{m}{d}} + X_{NL}(m) + 2m \sum_{d|m} \mu\left(\frac{m}{d}\right) X^{\pi}_{B,3}(d)\right], \tag{2.11}$$

$$BL^{\pi}_{\mathcal{A}_2}(m) = \frac{1}{4m}\left[\sum_{d|m,\, d\, \mathrm{odd}} \mu(d)\, 2^{\frac{m}{d}} + 2n \sum_{d|m} \mu\left(\frac{m}{d}\right) X^{\pi}_{B,2}(d)\right], \tag{2.12}$$

$$X_{NL}(m) = \begin{cases} 1, & m = 1, \\ -1, & m = 2^{\alpha},\ \alpha \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$X^{\pi}_{B,3}(d) = \begin{cases} \frac{4}{3}\cdot 3^{\frac{d}{2}}, & d \text{ is even}, \\ 2\cdot 3^{\frac{d-1}{2}}, & d \text{ is odd}, \end{cases} \qquad X^{\pi}_{B,2}(d) = \begin{cases} 2^{\frac{d}{2}}, & d \text{ is even}, \\ 2^{\frac{d-1}{2}}, & d \text{ is odd}. \end{cases} \tag{2.13}$$

The formulas from Theorem 2.9 are enumerated in Table 2.1.

n	3^n	2^n	$\#^{\leq}_{\mathcal{A}_3}(n)$	$BL^{\pi}_{\mathcal{A}_3}(n)$		$\#^{\leq}_{\mathcal{A}_2}(n)$	$BL^{\pi}_{\mathcal{A}_2}(n)$	
2	9	4	3	2	{01, 0a}	2	1	{01}
3	27	8	7	4	{00a, 001, 0a1, 0aa}	3	1	{001}
4	81	16	16	9	{000a, 0001, 00aa, 00a1, 0011, 0a0a, 0aaa, 0aa1, 0a1a}	5	2	{0001, 0011}
5	243	32	36	20	not listed	8	3	{00001, 00011, 00101 }
6	729	64	80	44	not listed	13	5	{000001, 000011, 000101, 000111, 001011}
7	2187	128	184	104	not listed	21	8	not listed
8	6561	256	437	253	not listed	35	14	not listed
9	19683	512	1061	624	not listed	56	21	not listed
10	59049	1024	2689	1628	not listed	95	39	not listed

Table 2.1: Enumerated formulas from Theorem 2.9. The columns for 3^n and 2^n are added for comparison since there are in total 3^n regions Ω_{w} with $\mathrm{w} \in \mathcal{A}_3^n$ and 2^n of them correspond to the asymptotically stable stationary solutions. The unlabelled columns list the lexicographically smallest representatives of the Lyndon bracelets of a given length created with the respective alphabets; further lists are omitted to prevent clutter. Note that $\#^{\leq}_{\mathcal{A}_k}(n+1) = \#^{\leq}_{\mathcal{A}_k}(n) + BL^{\pi}_{\mathcal{A}_k}(n+1)$ holds for $n \geq 2$ and $k = 2, 3$.

2.4 Known results

Here, we summarize known results relevant to the focus of this paper. This summary consists of two parts since our main result, Theorem 2.9, contributes to the knowledge of the periodic stationary solutions of the LDE (1.1) as well as to the theory of combinatorial enumeration.

The number of equivalence classes with respect to the action of the groups C_n and D_n and their connection to the stationary solutions of the GDE (1.6) and the LDE (1.1) were studied in the paper [21]. The results considered all stationary solutions (words from $\mathcal{A}_3^n$) as well as the stable solutions (words from $\mathcal{A}_2^n$). Möbius inversion formula was used therein to determine the numbers of the Lyndon necklaces and the Lyndon bracelets.

A more general case of the group C_n^Π which acted on the set of words created with a given number of symbols not necessarily less or equal to three was considered in [13]. The author also simplified the counting formulas for the permuted necklaces and the permuted Lyndon necklaces for the case of two symbols, i.e., the alphabet $\mathcal{A}_2$, to the form which also appears in this paper, Lemmas 3.3 and 3.10. However, none of the presented results could be directly applied to the case of the transformation π acting on the words from $\mathcal{A}_3^n$. Formally, the studied object was the group product of a cyclic group C_n and a symmetric group S_k (the group of all permutations of k symbols). This coincides with our case only if $k = 2$, i.e., the words are created with a two symbol alphabet $\mathcal{A}_2$. If $k = 3$, then the group C_n^Π is isomorphic to the group product $C_n \times G$ where G is only a specific subgroup of S_3. Let us also mention that the problem was studied from the combinatorial point of view.

The authors in [14] were among other results able to derive a general counting formula for the permuted bracelets and the permuted Lyndon bracelets of words created with an arbitrary number of symbols. As in the case of the necklaces in [13], the results relevant to this paper cover the case of the reduced alphabet $\mathcal{A}_2$ only. The generality of the presented formulas however comes with a cost of their complexity. Taking advantage of our more specific setting, we are able to utilize alternative approach which enables us to further simplify the formulas for the case of the words from $\mathcal{A}_2^n$. Also, the focus of the work lied mainly in clarifying certain combinatorial concepts.

3 Counting of equivalence classes

We continue with listing and deriving auxiliary counting formulas as well as those which are directly used to prove the main result, Theorem 2.9.

In this section, (m, n) denotes the greatest common divisor of $m, n \in \mathbb{N}$.

3.1 Counting of non-Lyndon words

We start with counting of the necklaces of length n made with k symbols.

Lemma 3.1 ([34, p. 162])**.** *Given $n \in \mathbb{N}$, the number of equivalence classes induced by the action of the group C_n on the set of all words of length n made with a k-symbol alphabet is*

$$N_k(n) = \frac{1}{n} \sum_{d|n} \varphi(d) k^{\frac{n}{d}}. \tag{3.1}$$

The function $\varphi(d)$ is the Euler totient function which counts relatively coprime numbers to d, see [2]. Another classical result concerns the number of the bracelets of length n made with k symbols.

Lemma 3.2 ([34, p. 150])**.** *Given $n \in \mathbb{N}$, the number of equivalence classes induced by the action of the group D_n on the set of all words of length n made with a k-symbol alphabet is*

$$B_k(n) = \frac{1}{2}\left[N_k(n) + X_{B,k}(n)\right], \tag{3.2}$$

where

$$X_{B,k}(n) = \begin{cases} \dfrac{k+1}{2}\, k^{\frac{n}{2}}, & n \text{ is even}, \\ k^{\frac{n+1}{2}}, & n \text{ is odd}. \end{cases} \tag{3.3}$$

The formulas for the necklaces and the bracelets can be derived for a general number of symbols k. If we take the symbol permutation π into the account, the formulas regarding the alphabets $\mathcal{A}_2$ and $\mathcal{A}_3$ are slightly different and thus, we treat both cases separately. The main difference is that there are no invariant words with respect to the value permutation π with the alphabet $\mathcal{A}_2$ and n odd. Indeed, the necessary condition for the invariance is that the word has the same number of 0's and 1's. This can be bypassed by the use of the symbol $\mathfrak{a}$ from the alphabet $\mathcal{A}_3$.

Lemma 3.3 ([13, p. 300])**.** *Given $n \in \mathbb{N}$, the number of equivalence classes induced by the action of the group C_n^{Π} on the set of all words of length n made with the alphabet $\mathcal{A}_2$ is*

$$N^{\pi}_{\mathcal{A}_2}(n) = \frac{1}{2n}\left[\sum_{d|n,\, d\, odd} \varphi(d)\, 2^{\frac{n}{d}} + 2 \sum_{d|n,\, d\, even} \varphi(d)\, 2^{\frac{n}{d}}\right]. \tag{3.4}$$

A somewhat similar formula can be derived for the necklaces made with the three-letter alphabet $\mathcal{A}_3$.

Lemma 3.4. *Given $n \in \mathbb{N}$, the number of equivalence classes induced by the action of the group C_n^{Π} on the set of all words of length n made with the alphabet $\mathcal{A}_3$ is*

$$N^{\pi}_{\mathcal{A}_3}(n) = \frac{1}{2n}\left[\sum_{d|n,\, d\, odd} \varphi(d)\left(1 + 3^{\frac{n}{d}}\right) + 2 \sum_{d|n,\, d\, even} \varphi(d)\, 3^{\frac{n}{d}}\right]. \tag{3.5}$$

Proof. The group C_n^{Π} contains the pure rotations r_i and the rotations with the symbol permutations $r\pi_i$ totalling $2n$ operations. A direct application of Burnside's lemma (Theorem 2.6) yields

$$N^{\pi}_{\mathcal{A}_3}(n) = \frac{1}{2n}\left[\sum_{l=0}^{n-1} I(r^l) + \sum_{l=0}^{n-1} I(\pi r^l)\right].$$

The expression (3.1) in the context of Lemma 3.1 shows that

$$\sum_{l=0}^{n-1} I(r^l) = \sum_{d|n} \varphi(d) 3^{\frac{n}{d}}.$$

Given $l = 0, 1, \ldots, n-1$, the aim is to express the general form of a word w invariant to the operation πr^l. A rotation by l positions induces a permutation of the word's w letters with (n,l) cycles of length $n/(n,l)$. The word w is then divided into $n/(n,l)$ disjoint subwords of length (n,l). Assume that the first (n,l) letters of the word w are given. A repeated application of the operation πr^l then determines the form of all the remaining subwords of length (n,l). Indeed, the rotation by l positions applied to a word of length n induces a rotation by $l/(n,l)$ positions of the $n/(n,l)$ subwords because $l/(n,l)$ and $n/(n,l)$ are relatively coprime. Here, the parity of the subwords' number $n/(n,l)$ must be considered. If $n/(n,l)$ is odd, then the only possible word invariant to πr^l is constant $\mathfrak{a}$'s. The even $n/(n,l)$ allows $3^{(n,l)}$ possible words.

Let us pick an arbitrary divisor d of n. Then, surely $d = n/(n,l)$ for some $l \in \{0,1,\dots,n-1\}$. The cyclic group C_d with d elements can be generated by $\varphi(d)$ different values relatively coprime to d.

The argumentation above results in

$$\begin{aligned} N^{\pi}_{\mathcal{A}_3}(n) &= \frac{1}{2n}\left[\sum_{l=0}^{n-1} I(r^l) + \sum_{l=0}^{n-1} I(\pi r^l)\right], \\ &= \frac{1}{2n}\left[\sum_{d|n}\varphi(d)\,3^{\frac{n}{d}} + \sum_{d|n,\,d\,\text{odd}}\varphi(d) + \sum_{d|n,\,d\,\text{even}}\varphi(d)\,3^{\frac{n}{d}}\right], \\ &= \frac{1}{2n}\left[\sum_{d|n,\,d\,\text{odd}}\varphi(d)\left(1+3^{\frac{n}{d}}\right) + 2\sum_{d|n,\,d\,\text{even}}\varphi(d)\,3^{\frac{n}{d}}\right]. \end{aligned}$$

□

We now approach to the formulas regarding the group D^{Π}_n; the permuted bracelets. As in the previous text, we treat the cases of the alphabets $\mathcal{A}_2$ and $\mathcal{A}_3$ separately. A general counting formula regarding the alphabet $\mathcal{A}_2$ as a special case was derived in [14]. We present an alternative proof which can be generalized to the case of the alphabet $\mathcal{A}_3$.

Lemma 3.5. *Given $n \in \mathbb{N}$, the number of equivalence classes induced by the action of the group D^{Π}_n on the set of all words of length n made with the alphabet $\mathcal{A}_2$ is*

$$B^{\pi}_{\mathcal{A}_2}(n) = \frac{1}{2}\left[N^{\pi}_{\mathcal{A}_2}(n) + X^{\pi}_{B,2}(n)\right], \tag{3.6}$$

where

$$X^{\pi}_{B,2}(n) = \begin{cases} 2^{\frac{n}{2}}, & n \text{ is even}, \\ 2^{\frac{n-1}{2}}, & n \text{ is odd}. \end{cases} \tag{3.7}$$

Proof. The group D^{Π}_n contains the rotations r^i, the rotations with the reflection sr^i, the rotations with the symbol permutation πr^i and the rotations with the reflection and the symbol permutation $\pi s r^i$. Burnside's lemma (Theorem 2.6) then yields

$$N^{\pi}_{\mathcal{A}_3}(n) = \frac{1}{4n}\left[\sum_{l=0}^{n-1} I(r^l) + \sum_{l=0}^{n-1} I(\pi r^l) + \sum_{l=0}^{n-1} I(sr^l) + \sum_{l=0}^{n-1} I(\pi s r^l)\right].$$

The equivalence classes induced by the transformations r^l and πr^l are enumerated in the expression (3.4) of Lemma 3.3. Each line in formula (3.3) counts the number of orbits with respect to the rotation with reflection sr^l.

First, we clarify certain concepts valid for the operation sr^l and subsequently apply them to the case of $\pi s r^l$. The composition of the rotation and the reflection is not commutative in general, but $sr^l = r^l \circ sr^0 = sr^0 \circ r^{n-l}$ holds for $l = 0,1,\dots,n-1$. This formula and the group associativity yields

$$sr^l \circ sr^l = (r^l \circ sr^0) \circ (sr^0 \circ r^{n-l}) = r^l \circ r^{n-l} = r^0 = e.$$

Thus, the induced permutation of the word's letters has cycles of the length 1 or 2 only.

Given $l = 0,1,\dots,n-1$, then

$$\left(sr^l(\mathrm{w})\right)_i = \mathrm{w}_{n-l-i+1}$$

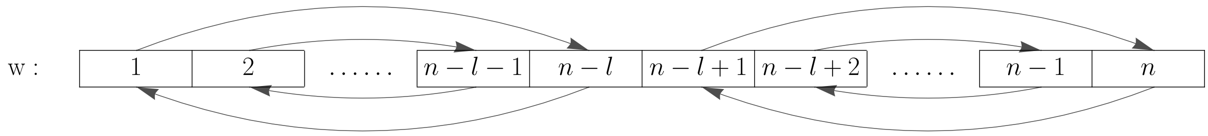


Figure 3.1: Illustration of operation of the group transformation rs_l on the word w of length n. The transformation sr^l divides the word w into two subwords whose elements starting from the edges map to each other.

for $i \leq \lceil l/2 \rceil$. Due to the composition formula, the positions from $n - l + 1$ to n transform accordingly. This induces a partition of the word w into two subwords, see Figure 3.1 for illustration. The combined parities of n and l determine the parity of the subwords' length and thus whether there is a middle letter mapped to itself. For any subword of odd length, there is exactly one loop. All possible combinations are

$n \backslash l$	even	odd
even	even, even	odd, odd
odd	odd, even	even, odd

.

Let us now assume the operation πsr^l. If n is odd, then one of the subwords induced by the action of sr^l is always odd and thus there are no words fixed by πsr^l. If n is even the only possibility for the word w to be fixed is when l is also even. There are then $n/2$ cycles of length 2 leading to $n/2 \cdot 2^{n/2}$ words fixed by the operation of the form πsr^i.

The summing of all cases and including (3.3) for $I(sr^l)$ results in

$$
\begin{aligned}
B^{\pi}_{\mathcal{A}_2}(n)\Big|_{n \text{ even}} &= \frac{1}{4n}\left[2n \cdot N^{\pi}_{\mathcal{A}_2}(n) + \frac{3n}{2} \cdot 2^{\frac{n}{2}} + \frac{n}{2} \cdot 2^{\frac{n}{2}}\right], \\
&= \frac{1}{2}\left[N^{\pi}_{\mathcal{A}_2}(n) + 2^{\frac{n}{2}}\right], \\
B^{\pi}_{\mathcal{A}_2}(n)\Big|_{n \text{ odd}} &= \frac{1}{4n}\left[2n \cdot N^{\pi}_{\mathcal{A}_2}(n) + n \cdot 2^{\frac{n+1}{2}}\right] \\
&= \frac{1}{2}\left[N^{\pi}_{\mathcal{A}_2}(n) + 2^{\frac{n-1}{2}}\right].
\end{aligned}
$$

□

A general idea presented in the proof of Lemma 3.5 can be applied to the case of the three letter alphabet $\mathcal{A}_3$.

Lemma 3.6. *Given $n \in \mathbb{N}$, the number of equivalence classes induced by the action of the group D^{Π}_n on the set of all words of length n made with the alphabet $\mathcal{A}_3$ is*

$$B^{\pi}_{\mathcal{A}_3}(n) = \frac{1}{2}\left[N^{\pi}_{\mathcal{A}_3}(n) + X^{\pi}_{B,3}(n)\right], \tag{3.8}$$

where

$$X^{\pi}_{B,3}(n) = \begin{cases} \frac{4}{3} \cdot 3^{\frac{n}{2}}, & n \text{ is even}, \\ 2 \cdot 3^{\frac{n-1}{2}}, & n \text{ is odd}. \end{cases} \tag{3.9}$$

Proof. As in the proof of Lemma 3.5, the only operations to be considered in detail are of the form πsr^i. For the sake of completeness, we note that there are $2n \cdot 3^{n/2}$ (for n even) and $n \cdot 3^{(n+1)/2}$ (for n odd) words invariant to the action of transformations of the form sr^i, see (3.3).

Let $l = 0, 1, \dots, n-1$ be given. The operation sr^l induces a letter permutation with cycles of length 1 or 2. In order for the word w to be fixed by the operation πsr^l, positions in the cycle of length 1 can contain the letter $\mathfrak{a}$ only.

If n is odd, then there are $(n-1)/2$ cycles of length 2 leading to $n \cdot 3^{(n-1)/2}$ fixed words. If n is even, then there are two cycles of length 1 only if l is odd. Summing over all $l = 0, \dots, n-1$ leads to $n/2 \cdot (3^{n/2-1} + 3^{n/2})$.

The summary of the results gives

$$
\begin{aligned}
B^{\pi}_{\mathcal{A}_3}(n)\Big|_{n \text{ even}} &= \frac{1}{4n}\left[2n \cdot N^{\pi}_{\mathcal{A}_3}(n) + 2n \cdot 3^{\frac{n}{2}} + \frac{n}{2} \cdot (3^{\frac{n}{2}-1} + 3^{\frac{n}{2}})\right], \\
&= \frac{1}{2}\left[N^{\pi}_{\mathcal{A}_3}(n) + \frac{4}{3} \cdot 3^{\frac{n}{2}}\right], \\
B^{\pi}_{\mathcal{A}_3}(n)\Big|_{n \text{ odd}} &= \frac{1}{4n}\left[2n \cdot N^{\pi}_{\mathcal{A}_2}(n) + n \cdot 3^{\frac{n+1}{2}} + n \cdot 3^{\frac{n-1}{2}}\right], \\
&= \frac{1}{2}\left[N^{\pi}_{\mathcal{A}_3}(n) + 2 \cdot 3^{\frac{n-1}{2}}\right].
\end{aligned}
$$

□

3.2 Counting of the Lyndon words

To derive the forthcoming formulas, we use a special property of the Möbius function μ and the Euler totient function φ; these functions are multiplicative. An arithmetic function $\psi : \mathbb{N} \to \mathbb{R}$ is multiplicative if and only if $\psi(1) = 1$ and $\psi(ab) = \psi(a)\psi(b)$ provided a and b are relatively coprime. To prove that two multiplicative functions ψ_1, ψ_2 are equal it is enough to show that $\psi_1(p^\alpha) = \psi_2(p^\alpha)$ for all prime p and $\alpha \in \mathbb{N}$. For further information about the multiplicative functions see, e.g., [2].

We start with a technical lemma which is used later.

Lemma 3.7. *The identity*

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{n}{d}\, \varphi(d) = \mu(n) \tag{3.10}$$

holds for any $n \in \mathbb{N}$. Furthermore, the following identities hold for any n even,

$$\sum_{d|n,\, d\, even} \mu\left(\frac{n}{d}\right) \frac{n}{d}\, \varphi(d)\ \Big|_{n\ even} = -\mu(n), \tag{3.11}$$

$$\sum_{d|n,\, d\, odd} \mu\left(\frac{n}{d}\right) \frac{n}{d}\, \varphi(d)\ \Big|_{n\ even} = 2\mu(n). \tag{3.12}$$

Proof. The expression (3.10) is an equality of two multiplicative functions. It is sufficient to verify the formula for $n = p^\alpha$, where p is a prime and $\alpha \in \mathbb{N}$, [2]. If $\alpha \geq 2$ then $\mu(p^\alpha) = 0$ and

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{n}{d}\, \varphi(d) = \mu(p)\, p\, \varphi(p^{\alpha-1}) + \mu(1)\, \varphi(p^\alpha) = -p^{\alpha-1}(p-1) + p^{\alpha-1}(p-1) = 0,$$

if $\alpha = 1$, then $\mu(p^\alpha) = -1$ and

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{n}{d}\, \varphi(d) = \mu(p)\, p\, \varphi(1) + \mu(1)\, \varphi(p) = -p + p - 1 = -1,$$

if $\alpha = 0$, then $\mu(p^\alpha) = 1$ and

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{n}{d}\, \varphi(d) = \mu(1)\, 1\, \varphi(1) = 1.$$

This proves (3.10).

Let us assume, that the even integer $n \in \mathbb{N}$ has the form $n = 2^\beta P$, where P is a product of odd primes. We can now rewrite (3.12) as

$$\sum_{d|n,\, d\,\text{odd}} \mu\left(\frac{n}{d}\right) \frac{n}{d}\, \varphi(d) = \sum_{d|n/2^\beta} \mu\left(\frac{n}{d}\right) \frac{n}{d}\, \varphi(d).$$

If $\beta \geq 1$, then the fraction n/d always contains a squared prime factor and thus $\mu(n/d) = 0$ which corresponds to $\mu(2^\beta P) = 0$. Suppose $\beta = 1$. We can now use the substitution $m = n/2$ together with the formula (3.10)

$$\begin{aligned}\sum_{d|n/2} \mu\left(\frac{n}{d}\right) \frac{n}{d}\, \varphi(d) &= \sum_{d|m} \mu\left(\frac{2m}{d}\right) \frac{2m}{d}\, \varphi(d)\\ &= -2 \sum_{d|m} \mu\left(\frac{m}{d}\right) \frac{m}{d}\, \varphi(d) = -2\mu(m) = -2\mu\left(\frac{n}{2}\right) = 2\mu(n).\end{aligned}$$

The first sign change is possible due to the fact that the fraction m/d is an odd integer and thus 2 is not part of its prime factorization. The second one utilizes the same idea. This concludes the proof of (3.12).

The identity (3.11) follows from (3.10) and (3.12) since

$$\sum_{d|n} f(d) = \sum_{d|n,\, d\,\text{even}} f(d) + \sum_{d|n,\, d\,\text{odd}} f(d),$$

holds for any $n \in \mathbb{N}$. □

The counting formula for the Lyndon necklaces can be derived by a direct argument as in [15] but we choose more technical approach whose idea is useful in later proofs.

Lemma 3.8. *Given $n \in \mathbb{N}$, the number of the Lyndon necklaces (the group C_n) with period n on the set of all words of length n made with k symbols is*

$$NL_k(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) k^d. \tag{3.13}$$

Proof. Since

$$L_k(n) = \sum_{d|n} NL_k(n)$$

holds for all $n \in \mathbb{N}$ the use of the Möbius inversion formula (Theorem 2.7) and the subsequent substitution $d = ml$ yields

$$\begin{aligned}
NL_k(n) &= \sum_{m|n} \mu(m)\, L_k\left(\frac{n}{m}\right), \\
&= \sum_{m|n} \mu(m)\, \frac{m}{n} \sum_{l|m/n} \varphi(l)\, k^{\frac{n}{ml}}, \\
&= \frac{1}{n} \sum_{d|n} k^{\frac{n}{d}} \sum_{l|d} \mu\left(\frac{d}{l}\right) \frac{d}{l}\, \varphi(l), \\
&= \frac{1}{n} \sum_{d|n} k^{\frac{n}{d}}\, \mu(d).
\end{aligned}$$

The last step uses (3.10). □

The counting formula for the Lyndon bracelets is a direct consequence of Möbius inversion formula (Theorem 2.7) and Lemmas 3.2 and 3.8.

Lemma 3.9. *Given $n \in \mathbb{N}$, the number of the Lyndon bracelets (the group D_n) with period n on the set of all words of length n made with k symbols is*

$$BL_k(n) = \frac{1}{2}\left[NL_k(n) + \sum_{d|n} \mu\left(\frac{n}{d}\right) X_{B,k}(d)\right], \tag{3.14}$$

where $X_{B,k}(d)$ is given by (3.3).

We continue with the counting formulas for the permuted Lyndon necklaces.

Lemma 3.10 ([13, p. 301])**.** *Let $n \in \mathbb{N}$ be given. The number of the permuted Lyndon necklaces (the group C_n^{Π}) with period n on the set of all words of length n made with the alphabet $\mathcal{A}_2$ is*

$$NL^{\pi}_{\mathcal{A}_2}(n) = \frac{1}{2n} \sum_{d|n,\, d \text{ odd}} \mu(d)\, 2^{\frac{n}{d}}. \tag{3.15}$$

As previously mentioned, the statement of Lemma 3.10 cannot be generalized to the case of the three-letter alphabet $\mathcal{A}_3$ in a straightforward manner.

Lemma 3.11. *Given $n \in \mathbb{N}$, the number of the permuted Lyndon necklaces (the group C_n^{Π}) with period n on the set of all words of length n made with the alphabet $\mathcal{A}_3$ is*

$$NL^{\pi}_{\mathcal{A}_3}(n) = \frac{1}{2n}\left[\sum_{d|n,\, d \text{ odd}} \mu(d)\, 3^{\frac{n}{d}} + X_{NL}(n)\right], \tag{3.16}$$

where

$$X_{NL}(n) = \begin{cases} 1, & n = 1, \\ -1, & n = 2^{\alpha},\ \alpha \in \mathbb{N}, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. We directly apply Möbius inversion formula (Theorem 2.7) to (3.5) in an adjusted form

$$N^{\pi}_{\mathcal{A}_3}(n) = \frac{1}{2n}\left[\sum_{d|n} \varphi(d)\, 3^{\frac{n}{d}} + \sum_{d|n,\, d \text{ even}} \varphi(d)\, 3^{\frac{n}{d}} + \sum_{d|n,\, d \text{ odd}} \varphi(d)\right].$$

Thanks to the linearity of Möbius inversion formula, we may threat the expression summand-wise. For the sake of simplicity, the first summand is readily rewritten in the virtue of Lemma 3.8

$$\begin{aligned} NL^{\pi}_{\mathcal{A}_3}(n) =& \sum_{m|n} \mu(m)\, N^{\pi}_{\mathcal{A}_3}\left(\frac{n}{m}\right), \\ =& \frac{1}{2n}\sum_{d|n}\mu(d)\,3^{\frac{n}{d}} + \sum_{m|n}\mu(m)\,\frac{m}{2n}\sum_{l|n/m,\, l\,\text{even}}\varphi(l)\,3^{\frac{n}{ml}} + \sum_{d|n}\mu\left(d\right)\frac{d}{2n}\sum_{l|n/d,\, l\,\text{odd}}\varphi(l). \end{aligned}$$

We now want to show that

$$\sum_{m|n}\mu(m)\,\frac{m}{2n}\sum_{l|n/m,\, l\,\text{even}}\varphi(l)\,3^{\frac{n}{ml}} = -\frac{1}{2n}\sum_{d|n,\, d\,\text{even}}\mu(d)\,3^{\frac{n}{d}},$$

which proves the first part of (3.16). Indeed, the use of the substitution $d = ml$ in the virtue of the proof of Lemma 3.8 and (3.11) yields

$$\sum_{m|n}\mu(m)\,\frac{m}{2n}\sum_{l|n/m,\, l\,\text{even}}\varphi(l)\,3^{\frac{n}{ml}} = \frac{1}{2n}\sum_{d|n}3^{\frac{n}{d}}\sum_{l|n/d,\, l\,\text{even}}\mu\left(\frac{d}{l}\right)\frac{d}{l}\,\varphi(l) = -\frac{1}{2n}\sum_{d|n,\, d\,\text{even}}\mu(d)\,3^{\frac{n}{d}}.$$

The rest of the proof is concluded by the evaluation of

$$\sum_{d|n}\mu\left(d\right)\frac{d}{2n}\sum_{l|n/d,\, l\,\text{odd}}\varphi(l).$$

Any number $m \in \mathbb{N}$ can be expressed as $m = 2^{\alpha}P$, where $\alpha \in \mathbb{N}_0$ and P is a product of odd primes. Then

$$\frac{1}{m}\sum_{d/m,\, d\,\text{odd}}\varphi(d) = \frac{P}{m} = \frac{1}{2^{\alpha}} \tag{3.17}$$

since

$$\sum_{d|m}\varphi(d) = m,$$

holds in general, [2].

Assume now that $n \in \mathbb{N}$ can be expressed as $n = 2^{\beta}Q$, where $\beta \in \mathbb{N}_0$ and Q is a product of odd primes. Let us turn our attention to the equality

$$\frac{1}{n}X_{NL(n)} = \sum_{d|n}\mu\left(d\right)\frac{d}{n}\sum_{l|n/d,\, l\,\text{odd}}\varphi(l).$$

Any $d|n$ can be represented as $2^{\gamma}R$, where $0 \le \gamma \le \beta$ and R is a product of odd primes. Decomposing the expression by the exponent γ and using (3.17) lead to

$$\frac{1}{n}X_{NL(n)} = \sum_{\gamma=0}^{\beta}\sum_{R|Q}\mu(2^{\gamma}R)\,\frac{1}{2^{\beta-\gamma}} = \sum_{\gamma=0}^{\beta}\frac{1}{2^{\beta-\gamma}}\sum_{R|Q}\mu(2^{\gamma}R). \tag{3.18}$$

Assume that $\beta = 0$ and $Q = 1$. A straightforward computation gives

$$\left.\frac{1}{n}X_{NL(n)}\right|_{n=1} = \mu(1)\cdot 1 = 1.$$

Assume that $\beta > 1$ and $Q = 1$. If we consider $\gamma \geq 2$ in (3.18), then $\mu(2^\gamma R) = 0$. The sum can be now evaluated

$$\frac{1}{n} X_{NL(n)}\bigg|_{n=2^\beta} = \mu(1)\,\frac{1}{2^\beta} + \mu(2)\,\frac{1}{2^{\beta-1}} = -\frac{1}{2^\beta} = -\frac{1}{n}.$$

Assume that $Q > 1$. Let us fix γ such that $0 \leq \gamma \leq \beta$. Without loss of generality, we can assume that $\gamma \leq 1$ and each prime factor in R is present at most once. Indeed, $\mu(2^\gamma R) = 0$ otherwise. The sign of the nonzero expression $\mu(2^\gamma R)$ is now dependent on the number of prime factors of R. If there are m prime factors in Q, then R with l factors can be chosen in $\binom{m}{l}$ possible ways. The sign of $\mu(2^\gamma R)$ alternates as l increases and we have

$$\sum_{l=0}^{m} (-1)^l \binom{m}{l} = 0.$$

This results in

$$\frac{1}{n} X_{NL(n)}\bigg|_{n=2^\beta Q} = 0. \qquad \square$$

We conclude this section with two lemmas that are direct consequences of Möbius inversion formula (Theorem 2.7), Lemma 3.5 (respectively 3.6) and Lemma 3.8.

Lemma 3.12. *Given $n \in \mathbb{N}$, the number of the permuted Lyndon bracelets (the group D_n^Π) with period n on the set of all words of length n made with the alphabet $\mathcal{A}_2$ is*

$$BL^\pi_{\mathcal{A}_2}(n) = \frac{1}{2}\left[NL^\pi_{\mathcal{A}_2}(n) + \sum_{d|n} \mu\left(\frac{n}{d}\right)\, X^\pi_{B,2}(d)\right], \tag{3.19}$$

where $NL^\pi_{\mathcal{A}_2}(n)$ and $X^\pi_{B,2}(d)$ are given by (3.15) and (3.7), respectively.

Lemma 3.13. *Given $n \in \mathbb{N}$, the number of the permuted Lyndon bracelets (the group D_n^Π) with period n on the set of all words of length n made with the alphabet $\mathcal{A}_3$ is*

$$BL^\pi_{\mathcal{A}_3}(n) = \frac{1}{2}\left[NL^\pi_{\mathcal{A}_3}(n) + \sum_{d|n} \mu\left(\frac{n}{d}\right)\, X^\pi_{B,3}(d)\right], \tag{3.20}$$

where $NL^\pi_{\mathcal{A}_3}(n)$ and $X^\pi_{B,3}(d)$ are given by (3.16) and (3.9), respectively.

4 Conclusion

We start the final part of this paper with the proof of the main result, Theorem 2.9.

Proof of Theorem 2.9. For given $n \geq 2$ the sequence $BL^\pi_{\mathcal{A}_3}(n)$ gives the number of the permuted Lyndon bracelets, i.e., the equivalence classes of words with respect to the rotations r^i, the reflection s, the value permutation π (group D_n^Π), their compositions and with primitive period of length n. As discussed in §2.2, regions Ω_w surely have identical (the rotations r_i, the reflections s) or similar, with respect to the operator $\mathcal{T}$ defined by (2.8), (the value permutation π) shape and are thus qualitatively equivalent (see Definition 2.4). The expression (2.11) is exactly (3.20) with (3.16) substituted and (2.13) corresponds to (3.9). We sum $BL^\pi_{\mathcal{A}_3}(n)$ from $m = 2$ to avoid including trivial existence regions for the constant words $\mathfrak{o}\mathfrak{o}\ldots\mathfrak{o}$, $\mathfrak{a}\mathfrak{a}\ldots\mathfrak{a}$ and $\mathfrak{1}\mathfrak{1}\ldots\mathfrak{1}$

which are represented by the additional 1, this yields (2.9). The formulas are upper estimates only since we cannot eliminate the possibility that there are two qualitatively equivalent regions whose respective words are not related by any of the symmetries. Similar argumentation holds for regions belonging to the stable stationary solutions since the corresponding words are made with the alphabet $\mathcal{A}_2$, Lemma 2.1 and Definition 2.2. The expression (2.12) is equal to (3.19) where $BL^{\pi}_{\mathcal{A}_2}(n)$ is given by (3.15). □

The approach presented here can be used to obtain similar results in other or more general settings. The two main extension directions are the change of a spatial structure and the change of dynamics. The extensions can be combined but we present them separately for the sake of clarity.

4.1 Change of spatial structure

4.1.1 Graphs with nontrivial automorphism

The main objects of interests were the LDE (1.1) and the GDE (1.6) in this paper. In general, given a graph $\mathcal{G} = (V, E)$, the Nagumo graph differential equation can be written as

$$\mathrm{u}_i'(t) = d \sum_{j \in \mathcal{N}(i)} \big(\mathrm{u}_j(t) - \mathrm{u}_i(t)\big) + f\big(\mathrm{u}_i(t); a\big),$$

where $i \in V$ and $\mathcal{N}(i)$ is the set of all neighbours of the vertex i, i.e., $j \in \mathcal{N}(i)$ if and only if $(i, j) \in E$. Provided the graph $\mathcal{G}$ has a nontrivial automorphism (a nontrivial self-map which preserves the edge-vertex connectivity) the approach used here can be extended. Indeed, the group D_n is the automorphism group of the cycle graph with n vertices and all computations can be carried out by replacing the dihedral group D_n with the automorphism group of the graph $\mathcal{G}$.

4.1.2 Multi-dimensional square lattices

The underlying spatial structure of the LDE (1.1) is a one-dimensional lattice, an infinite path graph. Examination of bistable reaction-diffusion systems on multi-dimensional square lattices has been carried out, see e.g., [11, 18, 22]. For example, let us have a bistable reaction-diffusion system on the two-dimensional square lattice

$$u_{i,j}'(t) = d\big(u_{i-1,j}(t) + u_{i+1,j}(t) + u_{i,j-1}(t) + u_{i,j+1}(t) - 4u_{i,j}(t)\big) + f\big(u_{i,j}(t); a\big), \tag{4.1}$$

for $i, j \in \mathbb{Z}$. A reproduction of the proof of Lemma 2.1 together with the comparison principle [16, Proposition 3.1] show that the stationary solutions of the LDE (4.1) in the form of a repeated 2×2 pattern are equivalent to the stationary solutions of the GDE (1.6) on four vertices with the doubled diffusion rate d, see Figure 4.1 for illustration.

4.2 Change of dynamics

Various changes in the nonlinear part of (1.1) are discussed here. The proof of Lemma 2.1 in [21, Lemma 1] is actually independent of the nonlinear term with one exception. Indeed, the only part of the proof dependent on the specific nonlinear term is the comparison principle from [11, Lemma 1] and the only assumption is the existence of two ordered steady states of the equation, constant 0 and constant 1 in our case. This is satisfied for bistable and multistable reaction terms presented here.

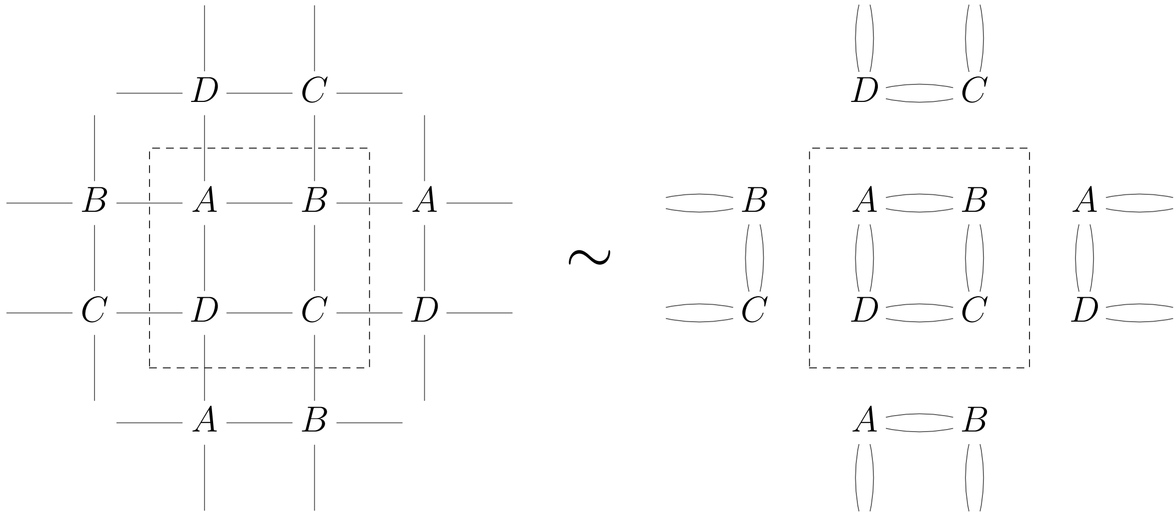


Figure 4.1: Illustration of a possible Lemma 2.1 extension for patterns on two-dimensional lattices. A general idea is that the edges crossing the dashed line are wrapped back inside from the opposite sides.

4.2.1 Scaled cubic nonlinearity

The cubic bistable nonlinear term (1.2) is dependent on one parameter only and moreover, the value of the parameter is actually one of its roots. Let us assume the function

$$f_{\mathrm{cub}}(s, \mathrm{p}) = s\big(s - \nu_-(\mathrm{p})\big)\big(\nu_+(\mathrm{p}) - s\big), \tag{4.2}$$

where $\mathrm{p} \in \Theta \subset \mathbb{R}^m$ is a detuning vector, Θ is an open set and we assume that $\nu_-, \nu_+ : \Theta \to \mathbb{R}^+$ and $0 < \nu_-(\mathrm{p}) < \nu_+(\mathrm{p})$ for all $\mathrm{p} \in \Theta$. The term f_{cub} has two bounding roots 0 and $\nu_+(\mathrm{p})$ for any given $\mathrm{p} \in \Theta$. The LDE (1.1) with f_{cub} thus admits the comparison principle and its n-periodic stationary solutions correspond to the stationary solutions of its respective GDE on a cycle graph with n vertices.

The stationary problem for the GDE can be written in the form

$$\widetilde{h}(\mathrm{u}; \mathrm{p}, d) = 0, \tag{4.3}$$

where

$$\widetilde{h}_i(\mathrm{u}; \mathrm{p}, d) = d(\mathrm{u}_{i-1} - 2\mathrm{u}_i + \mathrm{u}_{i+1}) + f_{\mathrm{cub}}(\mathrm{u}_i, \mathrm{p}).$$

We omitted the modulo wrapping at vertices 1 and n as in (2.2) to enlighten the notation. A direct computation yields

$$\widetilde{h}(\mathrm{u}; \mathrm{p}, d) = \nu_+^3(\mathrm{p})\, h\left(\frac{\mathrm{u}}{\nu_+(\mathrm{p})}; \frac{\nu_-(\mathrm{p})}{\nu_+(\mathrm{p})}, \frac{d}{\nu_+^2(\mathrm{p})}\right).$$

This enables us to define solution types for (4.3) via Definition 2.2 (the sign of the first derivative's determinant agrees) and to obtain corresponding existence regions $\widetilde{\Omega}_{\mathrm{w}}$ through the implicit transformation

$$\widetilde{\Omega}_{\mathrm{w}} = \left\{ (\mathrm{p}, d) \in \Theta \times \mathbb{R}_0^+ \;\middle|\; \left(\frac{\nu_-(\mathrm{p})}{\nu_+(\mathrm{p})}, \frac{d}{\nu_+^2(\mathrm{p})}\right) \in \Omega_{\mathrm{w}} \right\}. \tag{4.4}$$

An example of system leading to (4.3) is the reduced version of the model decribing potential propagation in myelinated axon with recovery [4]

$$\begin{aligned} u_i'(t) &= d\big(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\big) + f_{\text{Bell}}\big(u_i(t); a, b\big) - v_i(t), \\ v_i'(t) &= \sigma\, u_i(t) - \gamma\, v_i(t) \end{aligned} \tag{4.5}$$

such that

$$f_{\text{Bell}}\big(s; a, b\big) = s(s-a)(b-s).$$

Via approach similar to [4], we assume, that the change of the recovery value v_i is faster than the change in u_i and thus the second equation in (4.5) resides at its steady state. The problem can be then expressed as

$$u_i'(t) = d\big(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\big) + f_{\text{Bell}}\big(u_i(t); a, b\big) - \beta\, u_i(t)$$

with $\beta = \sigma/\gamma$ possibly small and the generalization of Lemma 2.1 ensures the equivalence of the periodic steady states of (4.5) and the system (4.3) solutions. We can directly determine

$$\mathbf{p} = (a, b, \beta), \quad \Theta = \left((a, b, \beta) \in \mathbb{R}^3 \,\middle|\, a, b, \beta > 0,\ b > a,\ \beta < \frac{(a-b)^2}{4} \right), \tag{4.6}$$

$$\nu_{\pm}(a, b, \beta) = \frac{1}{2}\left(a + b \pm \sqrt{(a-b)^2 - 4\beta}\right). \tag{4.7}$$

The inequality $b > a$ preserves the bistable behaviour in the original equation. See Figure 4.2 for illustration.

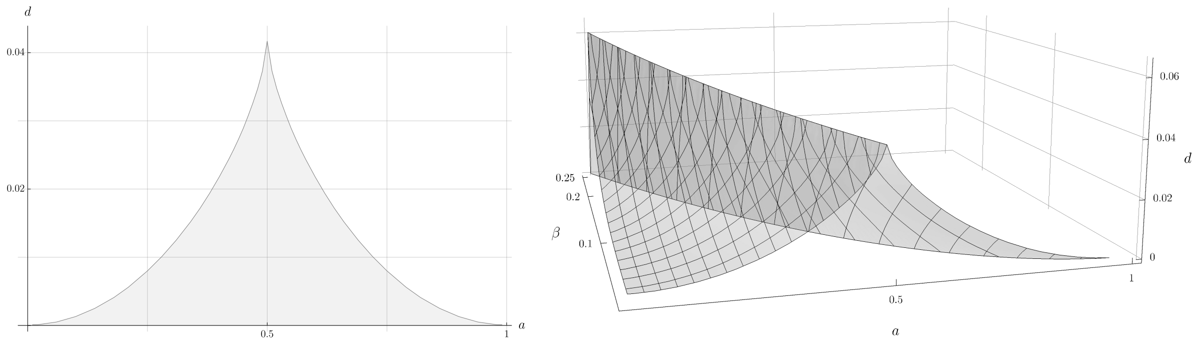


Figure 4.2: The left panel depicts the region Ω_{01} for the equation (1.1) and its comparison to the same region for (4.5) obtained via the transformation in (4.4). The parameter $b = 1$ was set.

4.2.2 Polynomial nonlinearity of higher order

This paper focused on the model (1.1) with the cubic bistable nonlinearity

$$f(s; a) = s(1-s)(s-a).$$

The idea presented in §2.1 can be extended to a general polynomial nonlinearity provided it allows a spatially nonhomogeneous steady state of the LDE (1.1) or the GDE (1.6)

$$f_{\text{ext}}(s; a_1, \ldots, a_q) = s(1-s)\prod_{i=1}^{q}(s - a_i)$$

for $q \geq 3$ odd, $a_i \in (0,1)$ and $a_i \neq a_j$ for all $i, j \in \{1, \ldots, q\}$ such that $i \neq j$. Note that

$$f_{\text{ext}}(s; a_1, \ldots, a_q) = -f_{\text{ext}}(1 - s; 1 - a_1, \ldots, 1 - a_q)$$

holds and the value permutation π can be thus redefined as

$$(\pi(\mathtt{w}))_i = \begin{cases} \mathtt{1}, & \mathtt{w}_i = \mathtt{0}, \\ \mathtt{a}_{q-i+1}, & \mathtt{w}_i = \mathtt{a}_i,\ i = 1, \ldots, q, \\ \mathtt{0}, & \mathtt{w}_i = \mathtt{1}. \end{cases}$$

The counting formulas for the necklaces (3.1), the bracelets (3.2) and the Lyndon words (3.13), (3.14) can be then straightforwardly applied with $k = q + 2$ for all solutions and $k = 2 + (q-1)/2$ for asymptotically stable solutions.

The cubic-quintic nonlinearity, [5],

$$f_{\text{cq}}(s, \mu) = \mu s + 2s^3 - s^5$$

has five distinct roots

$$\left\{0, \pm\sqrt{1 \pm \sqrt{1-\mu}}\right\}.$$

for $\mu \in (0,1)$. The LDE with f_{cq} can be rescaled for the stationary solutions to fit the interval $[0,1]$ and the approach described in the previous paragraph can be used. Note that sequence of $1/2$'s is then always a stationary solution regardless of μ and all the counting formulas would count not only shape-distinct regions $\Omega_{\mathtt{w}}$ but distinct periodic stationary solutions. This is true for (1.1) only if $a = 1/2$.

4.2.3 General bistable nonlinearity

A system with a general bistable nonlinearity f_{gen} as considered in [24]

1. $f_{\text{gen}}(0) = f_{\text{gen}}(a) = f_{\text{gen}}(1) = 0$, $0 < a < 1$ and $f_{\text{gen}}(x) \neq 0$ for $x \neq 0, a, 1$,
2. $f_{\text{gen}}(x) < 0$ for $0 < x < a$ and $f_{\text{gen}}(x) > 0$ for $a < x < 1$,
3. $f'_{\text{gen}}(x_0) = f'_{\text{gen}}(x_1) = 0$, $0 < x_0 < a < x_1 < 1$ and $f'_{\text{gen}}(x) \neq 0$ for $x \neq x_0, x_1$,

can be only partially treated by the methods presented here. The conditions above allow the application of the implicit function theorem. It is however possible for a general bistable nonlinearity to exhibit the "blue sky" bifurcation before any of the determinants in Definition 2.2 reaches zero, see [31, §1.2.2]. Moreover, the action of the value permutation group Π can be included only if f_{gen} can be expressed in a form such that

$$f_{\text{gen}}(x; a) = -f_{\text{gen}}(1 - x; 1 - a)$$

holds.

4.2.4 Multi-dimensional local dynamics

The local dynamics at an isolated vertex of models (1.1) and (1.6) are one-dimensional since the behaviour at a single vertex can be described by a single equation. This is not always the case in many reaction-diffusion models. For example, the Lotka–Volterra competition model on a graph as in [35]

$$\begin{aligned} u_i'(t) &= d_u \sum_{j \in N(i)} \big(u_j(t) - u_i(t)\big) + \rho_u u_i(t)\big(1 - u_i(t) - \alpha v_i(t)\big), \\ v_i'(t) &= d_v \sum_{j \in N(i)} \big(v_j(t) - v_i(t)\big) + \rho_v v_i(t)\big(1 - \beta u_i(t) - v_i(t)\big), \end{aligned} \tag{4.8}$$

where $N(i)$ is the set of all neighbours of the vertex i, locally possesses two asymptotically stable stationary solutions (originating from the points $(0,1)$ and $(1,0)$ which can be denoted by $\mathtt{0}, \mathtt{1}$) and one unstable nontrivial stationary solution (originating from the point $((1-\alpha)/(1-\alpha\beta), (1-\beta)/(1-\alpha\beta))$ here denoted by $\mathtt{a}$) at each separated vertex provided $\alpha, \beta > 1$. The solutions containing elements originating from $(0,0)$ are not considered since their immediate continuation is directed outside the positive quadrant. The implicit function theorem assures that the naming scheme from Definition 2.2 can be employed. A proper scaling results in $\rho_u = \rho_v = 1$ and the regions of existence are pathwise connected sets of points $(d_u, d_v, \alpha, \beta) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times (1, \infty) \times (1, \infty)$. As in [35], it is convenient to fix the ratio $\eta := d_u/d_v$ and consider the regions Ω_{w} in a three-dimensional space only. The stationary problem for (4.8) is now invariant with respect to the transformation $u_i \leftrightarrow v_i$, $\alpha \leftrightarrow \beta$ and all counting results can be thus applied provided the underlying graph has a nontrivial automorphism.

4.3 Open questions

The idea of Lemma 2.1 is such that the restriction to the periodic stationary solutions of the LDE (1.1) allows us to formally divide the lattice into a countable number of identical finite graphs. Similar approach was used in §4.1. General equivalence claim which helps to reduce the search for an arbitrary periodic patterns in sufficiently regular infinite graphs (e.g., triangular lattice, hexagonal lattice) into a finite-dimensional problem is still missing.

Acknowledgements

Author acknowledges the support of the project LO1506 of the Czech Ministry of Education, Youth and Sports under the program NPU I and the support of Grant Agency of the Czech Republic, project no. 18-03253S. The author is grateful to Petr Stehlík and Jonáš Volek for their valuable comments and patience. The author would also like to express his gratitude to the anonymous referee.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 24, 1–19; https://doi.org/10.14232/ejqtde.2021.1.24 www.math.u-szeged.hu/ejqtde/

On the BMO and $C^{1,\gamma}$-regularity for a weak solution of fully nonlinear elliptic systems in dimension three and four

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Received 4 October 2020, appeared 1 April 2021
Communicated by Patrizia Pucci

Abstract. We consider a nonlinear elliptic system of type

$$-D_\alpha A_i^\alpha(x, Du) = D_\alpha f_i^\alpha$$

and give conditions guaranteeing $C^{1,\gamma}$ interior regularity of weak solutions.

Keywords: nonlinear elliptic systems, regularity, Campanato–Morrey spaces.

2020 Mathematics Subject Classification: 35J60.

1 Introduction.

In this paper we give conditions guaranteeing that the first derivatives of weak solutions to the Dirichlet problem for a nonlinear elliptic system

$$\begin{cases} -D_\alpha A_i^\alpha(x, Du) = D_\alpha f_i^\alpha, & i = 1, \dots, N,\ \alpha \in \mathbb{R}^n,\ |\alpha| = 1,\ x \in \Omega, \\ \qquad u(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded $C^{1,1}$ domain with points $x = (x_1, \dots, x_n)$, $u : \Omega \to \mathbb{R}^N$, $u(x) = (u^1(x), \dots, u^N(x))$, $N \geq 2$ is a vector-valued function with gradient $Du = (D_1 u, \dots, D_n u)$, $D_\alpha = \partial/\partial x_\alpha$ and coefficients A_i^α are continuously differentiable with respect to Du and Hölder continuous with respect to x and in the following we will specify our assumptions imposed on the function (f_i^α) and boundary datum g (throughout the whole text we use the summation convention over repeated indexes).

It is well known that elliptic systems in general do not conserve the regularizing property of Laplace equation and the attempts to find conditions guaranteeing the smoothness of weak

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solutions as well as to construct counterexamples are rich and far reaching. The positive results, i.e. proof that weak solutions of systems of order $2k$ have (under suitable assumptions) continuous partial derivatives of order k, started already with pioneering work of Ch. B. Morrey in 1937 for domains Ω in $\mathbb{R}^2$ (see [16]) and continued by deep results of E. De Giorgi (see [7]) who proved that weak solutions of one equation of second order with linear growth and bounded and measurable coefficients on $\Omega \subset \mathbb{R}^n$ have continuous first derivatives. The case of nonlinear systems on plane domains was solved in paper by J. Stará (see [23]) in 1971 for systems of higher order.

In dimensions $n \geq 3$ analogous results do not hold as was shown by counterexamples of E. De Giorgi (see [8]) and E. Giusti and M. Miranda in 1968 (see [10]), J. Nečas in 1975 (see [19]) and L. Šverák and X. Yan in 2002 (see [24]).

The system (1.1) has been extensively studied in the papers [1, 2, 9, 12, 15, 17, 20] and for detailed and well-arranged informations, see [15]. If $n \geq 3$, it is known that Du can be discontinuous. Campanato in [3] proved for the system (1.1) that $Du \in \mathcal{L}^{2,\theta}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$ with $n-2 < \theta < n$, and also $u \in C^{0,(\theta-n+2)/2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ if $n = 3, 4$. More important for our work is a more general result from Kristensen–Melcher [13].

There are known many conditions on the coefficients which guarantee that solutions of nonlinear elliptic system of equations have required smoothness and, vice versa, counterexamples illustrating that generally such assertions do not hold.

In the present paper, that is extending the articles [4], [5] and [6], we introduce another conditions on coefficients of a nonlinear elliptic system (1.1) and we show that if the first derivatives of weak solutions u to Dirichlet problem for the system satisfy (1.11) with given $\mathcal{M}$ and $\widetilde{\Psi}$ from (1.10) then the gradient of weak solutions are locally BMO or Hölder continuous on domains Ω in $\mathbb{R}^3$ and $\mathbb{R}^4$. The condition (1.11) shows that the our result is applicable to broader class of problems for smaller value of $\mathcal{M}$. Finally, the reality of our theoretical result is illustrated by means of numerical examples.

By a weak solution to the Dirichlet problem for (1.1) we understand $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that $u - g \in W^{1,2}_0(\Omega, \mathbb{R}^N)$, $g \in W^{1,2}(\Omega, \mathbb{R}^N)$, $f \in L^2(\Omega, \mathbb{R}^{nN})$ and

$$\int_\Omega A^\alpha_i\,(x, Du(x))\, D_\alpha \varphi^i(x)\, dx = -\int_\Omega f^\alpha_i(x) D_\alpha \varphi^i(x)\, dx, \quad \forall\, \varphi \in W^{1,2}_0(\Omega, \mathbb{R}^N). \tag{1.2}$$

Further the symbol $\Omega_o \subset\subset \Omega$ stands for $\overline{\Omega}_o \subset \Omega$, $d_\Omega = \mathrm{diam}(\Omega)$ and for the sake of simplicity we denote by $|\cdot|$ the norm in $\mathbb{R}^n$ as well as in $\mathbb{R}^N$ and $\mathbb{R}^{nN}$. If $x \in \mathbb{R}^n$ and r is a positive real number, we set $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, (i.e., the open ball in $\mathbb{R}^n$), $\Omega_r(x) = \Omega \cap B_r(x)$. Denote by $u_{x,r} = u_r = \int_{\Omega_r(x)} u(y)\, dy / m_n(\Omega_r(x)) = ⨍_{\Omega_r(x)} u(y)\, dy$ the mean value of the function $u \in L(\Omega, \mathbb{R}^N)$ over the set $\Omega_r(x)$. Here $m_n(\Omega_r(x))$ is the n-dimensional Lebesgue measure of $\Omega_r(x)$ and we set $U_r(x) = \int_{\Omega_r(x)} |Du(y) - (Du)_{x,r}|^2\, dy / r^n = ⨍_{\Omega_r(x)} |Du(y) - (Du)_{x,r}|^2\, dy$, $\phi(x, r) = \int_{\Omega_r(x)} |Du(y) - (Du)_{x,r}|^2\, dy$.

The coefficients $(A^\alpha_i)_{i=1,\ldots,N,\alpha=1,\ldots,n}$ have linear controlled growth and satisfy strong uniform ellipticity condition. Without loss of generality we can suppose that $A^\alpha_i(x, 0) = 0$. We suppose that $A^\alpha_i(x, p) \in C^1(\mathbb{R}^{nN})$ for all $x \in \Omega$ and

(i) the strong ellipticity condition holds, i.e. there exist ν, $M > 0$ such that for every $x \in \Omega$ and p, $\xi \in \mathbb{R}^{nN}$

$$\nu|\xi|^2 \leq \frac{\partial A^\alpha_i}{\partial p^j_\beta}(x, p)\xi^i_\alpha \xi^j_\beta \leq M|\xi|^2, \tag{1.3}$$

(ii)

$$|A_i^\alpha(x,p)| \le M(1+|p|), \qquad \sum_{i,j,\alpha,\beta}\left|\frac{\partial A_i^\alpha}{\partial p_\beta^j}(x,p)\right| \le M, \tag{1.4}$$

for all $(x,p) \in \Omega \times \mathbb{R}^{nN}$,

(iii) for all $x, y \in \Omega$ and $p \in \mathbb{R}^{nN}$

$$|A_i^\alpha(x,p) - A_i^\alpha(y,p)| \le C_H|x-y|^\chi|p|, \qquad C_H > 0 \tag{1.5}$$

where $\chi = 1$ for $n = 3,4$,

(iv) there is a real function ω continuous on $[0,\infty)$, which is bounded, nondecreasing, concave, $\omega(0) = 0$ and such that for all $x \in \Omega$ and $p, q \in \mathbb{R}^{nN}$

$$\left|\frac{\partial A_i^\alpha}{\partial p_\beta^j}(x,p) - \frac{\partial A_i^\alpha}{\partial p_\beta^j}(x,q)\right| \le \omega\left(|p-q|\right). \tag{1.6}$$

We denote $\omega_\infty = \lim_{t\to\infty}\omega(t)$ and clearly $\omega(t) \le 2M$.

It is well known (see [9], p.169) that for uniformly continuous $\partial A_i^\alpha/\partial p_\beta^j$ there exists a real function ω satisfying (iv) and, viceversa, (1.6) implies uniform continuity of $\partial A_i^\alpha/\partial p_\beta^j$ and absolute continuity of ω on $[0,\infty)$. By pointwise derivative ω' we will understand the right derivative of ω which is finite on $(0,\infty)$.

Here we will consider the function ω from (1.6) given by the formula

$$\omega(t) = \begin{cases} \omega_o(t), & \text{for } 0 \le t \le t_o,\ t_o > 0 \\ \omega_1(t) = \dfrac{\sqrt{\varepsilon}}{t_o^\gamma}t^\gamma, & \text{for } t_o < t < t_1, \\ \omega_\infty & \text{for } t \ge t_1 \end{cases} \tag{1.7}$$

where ω_o is arbitrary continuous, concave, nondecreasing function such that $\omega_o(0) = 0$ and the constants $0 < \gamma \le 0.44$, $t_o > 0$ are selected in such a way that ω is continuous and concave on $[0,\infty)$.

For example we can choose

$$\omega_o(t) = \frac{2\sqrt{\varepsilon}}{2 + \ln\frac{t_o}{t}} \quad \text{for } 0 < t \le t_o,$$

and this function fail to satisfy Dini condition. It is obvious that in such case the coefficients $\partial A_i^\alpha/\partial p_\beta^j$ are only continuous.

It is well known that on the above assumptions the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(A(x,Du) + f\right) = 0 & \text{in } \Omega, \\ \qquad u - g \in W_0^{1,2}(\Omega,\mathbb{R}^N) \end{cases} \tag{1.8}$$

has for any function $f, g \in W^{1,2}(\Omega,\mathbb{R}^N)$ the unique solution u in the same space.

For the problem (1.8) the following estimate holds

$$
\begin{aligned}
&\fint_\Omega |Du - (Du)_\Omega|^2 \, dy \\
&\quad \le 12\left(\frac{M}{\nu}\right)^2 \fint_\Omega |Dg - (Dg)_\Omega|^2 \, dx + \left(\frac{10E}{\nu}\right)^2 \left(\frac{E}{\nu} + \frac{M}{\nu} + 3\left(\frac{M}{\nu}\right)^2\right) \fint_\Omega |Dg|^2 \, dx \\
&\qquad + 20\left(\frac{nN}{\nu}\right)^2 \left(1 + \left(\frac{4E}{\nu}\right)^2\right) \fint_\Omega |f - (f)_\Omega|^2 \, dx
\end{aligned} \tag{1.9}
$$

where $E = nNC_H d_\Omega^\chi$ (see Appendix A for the proof of (1.9)).

In the following we will use the function $\widetilde{\Psi}(u) = ue^{u^{2/(2\mu-1)}}$, here $u \geq 0$, $\mu \geq 17$ (for detailed information for $\widetilde{\Psi}$, see (2.6)) and we can define the value

$$
\mathcal{M} = \sup_{t_o < t < \infty} \frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) - \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t - t_o} < \infty \tag{1.10}
$$

where ω is from (1.7), $\varepsilon = \omega_\infty^2 / C_\mu^\alpha$, $\alpha > 1 - 2/n$ and $C_\mu = \left(\frac{(n-2)\mu}{2e}\right)^\mu$.

Now we can formulate the main theorem.

Theorem 1.1. *Let $\Omega_o \subset\subset \Omega \subset \mathbb{R}^n$, $d_o = \operatorname{dist}(\Omega_o, \partial\Omega)/2$, $n = 3, 4$. Assume that $g \in W^{1,2}(\Omega, \mathbb{R}^N)$, $Dg \in L^{2,\zeta}(\Omega, \mathbb{R}^{nN})$, $\zeta > 2$, $f \in W^{1,2} \cap \mathcal{L}^{2,\xi}(\Omega, \mathbb{R}^N)$, $n < \xi \leq n+2$, $n \leq \vartheta < \lambda = \min\{2\chi + \zeta, \xi\}$ and moreover $\operatorname{div} f \in L^\zeta(\Omega, \mathbb{R}^{nN})$. Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1.1) satisfying the conditions*

$$
\fint_\Omega |Du - (Du)_\Omega|^2 \, dy < \frac{1}{\mathcal{M}^2}, \tag{1.11}
$$

(1.13) and

$$
C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega, \mathbb{R}^{nN})} \leq \frac{|\Omega| \left(1 - (4\epsilon_o)^{\frac{\lambda}{\vartheta} - 1}\right) \epsilon_o^3 \nu^2}{8 d_o^n \max\{d_o^\lambda, d_o^{\lambda - n}\} \mathcal{M}^2} \tag{1.12}
$$

where $\epsilon_o = 1/4(2^{n+5}L)^{\frac{\vartheta}{n+2-\vartheta}}$, the constants L, C_H, C_M come from Lemma 2.5, (1.5) and (3.8), respectively. Then $Du \in C^{0,(\vartheta-n)/2}(\Omega_o, \mathbb{R}^{nN})$ in the case $\vartheta > n$ and $Du \in BMO(\Omega_o, \mathbb{R}^{nN})$ for $\vartheta = n$.

Remark 1.2. In the foregoing formulas the constants $\mu \geq 17$, $\alpha > 1 - 2/n$ have to be such that

$$
C_\mu^{\frac{n}{n-2}\alpha - 1} \geq 2^{\frac{6}{n-2}} \left(20 C_S \frac{M\omega_\infty}{\nu^2} \left(\frac{|\Omega|}{(2d_o)^n}\right)^{\frac{1}{2n}}\right)^{\frac{2n}{n-2}} \epsilon_o^{-\frac{n}{n-2}}. \tag{1.13}
$$

Here C_S is the Sobolev embedding constant.

The theorem we formulated above tells that, if coefficients of a nonlinear system satisfy (iv) with some ω given (1.7) and (1.11)–(1.13) are fulfilled, then the gradient of u is Hölder continuous on Ω_o.

In most partial regularity results for the system (1.1) the regular points $x \in \Omega$ of solution u are characterized in such a way that for some $r_x > 0$ the quantity $U_{r_x}(x)$ (for its definition see first section) has to be sufficiently small, but our condition regularity (1.11) allows $U_r(x)$ not to be necessarily small. Moreover, the condition (1.11) is global condition (we do not know an analogous condition from the literature) and has fundamental meaning for domain Ω in which it is possible ensure that the ratio $|\Omega|/(2d_o)^n$ is not extremely great (e.g. for the ball, see (1.13)).

For the function ω from (1.7) the right-hand side of (1.11) can be chosen in the following form

$$\frac{1}{\mathcal{M}^2} = \frac{t_o^2}{4}\left(\min\left\{\frac{1}{3\gamma}, \frac{C_\mu^{\left(-1+\frac{1}{2\gamma}\right)\alpha}}{e^{C_\mu^{\frac{2\alpha}{2\mu-1}}}}\right\}\right)^2. \tag{1.14}$$

(see Appendix B for more information and for μ and α see Remark 1.2).

Remark 1.3. We would like to point that, in the case of (1.8), the left-hand side of (1.11) could be substituted with the right-hand side of (1.9). We can present some consequences of our theorem that follow from estimate (1.9).

$$g = const. \wedge f = const. \implies \fint_\Omega |Du - (Du)_\Omega|^2\, dy = 0 \implies u = P_1$$

$$g = P_1 \wedge f = const. \wedge C_H = 0 \implies \fint_\Omega |Du - (Du)_\Omega|^2\, dy = 0 \implies u = P_1$$

$$g = P_1 \wedge f = const. \wedge d_\Omega \searrow 0 \implies \fint_\Omega |Du - (Du)_\Omega|^2\, dy \searrow 0$$

$$g = P_1 \wedge f \in \mathcal{L}^{2,\xi}(\Omega, \mathbb{R}^{nN}),\ \xi > n \wedge C_H = 0 \wedge d_\Omega \searrow 0 \implies \fint_\Omega |Du - (Du)_\Omega|^2\, dy \searrow 0$$

where P_1 is a polynom of at most first degree. We note that the last mentioned condition involves the data of the problem (1.8) only.

Remark 1.4. It is useful to point out that in the case when the ratio ω_∞/ν is small enough, the regularity of solution to the problem (1.8) is guaranteed by the Proposition 2.4 from [4].

2 Preliminaries and notations

Beside the usually used space $C_0^\infty(\Omega, \mathbb{R}^N)$, Hölder space $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^N)$, $W^{k,p}_{loc}(\Omega, \mathbb{R}^N)$, $W^{k,p}_0(\Omega, \mathbb{R}^N)$ (see, e.g.[22]) we use the following Campanato and Morrey spaces.

Definition 2.1 (Campanato and Morrey spaces)**.** Let $v \in [0, n]$. The Morrey space $L^{2,v}(\Omega, \mathbb{R}^N)$ is the subspaces of such functions $u \in L^2(\Omega, \mathbb{R}^N)$ for which

$$\|u\|^2_{L^{2,v}(\Omega,\mathbb{R}^N)} = \sup_{r>0, x\in\Omega} r^{-v} \int_{\Omega_r(x)} |u(y)|^2\, dy < \infty.$$

Let $v \in [0, n+2]$. The Campanato space $\mathcal{L}^{2,v}(\Omega, \mathbb{R}^N)$ is the subspace of such functions $u \in L^2(\Omega, \mathbb{R}^N)$ for which

$$[u]^2_{\mathcal{L}^{2,v}(\Omega,\mathbb{R}^N)} = \sup_{r>0, x\in\Omega} r^{-v} \int_{\Omega_r(x)} |u(y) - u_{x,r}|^2\, dy < \infty.$$

Remark 2.2. It is worth recalling the trivial but basic property that $\int_\Omega |u - u_\Omega|^2 dx = \min_{c\in\mathbb{R}^N} \int_\Omega |u - c|^2\, dx$ holds for each $u \in L^2(\Omega, \mathbb{R}^N)$.

For more details see [1], [9] and [22]. In particular, we will use:

Proposition 2.3. *For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary we have the following*

(a) *$\mathcal{L}^{2,v}(\Omega, \mathbb{R}^N)$ is isomorphic to the $C^{0,(v-n)/2}(\overline{\Omega}, \mathbb{R}^N)$, for $n < v \leq n+2$,*

(b) *$L^{2,v}(\Omega, \mathbb{R}^N)$ is isomorphic to the $\mathcal{L}^{2,v}(\Omega, \mathbb{R}^N)$, $0 \leq v < n$,*

(c) *the imbedding $\mathcal{L}^{2,v_1}(\Omega, \mathbb{R}^N) \subset \mathcal{L}^{2,v_2}(\overline{\Omega}, \mathbb{R}^N)$ is continuous for all $0 \leq v_2 < v_1 \leq n+2$,*

(d) *$L^{2,n}(\Omega, \mathbb{R}^N)$ is isomorphic to the $L^\infty(\Omega, \mathbb{R}^N) \subsetneq \mathcal{L}^{2,n}(\Omega, \mathbb{R}^N)$.*

The following lemma is a modification of a lemma from [5].

Lemma 2.4. *Let $A > 1$, d be positive numbers, C, B_1, $B_2 \geq 0$, $n \leq \delta < \beta$, $\delta < \alpha \leq n+2$ and $0 < s \leq 1$. Then there exist positive constants k_1, k_2 so that for any nonnegative nondecreasing function φ defined on $[0, d]$ and satisfying the inequalities*

$$\varphi(\sigma) \leq A \left(\frac{\sigma}{R}\right)^\alpha \varphi(R) + \frac{1}{2}\left(1 + A\left(\frac{\sigma}{R}\right)^\alpha\right)\left[(B_1 + B_2 U_{2R}^s)\,\varphi(2R) + CR^\beta\right], \qquad \forall\, 0 < \sigma < R \leq \frac{d}{2} \tag{2.1}$$

and

$$B_1 + B_2 U_d^s \leq \frac{1}{4}\tau^\delta, \qquad B_2 \left(\frac{Cm}{2^\beta \tau^\delta (1 - \tau^{\beta-\delta})}\right)^s \leq \frac{1}{4}\tau^\delta \tag{2.2}$$

where $U_R = \phi(R)/R^n$, $m = \max\{d^\beta, d^{\beta-n}\}$ and $\tau = 1/(2^{\alpha+1}A)^{\frac{1}{\alpha-\delta}}$. Then it holds

$$U_\sigma \leq \sigma^{\delta-n}(k_1 \varphi(d) + k_2), \qquad \forall\, \sigma \in (0, d]\,. \tag{2.3}$$

Proof. I. We will prove by induction that

$$\varphi(\tau^k d) \leq \tau^{k\delta}\left(\varphi(d) + \frac{Cm}{2^\beta \tau^\delta}\sum_{j=0}^{k-1} \tau^{(\beta-\delta)j}\right), \qquad U_{\tau^k d} \leq \tau^{k(\delta-n)}\left(U_d + \frac{Cm}{2^\beta \tau^\delta}\sum_{j=0}^{k-1}\tau^{(\beta-\delta)j}\right). \tag{2.4}$$

Let $k = 1$. Putting $\sigma = \tau d$, $R = d/2$ in (2.1) we obtain thanks to (2.2) and the assumption on τ

$$\begin{aligned}\varphi(\tau d) &\leq 2^\alpha A \tau^\alpha \varphi\left(\tfrac{d}{2}\right) + \tfrac{1}{2}(1 + 2^\alpha A \tau^\alpha)\left[(B_1 + B_2 U_d^s)\,\varphi(d) + C\left(\tfrac{d}{2}\right)^\beta\right] \\ &\leq (2^\alpha A\tau^\alpha + B_1 + B_2 U_d^s)\,\varphi(d) + C\left(\tfrac{d}{2}\right)^\beta = \tau^\delta\left(\varphi(d) + \tfrac{Cm}{2^\beta\tau^\delta}\right).\end{aligned}$$

Also by means of (2.2) we get

$$U_{\tau d} \leq \tau^{\delta-n}\left(U_d + \tfrac{Cm}{2^\beta \tau^\delta}\right), \qquad B_1 + B_2 U_{\tau d}^s \leq \frac{1}{2}\tau^\delta.$$

Next put $\sigma = \tau^{k+1}d$, $R = \tau^k d/2$ into (2.1) we get

$$\begin{aligned}\varphi(\tau^{k+1}d) &\leq 2^\alpha A\tau^\alpha \varphi\left(\tfrac{1}{2}\tau^k d\right) + \tfrac{1}{2}(1 + 2^\alpha A\tau^\alpha)\left[(B_1 + B_2 U_{\tau^k d}^s)\,\varphi(\tau^k d) + \tfrac{Cd^\beta}{2^\beta}\tau^{k\beta}\right] \\ &\leq (2^\alpha A\tau^\alpha + B_1 + B_2 U_{\tau^k d}^s)\,\varphi(\tau^k d) + \tfrac{Cd^\beta}{2^\beta \tau^\delta}\tau^{k\beta+\delta} \leq \tau^\delta \varphi(\tau^k d) + \tfrac{Cm}{2^\beta\tau^\delta}\tau^{(k+1)\delta}\end{aligned}$$

because $2^\alpha A\tau^\alpha + B_1 + B_2 U^s_{\tau^k d} \le \tau^\delta$. Using (2.4) we get

$$\varphi(\tau^{k+1}d) \le \tau^\delta \varphi(\tau^k d) + \tfrac{Cd^\beta}{2^\beta \tau^\delta}\tau^{(k+1)\delta} \le \tau^{(k+1)\delta}\left(\varphi(d) + \frac{Cm}{2^\beta\tau^\delta}\sum_{j=0}^{k-1}\tau^{(\beta-\delta)j}\right) + \frac{Cm}{2^\beta\tau^\delta}\tau^{(k+1)\delta}$$
$$= \tau^{(k+1)\delta}\left(\varphi(d) + \frac{Cm}{2^\beta\tau^\delta}\sum_{j=0}^{k}\tau^{(\beta-\delta)j}\right).$$

It immediately implies the estimate of $U_{\tau^{k+1}d}$.

II. Let now σ be an arbitrary positive number less than d. Then there is an integer k such that $\tau^{k+1}d \le \sigma < \tau^k d$. Using monotonicity of φ, this inequality and (2.4) we get

$$\varphi(\sigma) \le \varphi(\tau^k d) \le \tau^{k\delta}\left(\varphi(d) + \frac{Cm}{2^\beta\tau^\delta}\sum_{j=0}^{k-1}\tau^{j(\beta-\delta)}\right) \le \frac{\sigma^\delta}{(\tau d)^\delta}\left(\varphi(d) + \frac{Cm}{2^\beta\tau^\delta(1-\tau^{\beta-\delta})}\right)$$

and this estimate together with the choice of $k_1 = 1/(\tau d)^\delta$, $k_2 = Cm/(2^\beta d^\delta \tau^{2\delta}(1-\tau^{\beta-\delta}))$ completes the proof. □

For the statement of following Lemma see e.g. [1,9,20].

Lemma 2.5. *Consider system of the type (1.1) with $A_i^\alpha(x,p) = A_{ij}^{\alpha\beta}p_\beta^j$, $A_{ij}^{\alpha\beta} \in \mathbb{R}$ (i.e. linear system with constant coefficients) satisfying (i), (ii) and (iii). Then there exists a constant $L = L(n,N,M/\nu) \ge 1$ such that for every weak solution $v \in W^{1,2}(\Omega,\mathbb{R}^N)$ and for every $x \in \Omega$ and $0 < \sigma \le R \le \operatorname{dist}(x,\partial\Omega)$ the following estimate*

$$\int_{B_\sigma(x)} |Dv(y) - (Dv)_{x,\sigma}|^2\, dy \le L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R(x)} |Dv(y) - (Dv)_{x,R}|^2\, dy$$

holds.

Remark 2.6. The constant L from the previous lemma can be stated as

$$L = c(n,N)\left(\frac{M}{\nu}\right)^{2\left(2+\left[\frac{n}{2}\right]\right)}$$

and, because of a better presentment, choosing $n=3$, $N=2$ we can compute $L < 1.4\cdot 10^8(M/\nu)^6$.

In the paper [4, p. 108] a system for $n=N=3$ of type (1.1) was presented for which we can compute $L \approx 10^8$.

Lemma 2.7. *[25, p. 37] Let $\phi : [0,\infty] \to [0,\infty]$ be non decreasing function which is absolutely continuous on every closed interval of finite length, $\phi(0) = 0$. If $w \ge 0$ is measurable and $E(t) = \{y \in \mathbb{R}^n : w(y) > t\}$ then*

$$\int_{\mathbb{R}^n} \phi\circ w\, dy = \int_0^\infty m_n(E(t))\,\phi'(t)\,dt\,.$$

In the proof of Theorem 1.1 we will use an inequality which is a consequence of Natanson's lemma (see e.g. [18, p. 262]) and Fatou's lemma. It can be read as follows.

Lemma 2.8. *Let $f : [a, \infty) \to \mathbb{R}$ be a nonnegative function which is integrable on $[a, b]$ for all $a < b < \infty$ and*

$$\mathcal{N} = \sup_{0<h<\infty} \frac{1}{h} \int_a^{a+h} f(t)\, dt < \infty$$

is satisfied. Let $g : [a, \infty) \to \mathbb{R}$ be an arbitrary nonnegative, non-increasing and integrable function. Then

$$\int_a^\infty f(t) g(t)\, dt$$

exists and

$$\int_a^\infty f(t) g(t)\, dt \le \mathcal{N} \int_a^\infty g(t)\, dt$$

holds.

In the proof of Theorem 1.1 we use an inequality which can be read as follows.

Proposition 2.9 (see [4]). *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to (1.1) satisfying (i), (ii), (iii) and (iv). Then for every ball $B_{2R}(x) \subset \Omega$ and arbitrary constants $\mu \ge 2$, $b > 0$, $1 < q \le n/(n-2)$ and $c \in \mathbb{R}^{nN}$ we have*

$$\int_{B_R(x)} |Du - (Du)_{x,R}|^2 \ln_+^\mu \left(b|Du - (Du)_{x,R}|^2 \right) dy$$
$$\le 2^{n(q-1)} \left(5C_S \frac{M}{\nu} \right)^{2q} \left(\frac{\mu}{(q-1)\mathrm{e}} \right)^\mu \left(\frac{b}{(2R)^n} \int_{B_{2R}(x)} |Du - c|^2\, dy \right)^{q-1} \int_{B_{2R}(x)} |Du - c|^2\, dy \qquad (2.5)$$

where C_S is the Sobolev embedding constant.

Hereafter we shall use conjugate Young functions Φ, Ψ

$$\Phi(u) = u \ln_+^\mu (au) \quad \text{for } u \ge 0, \qquad \Psi(u) \le \overline{\Psi}(u) = \frac{1}{a} u \mathrm{e}^{u^{\frac{2}{2\mu-1}}} \quad \text{for } u \ge 0, \qquad (2.6)$$

where $a > 0$ and $\mu \ge 2$ are constants,

$$\ln_+(au) = \begin{cases} 0 & \text{for } 0 \le u < \frac{1}{a}, \\ \ln(au) & \text{for } u \ge \frac{1}{a}. \end{cases}$$

Then Young inequality for Φ, Ψ reads as

$$xy \le \Phi(x) + \Psi(y), \qquad \forall\, x, y \in \mathbb{R}. \qquad (2.7)$$

3 Proof of Theorem 1.1

Let x_o be any point of $\Omega_o \cap \mathcal{S}$ (it means that $\int_{B_R(x_o)} |Du - (Du)_{x_o,R}|^2\, dx > 0$) and $R \le d_o$. Where no confusion can result, we will use the notation B_R, U_R, $\phi(R)$ and $(Du)_R$ instead of $B_R(x_o)$, $U_R(x_o)$, $\phi(x_o, R)$ and $(Du)_{x_o,R}$. Denoting $A_{ij,0}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x_o, (Du)_R)$,

$$\tilde{A}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta} \left(x_o, (Du)_R + t\left(Du - (Du)_R\right) \right) dt\,,$$

we can rewrite the system (1.1) as

$$-D_\alpha \left(A_{ij,0}^{\alpha\beta} D_\beta u^j \right) = - D_\alpha \left(\left(A_{ij,0}^{\alpha\beta} - \tilde{A}_{ij}^{\alpha\beta} \right) \left(D_\beta u^j - \left(D_\beta u^j \right)_R \right) \right)$$
$$- D_\alpha \left(A_i^\alpha(x_o, Du) - A_i^\alpha(x, Du) \right) + D_\alpha \left(f_i^\alpha(x) - (f_i^\alpha)_R \right) .$$

Split u as $v+w$ where v is the solution of the Dirichlet problem

$$-D_\alpha\left(A_{ij,0}^{\alpha\beta}D_\beta v^j\right)=0 \quad \text{in } B(R)$$
$$v-u\in W_0^{1,2}\left(B_R,\mathbb{R}^N\right).$$

and $w\in W_0^{1,2}(B_R,\mathbb{R}^N)$ is the weak solution of the system

$$\begin{aligned}-D_\alpha\left(A_{ij,0}^{\alpha\beta}D_\beta w^j\right)=&-D_\alpha\left(\left(A_{ij,0}^{\alpha\beta}-\tilde{A}_{ij}^{\alpha\beta}\right)\left(D_\beta u^j-\left(D_\beta u^j\right)_R\right)\right)\\&-D_\alpha\left(A_i^\alpha(x_o,Du)-A_i^\alpha(x,Du)\right)+D_\alpha\left(f_i^\alpha(x)-(f_i^\alpha)_R\right).\end{aligned}$$

For every $0<\sigma\le R$ from Lemma 2.5 it follows

$$\int_{B_\sigma}|Dv-(Dv)_\sigma|^2\,dx\le L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R}|Dv-(Dv)_R|^2\,dx$$

hence

$$\begin{aligned}\int_{B_\sigma}|Du-(Du)_\sigma|^2\,dx&\le 2L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R}|Dv-(Dv)_R|^2\,dx+4\int_{B_R}|Dw|^2\,dx\\&\le 4L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R}|Du-(Du)_R|^2\,dx+4\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\int_{B_R}|Dw|^2\,dx.\end{aligned}\tag{3.1}$$

Now $w\in W_0^{1,2}(B_R,\mathbb{R}^N)$ satisfies

$$\begin{aligned}\int_{B_R}A_{ij,0}^{\alpha\beta}D_\beta w^jD_\alpha\varphi^i\,dx\le&\int_{B_R}\left|A_{ij,0}^{\alpha\beta}-\tilde{A}_{ij}^{\alpha\beta}\right|\left|D_\beta u^j-(D_\beta u^j)_R||D_\alpha\varphi^i\right|\,dx\\&+\int_{B_R}|A_i^\alpha(x_o,Du)-A_i^\alpha(x,Du)|\left|D_\alpha\varphi^i\right|\,dx\\\le&\left(\int_{B_R}\omega^2\left(|Du-(Du)_R|\right)|Du-(Du)_R|^2\,dx\right)^{1/2}\left(\int_{B_R}|D\varphi|^2\,dx\right)^{1/2}\\&+\left(\int_{B_R}|A_i^\alpha(x_o,Du)-A_i^\alpha(x,Du)|^2\,dx\right)^{1/2}\left(\int_{B_R}|D\varphi|^2\,dx\right)^{1/2}\\&+\left(\int_{B_R}|f-f_R|^2\,dx\right)^{1/2}\left(\int_{B_R}|D\varphi|^2\,dx\right)^{1/2}\end{aligned}$$

for any $\varphi\in W_0^{1,2}(B_R,\mathbb{R}^N)$. Hence, choosing $\varphi=w$, we get

$$\begin{aligned}\nu^2\int_{B_R}|Dw|^2\,dx\le&\,2\int_{B_R}\omega^2\left(|Du-(Du)_R|\right)|Du-(Du)_R|^2\,dx\\&+4\int_{B_R}|A_i^\alpha(x_o,Du)-A_i^\alpha(x,Du)|^2\,dx+4\int_{B_R}|f-f_R|^2\,dx.\end{aligned}\tag{3.2}$$

From (3.1) and (3.2) we have

$$\begin{aligned}\phi(\sigma)=\int_{B_\sigma}|Du-(Du)_\sigma|^2\,dx&\le 4L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R}|Du-(Du)_R|^2\,dx\\&+\frac{8\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2}\left[\int_{B_R}\omega^2\left(|Du-(Du)_R|\right)|Du-(Du)_R|^2\,dx\right.\\&\left.+2\int_{B_R}|A_i^\alpha(x_o,Du)-A_i^\alpha(x,Du)|^2\,dx+2\int_{B_R}|f-f_R|^2\,dx\right]\\&=4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R)+\frac{8\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2}\left(I_1+2I_2+2I_3\right)\end{aligned}\tag{3.3}$$

We use the Young inequality (2.7) (here complementary functions are defined through (2.6)) and for any $0 < \varepsilon < \omega_\infty^2$ we obtain

$$
\begin{aligned}
I_1 &= \int_{B_R} \omega^2 \left(|Du - (Du)_R|\right) |Du - (Du)_R|^2 \, dx \\
&\le \varepsilon \int_{B_R} |Du - (Du)_R|^2 \ln_+ \left(a\varepsilon \, |Du - (Du)_R|^2\right) dx + \int_{B_R} \Psi\left(\frac{\omega_R^2}{\varepsilon}\right) dx = \varepsilon J_1 + J_2
\end{aligned}
\tag{3.4}
$$

where $\omega_R^2(x) = \omega^2\left(|Du(x) - (Du)_R|\right)$.

The term J_1 can be estimated by means of Proposition 2.9 (here $q = n/(n-2)$) and we get

$$
J_1 \le C C_\mu \left(a\varepsilon U_{2R}\right)^{q-1} \phi(2R) \tag{3.5}
$$

where

$$
C = 2^{(q-1)n} \left(5C_S \frac{M}{\nu}\right)^{2q}, \qquad C_\mu = \left(\frac{n-2}{2\mathrm{e}} \mu\right)^{\mu}.
$$

Taking in Lemma 2.7 $w(y) = |v(y) - v_{x,R}|$ on $B_R(x)$ and $w = 0$ otherwise, we have $E_R(t) = \{y \in B_R(x) \,:\, |v(y) - v_{x,R}| > t\}$ and for the the second integral J_2 we get

$$
J_2 = \frac{1}{a} \int_0^\infty \frac{d}{dt} \widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) m_n\left(E_R(t)\right) dt \tag{3.6}
$$

where $\widetilde{\Psi} = a\overline{\Psi}$.

We have (we use Lemma 2.8) for $\forall\, \varepsilon > 0$

$$
\begin{aligned}
&\int_0^\infty \frac{d}{dt} \widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) m_n\left(E_R(t)\right) dt \\
&\quad \le \int_0^{t_o} \frac{d}{dt} \widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) m_n\left(E_R(t)\right) dt + \int_{t_o}^\infty \frac{d}{dt} \widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) m_n\left(E_R(t)\right) dt \\
&\quad \le \kappa_n R^n \int_0^{t_o} \frac{d}{dt} \widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) dt + \sup_{t_o < t < \infty} \left(\frac{1}{t - t_o} \int_{t_o}^{t} \frac{d}{ds} \widetilde{\Psi}\left(\frac{\omega^2(s)}{\varepsilon}\right) ds\right) \int_{t_o}^\infty m_n\left(E_R(s)\right) ds \\
&\quad \le \kappa_n \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right) R^n + \sup_{t_o < t < \infty} \left[\frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) - \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t - t_o}\right] \int_{B_R} |Du - (Du)_R| \, dx \\
&\quad \le \kappa_n \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right) R^n + \mathcal{M} \kappa_n^{1/2} R^{n/2} \phi^{1/2}(R) \\
&\quad \le \left[\frac{\kappa_n}{2^n} \frac{\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{U_{2R}} + \left(\frac{\kappa_n}{2^n}\right)^{1/2} \frac{\mathcal{M}}{\sqrt{U_{2R}}}\right] \phi(2R) \le \left[\frac{\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{U_{2R}} + \frac{\mathcal{M}}{\sqrt{U_{2R}}}\right] \phi(2R).
\end{aligned}
\tag{3.7}
$$

If for some $R > 0$ the average $U_R = 0$ then it is clear that x_o is the regular point. So next we can suppose U_R is positive for all $R > 0$.

From [2] and [13] we have that $Du \in L^{2,\zeta}(\Omega, \mathbb{R}^{nN})$, $\zeta \in (2,3)$ and also

$$
\begin{aligned}
\int_{B_R} |Du|^2 \, dx &\le \frac{c^2(\zeta, M/\nu, C_H, \chi, \Omega)}{\nu^2} \left(\|f\|^2_{L^{2,\zeta}(\Omega,\mathbb{R}^{nN})} + \|Dg\|^2_{L^{2,\zeta}(\Omega,\mathbb{R}^{nN})}\right) R^\zeta \\
&= C_M R^\zeta, \qquad \forall\, 0 < R \le d_o.
\end{aligned}
\tag{3.8}
$$

From the assumptions (iii) follows

$$I_2 \le C_M C_H^2 R^{2\chi} \int_{B_R} |Du|^2\, dx \le C_M C_H^2 R^{2\chi+\zeta} \tag{3.9}$$

and

$$I_3 \le [f]^2_{\mathcal{L}^{2,\xi}(\Omega_o,\mathbb{R}^{nN})} R^{\xi}\,. \tag{3.10}$$

We get from (3.3) and (3.4) by means of (3.5), (3.7), (3.9) and (3.10)

$$\begin{aligned}
\phi(\sigma) &\le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(2R)\\
&\quad + 8\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\left\{\left[\frac{CC_\mu\,\varepsilon}{\nu^2}(a\varepsilon U_{2R})^{q-1} + \frac{1}{a\nu^2}\left(\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)\frac{1}{U_{2R}} + \frac{\mathcal{M}}{\sqrt{U_{2R}}}\right)\right]\phi(2R)\right.\\
&\quad \left. + 2C_M\left(\frac{C_H}{\nu}\right)^2 R^{2+\zeta} + \frac{2}{\nu^2}[f]^2_{\mathcal{L}^{2,\xi}(\Omega_o,\mathbb{R}^{nN})}R^{\xi}\right\}\\
&\le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(2R) + 8\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\times\\
&\quad \times\left\{\left[\frac{CC_\mu\varepsilon}{\nu^2}(a\varepsilon U_{2R})^{q-1} + \frac{1}{a\nu^2}\left(\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)\frac{1}{U_{2R}} + \frac{\mathcal{M}}{\sqrt{U_{2R}}}\right)\right]\phi(2R)\right.\\
&\quad \left. + \frac{2}{\nu^2}\left(C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega_o,\mathbb{R}^{nN})}\right)R^{\lambda}\right\}
\end{aligned} \tag{3.11}$$

where $\lambda = \min\{2\chi+\zeta, \xi\}$.

In (3.11) we can choose

$$\varepsilon = \frac{\omega_\infty^2}{C_\mu^\alpha}, \qquad a = \frac{128|\Omega|^{1/2}}{(2d_o)^{n/2}\,\nu^2\epsilon_o\, U_{2R}} \quad \text{for } U(2R) > 0 \tag{3.12}$$

where $\epsilon_o = \frac{1}{4(2^{n+5}L)^{\vartheta/(n+2-\vartheta)}}$ and $\mu \ge 17$, $\alpha > 1 - 2/n$ are suitable constants.

We set $P = \omega_\infty/\nu$. Then we obtain for $U_{2R} > 0$

$$\begin{aligned}
\phi(\sigma) &\le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + \frac{1}{2}\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\\
&\quad \times\left\{\left[\frac{16CP^2}{C_\mu^{\alpha-1}}\left(\frac{128|\Omega|^{1/2}P^2}{(2d_o)^{n/2}\,C_\mu^\alpha\,\epsilon_o}\right)^{q-1} + \frac{(2d_o)^{n/2}\nu^2}{8|\Omega|^{1/2}}\left(\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right) + \mathcal{M}\sqrt{U_{2R}}\right)\epsilon_o\right]\phi(2R)\right.\\
&\quad \left. + \frac{32}{\nu^2}\left(C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega_o,\mathbb{R}^{nN})}\right)R^{\lambda}\right\}\\
&\le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + \frac{1}{2}\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\\
&\quad \times\left\{\left[\frac{2^{7q-3}CP^{2q}}{C_\mu^{q\alpha-1}\,\epsilon_o^{q-1}}\left(\frac{|\Omega|}{(2d_o)^n}\right)^{\frac{q-1}{2}} + \frac{1}{8}\left(3 + \frac{(2d_o)^{n/2}}{|\Omega|^{1/2}}\mathcal{M}\sqrt{U_{2R}}\right)\epsilon_0\right]\phi(2R)\right.\\
&\quad \left. + \frac{32}{\nu^2}\left(C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega_o,\mathbb{R}^{nN})}\right)R^{\lambda}\right\}
\end{aligned}$$

for all $0 < \sigma \le R \le d_o$ (for the estimate $\widetilde{\Psi}(\omega^2(t_o)/\varepsilon) \le 3$, see Appendix).

In the last term of the foregoing inequality we employed the estimate from (1.11). The constants $\alpha > 1 - 2/n$ and $\mu \ge 17$ can be always chosen in such a way that

$$\frac{2^{7q-3}CP^{2q}}{C_\mu^{q\alpha-1}\,\epsilon_o^{q-1}}\left(\frac{|\Omega|}{(2d_o)^n}\right)^{\frac{q-1}{2}} \le \frac{1}{2}\epsilon_o \quad\Longleftrightarrow\quad C_\mu^{q\alpha-1} \ge \frac{2^{7q-3}CP^{2q}}{\epsilon_o^q}\left(\frac{|\Omega|}{(2d_o)^n}\right)^{\frac{q-1}{2}} \tag{3.13}$$

and we get

$$\phi(\sigma) \le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + \frac{1}{2}\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)$$
$$\times\left\{\left[\frac{7}{8}\epsilon_o + \frac{1}{8}\,\epsilon_o\mathcal{M}\,\frac{(2d_o)^{n/2}}{|\Omega|^{1/2}}\sqrt{U_{2R}}\right]\phi(2R) + \frac{32\left(C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega,\mathbb{R}^{nN})}\right)}{\nu^2}R^\lambda\right\}$$

for all $0 < \sigma \le R \le d_o$.

We can put $A = 4L$, $\alpha = n + 2$,

$$B_1 = \frac{7}{8}\,\epsilon_o, \qquad B_2 = \frac{\mathcal{M}\,(2d_o)^{n/2}}{8|\Omega|^{1/2}}\,\epsilon_o, \qquad C = \frac{32\left(C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega,\mathbb{R}^{nN})}\right)}{\nu^2},$$

$s = 1/2$, $\beta = \lambda$, $\delta = \vartheta$, $\tau^\delta/4 = \epsilon_o$ and $d = d_o$. Now from (1.11) follows that $B_2\sqrt{U_{2d_o}(x)} \le \epsilon_o/8$ and if (1.12) is satisfied we can using Lemma 2.4. In conclusion we get

$$\phi(\sigma) \le \sigma^\vartheta(k_1\phi(2d_o) + k_2), \qquad \forall\, 0 < \sigma \le d_o,\ \ n \le \vartheta < \lambda\,.$$

□

4 Illustrating examples and comments

Example 4.1. We will consider the system (1.1) with ω from Example 1.7 for $\Omega = B_R(0)$, $\Omega_o = B_{R/2}(0)$ and also $d_o = R/4$. Supposing $n = 3$, $N = 2$, $q = 3$, $\vartheta = 3.1$, $\omega_\infty = \nu$, $M/\nu = 10$, $C_S = 10$, $\epsilon_o \approx 10^{-28}$ (the value ϵ_o seems to be realistic, see Remark 2.6, here $L \approx 10^{14}$), $\chi = 1$ and $\lambda = 4$ we can get as follows:

ω_∞	$=$	10^{30}	10^{50}	10^{70}	10^{90}	10^{110}	10^{130}
t_o	$=$	10^{3}	10^{11}	10^{18}	10^{24}	10^{30}	10^{36}
$\omega(t_o)$	$\approx$	10^{8}	10^{28}	10^{48}	10^{68}	10^{88}	10^{108}
t_1	$\approx$	10^{58}	10^{67}	10^{73}	10^{79}	10^{85}	10^{91}
$\omega(\omega_\infty)$	$\approx$	10^{19}	10^{44}	10^{69}	10^{90}	10^{110}	10^{130}
real value $\frac{1}{\mathcal{M}^2}$	$\approx$	10^{5}	10^{21}	10^{35}	10^{47}	10^{59}	10^{71}
estimate $\frac{1}{\mathcal{M}^2}$ by means of (1.14)	$\approx$	10^{5}	10^{21}	10^{35}	10^{47}	10^{59}	10^{71}
$\omega\left(\frac{1}{\mathcal{M}^2}\right)$	$\approx$	10^{10}	10^{32}	10^{55}	10^{78}	10^{100}	10^{122}
α	$=$	1.9	1.92	1.92	1.92	1.91	1.9
γ	$=$	0.39	0.39	0.39	0.39	0.39	0.39
μ	$=$	30.3	30.1	30.1	30.1	30.2	30.3
the right-hand side of (1.12)	$\approx$	$\frac{10^{-16}}{Z_R}$	$\frac{10^{40}}{Z_R}$	$\frac{10^{94}}{Z_R}$	$\frac{10^{146}}{Z_R}$	$\frac{10^{198}}{Z_R}$	$\frac{10^{250}}{Z_R}$

Here t_1 is the point for which $\omega(t_1) = \omega_\infty$ and $\mathcal{Z}_R = \max\{(R/4)^4, R/4\}$. It is necessary to remember that the condition (1.13) from the main Theorem is satisfied for the above-mentioned parameters.

In conclusion is possible to say, that the theorem gives good results if $\nu \geq 1/\epsilon_o = 4(2^{n+5}L)^{\frac{\vartheta}{n+2-\vartheta}}$.

Appendix A

First we have to estimate of $\int_\Omega |Du|^2\,dx$. We can rewrite the system (1.1) as

$$\int_\Omega \left[\widetilde{A}_{ij}^{\alpha\beta}\left(D_\beta u^j - D_\beta g^j\right) + \left(A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)\right) + \widetilde{\widetilde{A}}_{ij}^{\alpha\beta} D_\beta g^j\right] D_\alpha \varphi^i\,dx = \int_\Omega \left(f_i^\alpha - (f_i^\alpha)_\Omega\right) D_\alpha \varphi^i\,dx \quad \text{(A.1)}$$

where $\widetilde{A}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}\left(x_o, Dg + t(Du - Dg)\right)dt$ and $\widetilde{\widetilde{A}}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}(x_o, tDg)\,dt$.

We put in (A.1) $\varphi^i = u^i - g^i$ and we get as follows

$$\begin{aligned}
&\int_\Omega \widetilde{A}_{ij}^{\alpha\beta} D_\beta u^j D_\alpha u^i\,dx + \int_\Omega \widetilde{A}_{ij}^{\alpha\beta} D_\beta g^j D_\alpha g^i\,dx \\
&\quad = \int_\Omega \widetilde{A}_{ij}^{\alpha\beta}\left(D_\beta u^j D_\alpha g^i + D_\beta g^j D_\alpha u^i\right)dx - \int_\Omega \widetilde{\widetilde{A}}_{ij}^{\alpha\beta} D_\beta g^j D_\alpha u^i\,dx + \int_\Omega \widetilde{\widetilde{A}}_{ij}^{\alpha\beta} D_\beta g^j D_\alpha g^i\,dx \\
&\qquad - \int_\Omega \left(A_i^\alpha(x,Dg) - A_i^\alpha(x_o,Dg)\right)\left(D_\alpha u^i - D_\alpha g^i\right)dx \\
&\qquad + \int_\Omega \left(f_i^\alpha - (f_i^\alpha)_\Omega\right) D_\alpha u^i\,dx - \int_\Omega \left(f_i^\alpha - (f_i^\alpha)_\Omega\right) D_\alpha g^i\,dx
\end{aligned}$$

From ellipticity (1.3) we have

$$\begin{aligned}
\nu &\int_\Omega |Du|^2\,dx + \nu\int_\Omega |Dg|^2\,dx \\
&\leq \int_\Omega \left|\widetilde{A}_{ij}^{\alpha\beta}\right|\left(|D_\beta u^j||D_\alpha g^i| + |D_\beta g^j||D_\alpha u^i|\right)dx \\
&\quad + \int_\Omega \left|\widetilde{\widetilde{A}}_{ij}^{\alpha\beta}\right| |D_\beta g^j||D_\alpha u^i|\,dx + \int_\Omega \left|\widetilde{\widetilde{A}}_{ij}^{\alpha\beta}\right| |D_\beta g^j||D_\alpha g^i|\,dx \\
&\quad + \int_\Omega |A_i^\alpha(x,Dg) - A_i^\alpha(x_o,Dg)|\,|D_\alpha g^i|\,dx \\
&\quad + \int_\Omega |f_i^\alpha - (f_i^\alpha)_\Omega|\,|D_\alpha u^i|\,dx + \int_\Omega |f_i^\alpha - (f_i^\alpha)_\Omega|\,|D_\alpha g^i|\,dx \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6\,.
\end{aligned} \quad \text{(A.2)}$$

By means of Young's inequality we get by choosing $\varepsilon = \nu/2M$

$$\begin{aligned}
I_1 &\leq 2\int_\Omega |Du||Dg|\sum\left|\widetilde{A}_{ij}^{\alpha\beta}\right|dx \leq \frac{1}{2}\nu\int_\Omega |Du|^2\,dx + \frac{2M^2}{\nu}\int_\Omega |Dg|^2\,dx\,, \\
I_2 &\leq \int_\Omega |Du||Dg|\sum\left|\widetilde{\widetilde{A}}_{ij}^{\alpha\beta}\right|dx \leq \frac{1}{4}\nu\int_\Omega |Du|^2\,dx + \frac{M^2}{\nu}\int_\Omega |Dg|^2\,dx\,, \\
I_3 &\leq \int_\Omega |Dg||Dg|\sum\left|\widetilde{\widetilde{A}}_{ij}^{\alpha\beta}\right|dx \leq M\int_\Omega |Dg|^2\,dx\,, \\
I_4 &\leq C_H\int_\Omega |x - x_o|^\chi |Dg|\sum|D_\alpha g^i|\,dx \leq nNC_H d_\Omega^\chi\int_\Omega |Dg|^2\,dx\,,
\end{aligned}$$

$\varepsilon = \nu/4$

$$I_5 = \int_\Omega |f_i^\alpha - (f_i^\alpha)_\Omega||D_\alpha u^i|\,dx \le \int_\Omega |Du| \sum |f_i^\alpha - (f_i^\alpha)_\Omega|\,dx$$
$$\le \frac{1}{8}\nu \int_\Omega |Du|^2\,dx + \frac{2n^2N^2}{\nu} \int_\Omega |f-(f)_\Omega|^2\,dx\,,$$

$\varepsilon = 2\nu$

$$I_6 = \int_\Omega \sum |f_i^\alpha - (f_i^\alpha)_\Omega||D_\alpha g^i|\,dx \le \int_\Omega |Dg| \sum |f_i^\alpha - (f_i^\alpha)_\Omega|\,dx$$
$$\le 2\nu \int_\Omega |Dg|^2\,dx + \frac{n^2N^2}{8\nu} \int_\Omega |f-(f)_\Omega|^2\,dx\,.$$

Together from (A.2) we have

$$\int_\Omega |Du|^2\,dx \le 8\left(\frac{nNC_H d_\Omega^\chi}{\nu} + \frac{M}{\nu} + 3\left(\frac{M}{\nu}\right)^2\right)\int_\Omega |Dg|^2\,dx + \frac{18n^2N^2}{\nu^2}\int_\Omega |f-(f)_\Omega|^2\,dx\,. \tag{A.3}$$

Now we can rewrite the system (1.1) as

$$\int_\Omega \left[\widetilde{A}_{ij}^{\alpha\beta}\left(D_\beta u^j - (D_\beta g^j)_\Omega\right) + \left(A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)\right)\right] D_\alpha\varphi^i\,dx$$
$$= \int_\Omega \left(f_i^\alpha - (f_i^\alpha)_\Omega\right) D_\alpha \varphi^i\,dx \tag{A.4}$$

where $\widetilde{A}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}\left(x_o, (Dg)_\Omega + t\,(Du - (Dg)_\Omega)\right)\,dt$.

We put in (A.4) $\varphi^i = \left(u^i - (D_\alpha g^i)_\Omega x_\alpha\right) - \left(g^i - (D_\alpha g^i)_\Omega x_\alpha\right)$ and we get as follows

$$\begin{aligned}
&\int_\Omega \widetilde{A}_{ij}^{\alpha\beta}\left(D_\beta u^j - (D_\beta g^j)_\Omega\right)\left(D_\alpha u^i - (D_\alpha g^i)_\Omega\right)\\
&\quad - \int_\Omega \widetilde{A}_{ij}^{\alpha\beta}\left(D_\beta u^j - (D_\beta g^j)_\Omega\right)\left(D_\alpha g^i - (D_\alpha g^i)_\Omega\right)\,dx\\
&\quad + \int_\Omega \left(A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)\right)\left(D_\alpha u^i - (D_\alpha g^i)_\Omega\right)\,dx\\
&\quad - \int_\Omega \left(A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)\right)\left(D_\alpha g^i - (D_\alpha g^i)_\Omega\right)\,dx\\
&= \int_\Omega \left(f_i^\alpha(x) - (f_i^\alpha)_\Omega\right)\left(D_\alpha u^i - (D_\alpha g^i)_\Omega\right)\,dx - \int_\Omega \left(f_i^\alpha(x) - (f_i^\alpha)_\Omega\right)\left(D_\alpha g^i - (D_\alpha g^i)_\Omega\right)\,dx\,.
\end{aligned}$$

From ellipticity (1.3) we have

$$\begin{aligned}
\nu \int_\Omega |Du - (Dg)_\Omega|^2\,dx &\le \int_\Omega \left|\widetilde{A}_{ij}^{\alpha\beta}\right| \left|D_\alpha u^i - (D_\alpha g^i)_\Omega\right| \left|D_\alpha g^i - (D_\alpha g^i)_\Omega\right|\,dx\\
&\quad + \int_\Omega |A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)| \left|D_\alpha u^i - (D_\alpha g^i)_\Omega\right|\,dx\\
&\quad + \int_\Omega |A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)| \left|D_\alpha g^i - (D_\alpha g^i)_\Omega\right|\,dx\\
&\quad + \int_\Omega |f_i^\alpha(x) - (f_i^\alpha)_\Omega| \left|D_\alpha u^i - (D_\alpha g^i)_\Omega\right|\,dx\\
&\quad + \int_\Omega |f_i^\alpha(x) - (f_i^\alpha)_\Omega| \left|D_\alpha g^i - (D_\alpha g^i)_\Omega\right|\,dx\\
&= I_1 + I_2 + I_3 + I_4 + I_5\,.
\end{aligned} \tag{A.5}$$

By means of Young inequality we get by choosing $\varepsilon = \nu/M$

$$
\begin{aligned}
I_1 &\le \int_\Omega |Du-(Dg)_\Omega|\,|Dg-(Dg)_\Omega| \sum \left|\widetilde{A}_{ij}^{\alpha\beta}\right| dx \\
&\le \frac{1}{2}\nu \int_\Omega |Du-(Dg)_\Omega|^2\,dx + \frac{M^2}{2\nu}\int_\Omega |Dg-(Dg)_\Omega|^2\,dx\,,
\end{aligned}
$$

$\varepsilon = \nu/2$

$$
\begin{aligned}
I_2 &\le \int_\Omega |Du-(Dg)_\Omega| \sum |A_i^\alpha(x,Du)-A_i^\alpha(x_o,Du)|\,dx \\
&\le \frac{1}{4}\nu \int_\Omega |Du-(Dg)_\Omega|^2\,dx + \frac{n^2N^2}{\nu}\int_\Omega |A(x,Du)-A(x_o,Du)|^2\,dx \\
&\le \frac{1}{4}\nu \int_\Omega |Du-(Dg)_\Omega|^2\,dx + \frac{n^2N^2C_H^2 d_\Omega^{2\chi}}{\nu}\int_\Omega |Du|^2\,dx\,,
\end{aligned}
$$

$\varepsilon = \nu$

$$
\begin{aligned}
I_3 &\le \int_\Omega |Dg-(Dg)_\Omega| \sum |A_i^\alpha(x,Du)-A_i^\alpha(x_o,Du)|\,dx \\
&\le \frac{1}{2}\nu \int_\Omega |Dg-(Dg)_\Omega|^2\,dx + \frac{n^2N^2}{2\nu}\int_\Omega |A(x,Du)-A(x_o,Du)|^2\,dx \\
&\le \frac{1}{2}\nu \int_\Omega |Dg-(Dg)_\Omega|^2\,dx + \frac{n^2N^2C_H^2 d_\Omega^{2\chi}}{2\nu}\int_\Omega |Du|^2\,dx\,,
\end{aligned}
$$

$$
\begin{aligned}
I_5 &\le \int_\Omega |f_i^\alpha(x)-(f_i^\alpha)_\Omega|\,\left|D_\alpha g^i-(D_\alpha g^i)_\Omega\right| dx \\
&\le \frac{1}{2}\nu \int_\Omega |Dg-(Dg)_\Omega|^2\,dx + \frac{n^2N^2}{2\nu}\int_\Omega |f-(f)_\Omega|^2\,dx\,,
\end{aligned}
$$

$\varepsilon = \nu/4$

$$
\begin{aligned}
I_4 &\le \int_\Omega |f_i^\alpha(x)-(f_i^\alpha)_\Omega|\,\left|D_\alpha u^i-(D_\alpha g^i)_\Omega\right| dx \\
&\le \frac{1}{8}\nu \int_\Omega |Du-(Dg)_\Omega|^2\,dx + \frac{2n^2N^2}{\nu}\int_\Omega |f-(f)_\Omega|^2\,dx\,.
\end{aligned}
$$

Together we get

$$
\begin{aligned}
\int_\Omega |Du-(Du)_\Omega|^2\,dx \le \int_\Omega |Du-(Dg)_\Omega|^2\,dx &\le 4\left(2+\left(\frac{M}{\nu}\right)^2\right)\int_\Omega |Dg-(Dg)_\Omega|^2\,dx \\
&+ \frac{20n^2N^2}{\nu^2}\int_\Omega |f-(f)_\Omega|^2\,dx + \frac{12n^2N^2C_H^2 d_\Omega^{2\chi}}{\nu^2}\int_\Omega |Du|^2\,dx\,. \quad \text{(A.6)}
\end{aligned}
$$

By means of (A.3) we are getting from (A.6) final estimate

$$
\begin{aligned}
\int_\Omega |Du-(Du)_\Omega|^2\,dx \le 4&\left(2+\left(\frac{M}{\nu}\right)^2\right)\int_\Omega |Dg-(Dg)_\Omega|^2\,dx \\
&+ \frac{96n^2N^2C_H^2 d_\Omega^{2\chi}}{\nu^2}\left[\frac{nNC_H d_\Omega^{\chi}}{\nu} + \frac{M}{\nu} + 3\left(\frac{M}{\nu}\right)^2\right]\int_\Omega |Dg|^2\,dx \\
&+ \frac{4n^2N^2}{\nu^2}\left[5 + \frac{54n^2N^2C_H^2 d_\Omega^{2\chi}}{\nu^2}\right]\int_\Omega |f-(f)_\Omega|^2\,dx\,.
\end{aligned}
$$

Appendix B

We give estimates of the constant $\mathcal{M}$ defined by (1.10) where ω is defined by (1.7). We consider $t_o > 0$, $\alpha > 1 - 2/n$, $\mu \geq 17$ and $0 < \gamma \leq 0.44$.

$$
\begin{aligned}
h'(t) &= \left(\frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) - \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t - t_o} \right)' \\
&= \frac{\omega(t)\left[2\omega'(t)\left(1+\frac{2}{2\mu-1}\left(\frac{\omega^2(t)}{\varepsilon}\right)^{\frac{2}{2\mu-1}}\right)(t-t_o) - \omega(t)\right]\mathrm{e}^{\left(\frac{\omega^2(t)}{\varepsilon}\right)^{\frac{2}{2\mu-1}}} + \omega^2(t_o)\mathrm{e}^{\left(\frac{\omega^2(t_0)}{\varepsilon}\right)^{\frac{2}{2\mu-1}}}}{\varepsilon(t-t_o)^2} \\
&= \frac{\omega(t)\left[2\omega'(t)\left(1+a\left(\frac{\omega^2(t)}{\varepsilon}\right)^{a}\right)(t-t_o) - \omega(t)\right]\mathrm{e}^{\left(\frac{\omega^2(t)}{\varepsilon}\right)^{a}} + \omega^2(t_o)\mathrm{e}^{\left(\frac{\omega^2(t_0)}{\varepsilon}\right)^{a}}}{\varepsilon(t-t_o)^2}, \quad 0 < t_o < t < t_1.
\end{aligned}
$$

where $a = 2/(2\mu - 1) < 0.06$ $(\mu \geq 17)$.

For $\omega(t) = \omega_1(t) = kt^\gamma$, $k = \sqrt{\varepsilon}/t_o^\gamma$ we get

$$
\begin{aligned}
h'(t) &= \frac{k^2\left\{t^{2\gamma}\left[2\gamma\left(1+a\left(\frac{k^2t^{2\gamma}}{\varepsilon}\right)^a\right)\left(1-\frac{t_o}{t}\right)-1\right]\mathrm{e}^{\left(\frac{k^2t^{2\gamma}}{\varepsilon}\right)^a} + t_o^{2\gamma}\mathrm{e}^{\left(\frac{k^2t_o^{2\gamma}}{\varepsilon}\right)^a}\right\}}{\varepsilon(t-t_o)^2} \\
&= \frac{2\gamma\left(1+a\left(\frac{t}{t_o}\right)^{2a\gamma}\right)\left(1-\frac{t_o}{t}\right) + \left(\frac{t_o}{t}\right)^{2\gamma}\mathrm{e}^{\left(1-\left(\frac{t}{t_o}\right)^{2a\gamma}\right)} - 1}{(t-t_o)^2}\left(\frac{t}{t_o}\right)^{2\gamma}\mathrm{e}^{\left(\frac{t}{t_o}\right)^{2a\gamma}} \\
&= \frac{g_1(t)+g_2(t)-1}{(t-t_o)^2}\left(\frac{t}{t_o}\right)^{2\gamma}\mathrm{e}^{\left(\frac{t}{t_o}\right)^{2a\gamma}}, \qquad 0 < t_o < t < t_1\,. \qquad \text{(B.1)}
\end{aligned}
$$

We prove that there exists at most one point $t_o < t_m \leq t_1$ such that $h'(t) < 0$ on (t_o, t_m) and $h'(t) > 0$ on (t_m, ∞). For the proof, that $h'(t) < 0$ on (t_o, t_m) is sufficiently show, that

$$
g_1(t) + g_2(t) - 1 < 0, \qquad \forall\ t_o < t < t_m.
$$

If we put $t = t_o + h$, $h > 0$ and $\xi = 2a\gamma$ we have

$$
g_1(t_o+h) + g_2(t_o+h) = 2\gamma\left(1+a\left(1+\frac{h}{t_o}\right)^{\xi}\right)\frac{h}{t_o+h} + \left(1-\frac{h}{t_o+h}\right)^{2\gamma}\mathrm{e}^{1-\left(1+\frac{h}{t_o}\right)^{\xi}} < 1.
$$

Now we development the functions $(1+h/t_o)^\xi$, $(1-h/(t_o+h))^{2\gamma}$ and $\mathrm{e}^{1-(1+h/t_o)^\xi}$ to power series we can rewrite the previous ones into the form

$$
\begin{aligned}
&\frac{2\gamma+\xi}{t_o+h}h + \frac{\xi^2}{t_o(t_o+h)}h^2 + o_1(h^2) + \left(1 - \frac{2\gamma}{t_o+h}h - \frac{\gamma(1-2\gamma)}{(t_o+h)^2}h^2 + o_2(h^2)\right) \\
&\quad\times\left(1 - \frac{\xi}{t_o}h + \frac{\xi(1-\xi)}{2t_o^2}h^2 + o_1(h^2) + \frac{1}{2}\left(-\frac{\xi}{t_o}h + \frac{\xi(1-\xi)}{2t_o^2}h^2 + o_1(h^2)\right)^2 + o_3(h^2)\right) < 1\,.
\end{aligned}
$$

After some adjustment and if we suppose that $h \leq t_o$ we can write

$$
\left(\frac{\xi^2+2\gamma\xi}{t_o(t_o+h)} - \frac{\gamma(1-2\gamma)}{(t_o+h)^2}\right)h^2 + \frac{c(a,\gamma)}{t_o^3}h^3 < 0\,.
$$

It is also sufficient to prove

$$\frac{\gamma}{t_o+h}\left(\frac{4a^2\gamma+4a\gamma}{t_o}-\frac{1-2\gamma}{t_o+h}\right)h^2+\frac{c(a,\gamma)}{t_o^3}h^3<0$$
$$\iff \quad \left[2(1+2a+2a^2)\gamma-1\right]t_o+c_1(a,\gamma)h<0$$

and because $\lim_{h\to 0+}c_1(a,\gamma)h=0$ we can rewrite for sufficiently small $0<h\le t_o$ preceding inequality as follows

$$\gamma<\frac{1}{2\left(1+\frac{4}{2\mu-1}+\frac{8}{(2\mu-1)^2}\right)}>0.44, \qquad \forall\,\mu\ge 17\,. \tag{B.2}$$

From this consideration we have

$$\begin{aligned}\mathcal{M}&=\sup_{t_o<t<t_1}\frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right)-\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t-t_o}=\max\left\{\left(\frac{d}{dt}\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right)\right)_{t=t_o},\frac{\widetilde{\Psi}\left(\frac{\omega^2(t_1)}{\varepsilon}\right)-\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t_1-t_o}\right\}\\&=\max\left\{\frac{6\gamma}{t_o},\frac{C_\mu^\alpha e^{C_\mu^{\frac{2\alpha}{2\mu-1}}}-e}{t_o(C_\mu^{\frac{\alpha}{2\gamma}}-1)}\right\}\le\frac{1}{t_o}\max\left\{6\gamma,\frac{C_\mu^\alpha e^{C_\mu^{\frac{2\alpha}{2\mu-1}}}-e}{C_\mu^{\frac{\alpha}{2\gamma}}-1}\right\}\\&\le\frac{2}{t_o}\max\left\{3\gamma,\frac{e^{C_\mu^{\frac{2\alpha}{2\mu-1}}}}{C_\mu^{\left(-1+\frac{1}{2\gamma}\right)\alpha}}\right\}.\end{aligned}\tag{B.3}$$

For the term $\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)$ from the definition of $\mathcal{M}$ we get

$$\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)=\frac{\omega^2(t_o)}{\varepsilon}e^{\left(\frac{\omega^2(t_o)}{\varepsilon}\right)^{2/(2\mu-1)}}=e, \qquad \forall\,t_o>0\,.$$

Acknowledgements

We would like to thank the anonymous referee for valuable suggestions and comments which contribute to improve the manuscript.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **25**, 1–22; https://doi.org/10.14232/ejqtde.2021.1.25 www.math.u-szeged.hu/ejqtde/

Optimal harvesting for a stochastic competition system with stage structure and distributed delay

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Received 25 September 2020, appeared 1 April 2021
Communicated by Tibor Krisztin

Abstract. A stochastic competition system with harvesting and distributed delay is investigated, which is described by stochastic differential equations with distributed delay. The existence and uniqueness of a global positive solution are proved via Lyapunov functions, and an ergodic method is used to obtain that the system is asymptotically stable in distribution. By using the comparison theorem of stochastic differential equations and limit superior theory, sufficient conditions for persistence in mean and extinction of the stochastic competition system are established. We thereby obtain the optimal harvest strategy and maximum net economic revenue by the optimal harvesting theory of differential equations.

Keywords: stochastic differential equation, distributed delay, competition system, stability in distribution, optimal harvesting strategy.

2020 Mathematics Subject Classification: 60H10, 92B05, 93E20.

1 Introduction

In nature, relationships between species can be classified as either competition, predator-prey, or mutualism. Because of limited natural resources, competition among populations is widespread. Many scholars have researched competition models. Early studies mainly considered deterministic models [5, 16]. Individual organisms experience a growth process, from infancy to adulthood, immaturity to maturity, and adulthood to old age, with viability varying by age. Young individuals have a weaker ability to cope with environmental disturbances, predators, and competitors' survival pressure, while the survival ability of adult individuals is strong, and they are able to conceive the next generation. The stage-structured model is popular among scholars, and the study of the stage-structured deterministic model, as a single-species model [7] or two-species competitive model [14], is comprehensive. Predator-prey models with stage structures have been discussed in the literature [4, 17, 18]. X. Y. Huang et al. presented the sufficient conditions of extinction for a two-species competitive stage-structured system with harvesting [6].

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The effects of population competition are not immediate, hence, it is necessary to consider time delays in the governing equations [9, 15, 20]. We propose a competitive model with distributed delay and harvesting,

$$\begin{cases} dx_1 = (a_{11}x_2 - a_{12}x_1^2 - sx_1)dt, \\ dx_2 = \left(a_{21}x_1 - a_{22}x_2^2 - d_1x_2 \int_{-\infty}^{t} f_1(t-v)x_3(v)dv - \beta x_2\right)dt, \\ dx_3 = \left(x_3\left(r\left(1-\frac{x_3}{k_3}\right) - d_2\int_{-\infty}^{t} f_2(t-v)x_2(v)dv - qE\right)\right)dt, \end{cases} \tag{1.1}$$

where x_i is the density of the ith species, $i = 1,2,3$, where x_1, x_2, respectively represent the juveniles and adults of one of two species. a_{11} is the birth rate of juveniles and a_{21} is the transformation rate from juveniles to adults. a_{12}, a_{22} denote inter-specific competitive coefficients of x_1 and x_2. Considering x_1 is young and not competitive, we assume that only x_2 and x_3 are competitive. d_1 and d_2 are the loss rates of populations x_2 and x_3 in competition. r and k_3 are respectively the intrinsic growth rate and environmental capacity of species x_3. The sum of the death and conversion rates of juveniles x_1 and the sum of the death rates of adults x_2 are expressed by s and β, respectively. q is the catchability coefficient of species x_3. E denotes the effort used to harvest the population x_3. All of the parameters are assumed to be positive constants. The kernel $f_i : [0,\infty) \to [0,\infty)$ is normalized as

$$\int_0^\infty f_i(v)dv = 1, \qquad i = 1,2.$$

For the distributed delay, MacDonald [10] initially proposed that it is reasonable to use a Gamma distribution,

$$f_i(t) = \frac{t^n \alpha_i^{n+1} e^{-\alpha_i t}}{n!}, \qquad i = 1,2,$$

as a kernel, where $\alpha_i > 0, i = 1,2$ denote the rate of decay of effects of past memories, and n is called the order of the delay kernel $f_i(t)$. They are nonnegative integers.

This article mainly considers the weak kernel case, i.e., $f_i = \alpha_i e^{-\alpha_i t}$ for $n = 0$. The strong kernel case can be considered similarly. Let

$$u_1 = \int_{-\infty}^{t} f_1(t-v)x_3(v)dv, \qquad u_2 = \int_{-\infty}^{t} f_2(t-v)x_2(v)dv.$$

Then, by the linear chain technique [13], the system (1.1) is transformed to the following equivalent system:

$$\begin{cases} dx_1 = (a_{11}x_2 - a_{12}x_1^2 - sx_1)dt, \\ dx_2 = (a_{21}x_1 - a_{22}x_2^2 - d_1x_2u_1 - \beta x_2)dt, \\ dx_3 = \left(x_3\left(r\left(1-\frac{x_3}{k_3}\right) - d_2u_2 - qE\right)\right)dt, \\ du_1 = \alpha_1(x_3 - u_1)dt, \\ du_2 = \alpha_2(x_2 - u_2)dt, \end{cases} \tag{1.2}$$

In addition, the population must be disturbed by realistic environmental noise, which is important in the study of bio-mathematical models [12, 15, 19], such as rainfall, wind, and drought. White noise is introduced to indicate the effects on the system disturbance. It is assumed that environmental disturbances will manifest themselves mainly as disturbances in population density x_i $(i = 1, 2, 3)$ of a system (1.2). Further, the following system of stochastic differential equations is obtained:

$$\begin{cases} dx_1 = (a_{11}x_2 - a_{12}x_1^2 - sx_1)dt + \sigma_1 x_1 dB_1(t), \\ dx_2 = (a_{21}x_1 - a_{22}x_2^2 - d_1x_2u_1 - \beta x_2)dt + \sigma_1 x_2 dB_1(t), \\ dx_3 = \left(x_3\left(r\left(1 - \dfrac{x_3}{k_3}\right) - d_2u_2 - qE\right)\right)dt + \sigma_2 x_3 dB_2(t), \\ du_1 = \alpha_1(x_3 - u_1)dt, \\ du_2 = \alpha_2(x_2 - u_2)dt, \end{cases} \tag{1.3}$$

where $B_i(t)$, $i = 1, 2$, are independent standard Brownian motions and σ_i^2, $i = 1, 2$, represent the intensity of the white noise. Because x_1 and x_2 live together, they are affected by the same noise.

The following assumption applies throughout this paper.

Assumption 1.1. Because of limited environmental supply and interspecific and intra-specific constraints, species x_i must have environmental capacity k_i.

2 Existence and uniqueness of the global positive solution

Theorem 2.1. *For any initial value $x(0) = \big(x_1(0), x_2(0), x_3(0), u_1(0), u_2(0)\big) \in R_+^5$, there is a unique solution $x(t) = \big(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t)\big)$ of system (1.3) on $t \geqslant 0$. Furthermore, the solution will remain in R_+^5 with probability 1.*

Proof. System (1.3) is locally Lipschitz continuous, so for any initial value $x(0) = \big(x_1(0), x_2(0), x_3(0), u_1(0), u_2(0)\big) \in R_+^5$, there is a unique maximal local solution $x(t) = \big(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t)\big)$ for $t \in [0, \tau_e)$ a.s., where τ_e is the explosion time [1].

We must show that $\tau_e = \infty$ a.s. Let $m_0 > 0$ be sufficiently large that the initial value $x_i(0)$ is in the interval $\big[\frac{1}{m_0}, m_0\big]$. For each $m > m_0$, define a stopping time,

$$\tau_m = \inf\left\{t \in [0, \tau_e) : x_i(t) \notin \left(\frac{1}{m}, m\right),\ i = 1, 2, 3\right\}.$$

Obviously, τ_m increases as $m \to \infty$. Let $\tau_\infty = \lim_{m\to\infty} \tau_m$. Hence $\tau_\infty \leq \tau_e$ a.s., which is enough to certify $\tau_\infty = \infty$ a.s.

In contrast, there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$, such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

Hence an integer $m_1 > m_0$ exists, and for arbitrary $m > m_1$,

$$P\{\tau_m \leq T\} \leq \varepsilon.$$

A Lyapunov function $V : R_+^5 \to R_+$ is defined as

$$V(x) = x_1 - 1 - \ln x_1 + x_2 - a - a\ln\frac{x_2}{a} + x_3 - b - b\ln\frac{x_3}{b} + \frac{1}{\alpha_1}(u_1 - 1 - \ln u_1) + \frac{1}{\alpha_2}(u_2 - 1 - \ln u_2),$$

where a, b are positive constants to be determined later. The nonnegativity of this function can be seen because

$$\omega - 1 - \ln \omega \geq 0 \quad \text{for any } \omega > 0.$$

Let $T > 0$ be a random positive constant. For any $0 \leq t \leq \tau_m \wedge T$, using Itô's formula, one obtains

$$dV(x) = LV(x)dt + \sigma_1(x_1 - 1)dB_1(t) + \sigma_1(x_2 - 1)dB_1(t) + \sigma_2(x_3 - 1)dB_2(t), \tag{2.1}$$

where

$$\begin{aligned}
LV(x) = &\left(1 - \frac{1}{x_1}\right)(a_{11}x_2 - a_{12}x_1^2 - sx_1) + \left(1 - \frac{a}{x_2}\right)(a_{21}x_1 - a_{22}x_2^2 - d_1x_2u_1 - \beta x_2) \\
&+ (x_3 - b)\left(r\left(1 - \frac{x_3}{k_3}\right) - d_2u_2 - qE\right) + \sigma_1^2 + \frac{1}{2}\sigma_2^2 + \left(1 - \frac{1}{u_1}\right)(x_3 - u_1) \\
&+ \left(1 - \frac{1}{u_2}\right)(x_2 - u_2) \\
\leq &\ (a_{11} - \beta + a_{22}a + 1)x_2 - a_{22}x_2^2 - a_{12}x_1^2 + (a_{21} - s + a_{12})x_1 \\
&+ \left(r - qE + \frac{r}{k_3}b + 1\right)x_3 - \frac{r}{k_3}x_3^2 + (ad_1 - 1)u_1 + (bd_2 - 1)u_2 \\
&+ s + a\beta - br + bqE + 2 + \sigma_1^2 + \frac{1}{2}\sigma_2^2 \\
\leq &\ M + (ad_1 - 1)u_1 + (bd_2 - 1)u_2 + s + \frac{\beta}{d_1} - \frac{r}{d_2} + \frac{qE}{d_2} + 2 + \sigma_1^2 + \frac{1}{2}\sigma_2^2,
\end{aligned} \tag{2.2}$$

where $M = \sup\{-a_{22}x_2^2 + (a_{11} - \beta + \frac{a_{22}}{d_1} + 1)x_2 - a_{12}x_1^2 + (a_{21} - s + a_{12})x_1\} - \frac{r}{k_3}x_3^2 + (r - qE + \frac{r}{k_3d_2} + 1)x_3\}$.
Choose $a = \frac{1}{d_1}$, $b = \frac{1}{d_2}$ such that $ad_1 - 1 = 0$, $bd_2 - 1 = 0$. Then one obtains

$$LV(x) \leq M + s + \frac{\beta}{d_1} - \frac{r}{d_2} + \frac{qE}{d_2} + 2 + \sigma_1^2 + \frac{1}{2}\sigma_2^2 = K_1. \tag{2.3}$$

The following proof is similar to that of Bao and Yuan [2].

Apply inequality (2.3) to equation (2.1), and integrate from 0 to $\tau_m \wedge T$ to obtain

$$\begin{aligned}
\int_0^{\tau_m\wedge T} d\big(V(x(v))\big)dv \leq \int_0^{\tau_m\wedge T} Kdv + \int_0^{\tau_m\wedge T} \sigma_1(x_1 - 1)dB_1(v) + \int_0^{\tau_m\wedge T} \sigma_1(x_2 - 1)dB_1(v) \\
+ \int_0^{\tau_m\wedge T} \sigma_2(x_3 - 1)dB_2(v).
\end{aligned}$$

Taking the expectations, the above inequality becomes

$$E\big(V\big(x(\tau_m \wedge T)\big)\big) \leq V\big(x(0)\big) + E\big(K_1(\tau_m \wedge T)\big),$$

i.e.,

$$E\big(V\big(x(\tau_m \wedge T)\big)\big) \leq V\big(x(0)\big) + K_1T.$$

For each $u \geq 0$, define $\mu(u) = \inf\{V(x), |x_i| \geq u, i = 1, 2, 3\}$. Clearly, if $u \to \infty$, then $\mu(u) \to \infty$. One can see that

$$\mu(m)P(\tau_m \leq T) \leq E\big(V\big(x(\tau_m)\big)I_{\{\tau_m \leq T\}}\big) \leq V\big(x(0)\big) + K_1T.$$

When $m \to \infty$, it is easy to see that $P(\tau_\infty \leq T) = 0$. Owing to the arbitrariness of T, $P(\tau_\infty = \infty) = 1$. The proof is completed. □

3 Stability in distribution

Lemma 3.1. *Suppose $x(t) = \big(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t)\big)$ is a solution of system (1.3) with any given initial value. Then there exists a constant $K_2 > 0$, such that $\limsup_{t\to+\infty} E|x(t)| \le K_2$.*

Proof. The proof is similar to Theorem 3.1 in paper [2], and hence is omitted here. □

Then one can further prove the following theorem.

Theorem 3.2. *If $a_{12} > 2(a_{11} \vee a_{21})$, $a_{22} > \alpha_2 + (a_{11} \vee a_{21})$, $r > \alpha_1 k_3$, $\alpha_1 > d_1$, $\alpha_2 > d_2$, then system (1.3) will be asymptotically stable in distribution, i.e., when $t \to +\infty$, there is a unique probability measure $\mu(\cdot)$ such that the transition probability density $p(t, \phi, \cdot)$ of $x(t)$ converges weakly to $\mu(\cdot)$ with any given initial value $\phi(t) \in R^5_+$.*

Proof. Let $x^\phi(t)$ and $x^\varphi(t)$ be two solutions of system (1.3), with initial values $\phi(\theta) \in R^5_+$ and $\varphi(t) \in R^5_+$, respectively. Applying Itô's formula to

$$V(t) = \sum_{i=1}^{3} |\ln x_i^\phi(t) - \ln x_i^\varphi(t)| + \sum_{j=1}^{2} |\ln u_i^\phi(t) - \ln u_i^\varphi(t)|$$

yields

$$\begin{aligned}
d^+V(t) = &\sum_{i=1}^{3} \operatorname{sgn}\big(x_i^\phi(t) - x_i^\varphi(t)\big) d\big(\ln x_i^\phi(t) - \ln x_i^\varphi(t)\big) \\
&+ \sum_{j=1}^{2} \operatorname{sgn}\big(u_i^\phi(t) - u_i^\varphi(t)\big) d\big(\ln u_i^\phi(t) - \ln u_i^\varphi(t)\big) \\
\le\ & a_{11} \left| \frac{x_2^\phi(t)}{x_1^\phi(t)} - \frac{x_2^\varphi(t)}{x_1^\varphi(t)} \right| dt - a_{12}|x_1^\phi(t) - x_1^\varphi(t)|dt - (a_{22} - \alpha_2)|x_2^\phi(t) - x_2^\varphi(t)|dt \\
&+ a_{21} \left| \frac{x_1^\phi(t)}{x_2^\phi(t)} - \frac{x_1^\varphi(t)}{x_2^\varphi(t)} \right| dt - (\alpha_1 - d_1)|u_1^\phi(t) - u_1^\varphi(t)|dt - \left(\frac{r}{k_3} - \alpha_1\right)|x_3^\phi(t) - x_3^\varphi(t)|dt \\
&- (\alpha_2 - d_2)|u_2^\phi(t) - u_2^\varphi(t)|dt \\
\le & -\big(a_{12} - 2(a_{11} \vee a_{21})\big)|x_1^\phi(t) - x_1^\varphi(t)|dt - \big(a_{22} - \alpha_2 - 2(a_{11} \vee a_{21})\big)|x_2^\phi(t) - x_2^\varphi(t)|dt \\
&- \left(\frac{r}{k_3} - \alpha_1\right)|x_3^\phi(t) - x_3^\varphi(t)|dt - (\alpha_1 - d_1)|u_1^\phi(t) - u_1^\varphi(t)|dt \\
&- (\alpha_2 - d_2)|u_2^\phi(t) - u_2^\varphi(t)|dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E(V(t)) \le V(0) &- \big(a_{12} - 2(a_{11} \vee a_{21})\big) \int_0^t E|x_1^\phi(v) - x_1^\varphi(v)|dv \\
&- \big(a_{22} - \alpha_2 - 2(a_{11} \vee a_{21})\big)\int_0^t E|x_2^\phi(v) - x_2^\varphi(v)|dv - \left(\frac{r}{k_3} - \alpha_1\right)\int_0^t E|x_3^\phi(v) - x_3^\varphi(v)|dv \\
&- (\alpha_1 - d_1) \int_0^t E|u_1^\phi(v) - u_1^\varphi(v)|dv - (\alpha_2 - d_2) \int_0^t E|u_2^\phi(v) - u_2^\varphi(v)|dv.
\end{aligned}$$

Because $V(t) \geq 0$, according to the inequality above,

$$\begin{aligned}
&\big(a_{12} - 2(a_{11} \vee a_{21})\big) \int_0^t E|x_1^\phi(v) - x_1^\varphi(v)|dv + \big(a_{22} - \alpha_2 - 2(a_{11} \vee a_{21})\big) \int_0^t E|x_2^\phi(v) - x_2^\varphi(v)|dv \\
&\quad + \left(\frac{r}{k_3} - \alpha_1\right) \int_0^t E|x_3^\phi(v) - x_3^\varphi(v)|dv + (\alpha_1 - d_1) \int_0^t E|u_1^\phi(v) - u_1^\varphi(v)|dv \\
&\quad + (\alpha_2 - d_2) \int_0^t E|u_2^\phi(v) - u_2^\varphi(v)|dv \leq V(0) < \infty.
\end{aligned}$$

That is,

$$E|x_i^\phi(v) - x_i^\varphi(v)| \in L^1[0, +\infty), \quad i = 1, 2, 3 \quad \text{and} \quad E|u_j^\phi(v) - u_j^\varphi(v)| \in L^1[0, +\infty), \quad j = 1, 2.$$

Moreover, it can be seen from the first equation of system (1.3) that

$$E\big(x_1(t)\big) = x_1(0) + \int_0^t \big[a_{11}E\big(x_2(v)\big) - a_{12}E\big(x_1^2(v)\big) - sE\big(x_1(v)\big)\big]dv.$$

Thus $E\big(x_1(t)\big)$ is a continuously differentiable function. By Lemma 3.1,

$$\frac{dE\big(x_1(t)\big)}{dt} \leq a_{11}E\big(x_2(t)\big) \leq K_2.$$

Hence $E\big(x_1(t)\big)$ is uniformly continuous. Using the same method on the other equations of system (1.3), one can obtain that $E\big(x_2(t)\big)$, $E\big(x_3(t)\big)$, $E\big(u_1(t)\big)$, and $E\big(u_2(t)\big)$ are uniformly continuous. According to [3],

$$\lim_{t\to\infty} E|x_i^\phi(t) - x_i^\varphi(t)| = 0 \quad a.s., \qquad \lim_{t\to\infty} E|u_j^\phi(t) - u_j^\varphi(t)| = 0 \quad a.s. \tag{3.1}$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets). Suppose $p(t, \phi, dy)$ is the transition probability density of the process $x(t)$, and $p(t, \phi, A)$ is the probability of event $x^\phi(t) \in A$ with initial value $\phi(\theta) \in R_+^5$. By Lemma 3.1 and Chebyshev's inequality, the family of transition probability $p(t, \phi, A)$ is tight. So, a compact subset $\mathcal{K} \in R_+^5$ can be obtained such that $p(t, \phi, \mathcal{K}) \geq 1 - \epsilon^*$ for any $\epsilon^* > 0$.

Let $\mathcal{P}(R_+^5)$ be probability measures on R_+^5. For any two measures $P_1, P_2 \in \mathcal{P}$, we define the metric

$$d_{\mathbb{L}}(P_1, P_2) = \sup_{g\in\mathbb{L}} \left| \int_{R_+^5} g(x)P_1(dx) - \int_{R_+^5} g(x)P_2(dx) \right|,$$

where

$$\mathbb{L} = \{g : R_+^5 \to R : \|g(x) - g(y)\| \leq \|x - y\|, |g(\cdot)| \leq 1\}.$$

For any $g \in \mathbb{L}$ and $t, \iota > 0$, one obtains

$$\begin{aligned}
|Eg\big(x^\phi(t+\iota)\big) - Eg\big(x^\phi(t)\big)| &= |E[E\big(g\big(x^\phi(t+\iota)\big)|\mathcal{F}_\vartheta\big)] - Eg\big(x^\phi(t)\big)| \\
&= \left| \int_{R_+^5} E\big(g\big(x^\xi(t)\big)p(\vartheta, \phi, d\xi)\big) - Eg\big(x^\phi(t)\big) \right| \\
&\leq 2p(\vartheta, \phi, U_K^c) + \int_{U_K} |E\big(g\big(x^\xi(t)\big)\big) - E\big(g\big(x^\phi(t)\big)\big)|p(\vartheta, \phi, d\xi),
\end{aligned}$$

where $U_K = \{x \in R^5_+ : |x| \le K\}$, and U^c_K is a complementary set of U_K. Since the family of $p(t,\phi,dy)$ is tight, for any given $\iota \ge 0$, there exists sufficiently large K such that $p(\iota,\phi,U^c_K) < \frac{\epsilon^*}{4}$. From (3.1), there exists $T > 0$ such that for $t \ge T$,

$$\sup_{g\in\mathbb{L}} |E(g(x^{\xi}(t))) - E(g(x^{\phi}(t)))| \le \frac{\epsilon^*}{2}.$$

Consequently, it is easy to find that $|Eg(x^{\phi}(t+\iota)) - Eg(x^{\phi}(t))| \le \epsilon^*$. By the arbitrariness of g, we have

$$\sup_{g\in\mathbb{L}} |Eg(x^{\phi}(t+\iota)) - Eg(x^{\phi}(t))| \le \epsilon^*.$$

That is,

$$d_{\mathbb{L}}(p(t+\iota,\phi,\cdot),p(t,\phi,\cdot)) \le \epsilon^*, \quad \forall t \ge T,\ \iota > 0.$$

Therefore, $\{p(t,0,\cdot) : t \ge 0\}$ is Cauchy in $\mathcal{P}$ with metric $d_{\mathbb{L}}$. There is a unique $\mu(\cdot) \in \mathcal{P}(R^5_+)$ such that $\lim_{t\to\infty} d_{\mathbb{L}}(p(t,0,\cdot),\mu(\cdot)) = 0$. In addition, it follows from (3.1) that

$$\lim_{t\to\infty} d_{\mathbb{L}}(p(t,\phi,\cdot),p(t,0,\cdot)) = 0.$$

Hence

$$\lim_{t\to\infty} d_{\mathbb{L}}(p(t,\phi,\cdot),\mu(\cdot)) \le \lim_{t\to\infty} d_{\mathbb{L}}(p(t,\phi,\cdot),p(t,0,\cdot)) + \lim_{t\to\infty} d_{\mathbb{L}}(p(t,0,\cdot),\mu(\cdot)) = 0.$$

The proof is completed. □

4 Optimal harvesting

For convenience, we introduce the following notation:

$$b_1 = s + \frac{1}{2}\sigma_1^2, \quad b_2 = \beta + \frac{1}{2}\sigma_1^2, \quad b_3 = r - \frac{1}{2}\sigma_2^2,$$
$$\Gamma_1 = a_{12}a_{22}b_3 - d_2(a_{21}(a_{11}k_2 - b_1) - a_{12}b_2),$$
$$f^* = \limsup_{t\to\infty} f(t), \quad f_* = \liminf_{t\to\infty} f(t), \quad \langle f\rangle = t^{-1}\int_0^t f(s)ds.$$

Lemma 4.1 ([8]). *For $x(t) \in R_+$, the following holds:*

(i) If there are positive constants T and δ_0 such that

$$\ln x(t) \le \delta t - \delta_0 \int_0^t x(v)dv + \alpha B(t), \quad a.s.$$

for any $t \ge T$, where $\alpha, \delta_1, \delta_2$ are constants, then

$$\begin{cases} \langle x\rangle^* \le \frac{\delta}{\delta_0},\ a.s. & \text{if } \delta \ge 0, \\ \lim_{t\to\infty} x(t) = 0,\ a.s. & \text{if } \delta \le 0. \end{cases}$$

(ii) If there are positive constants T, δ, and δ_0 such that

$$\ln x(t) \ge \delta t - \delta_0 \int_0^t x(v)dv + \alpha B(t), \quad a.s.$$

for any $t \ge T$, then $\langle x\rangle_* \ge \frac{\delta}{\delta_0}$ a.s.

Lemma 4.2 (Strong law of large numbers [11]). *Let $M = \{M_t\}_{t\geq 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t\to\infty} \langle M, M\rangle_t = \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{\langle M, M\rangle_t} = 0 \quad a.s.$$

and

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} < \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{t} = 0 \quad a.s.$$

Lemma 4.3 (Strong law of large numbers for local martingales [11]). *Let $M(t), t \geq 0$, be a local martingale vanishing at time $t = 0$ and define*

$$\rho_M(t) = \int_0^t \frac{d\langle M\rangle(s)}{(1+s)^2}, \qquad t \geq 0,$$

where $M(t) = \langle M, M\rangle(t)$ is a Meyers angle bracket process. Then

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad a.s.,$$

provided

$$\lim_{t\to\infty} \rho_M(t) < \infty \quad a.s.$$

Lemma 4.4. *Let $(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t))$ be the solution of system (1.3) with any initial value $(x_1(0), x_2(0), x_3(0), u_1(0), u_2(0)) \in R^5_+$. Then, if $\alpha_1 > \alpha_2$, then*

$$\lim_{t\to\infty} \frac{u_1(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{u_2(t)}{t} = 0, \quad a.s.$$

and

$$\langle u_1(t)\rangle = \langle x_3(t)\rangle - \frac{u_1(t) - u_1(0)}{\alpha_1 t}, \qquad \langle u_2(t)\rangle = \langle x_2(t)\rangle - \frac{u_2(t) - u_2(0)}{\alpha_2 t}.$$

Proof. Define $V^*(w) = (1+w)^\theta$, where θ is a positive constant to be determined later, and

$$w(t) = x_1(t) + x_2(t) + x_3(t) + \frac{r}{2k_3\alpha_1} u_1^2(t) + \frac{a_{12}}{2\alpha_2} u_2^2(t).$$

By Itô's formula,

$$dV^*(w) = LV^*(w)dt + \sigma_1(1+w)^{\theta-1} x_1 dB_1(t) + \sigma_1(1+w)^{\theta-1} x_2 dB_1(t) + \sigma_2(1+w)^{\theta-1} x_3 dB_2(t),$$

where

$$
\begin{aligned}
LV^*(w) &= \theta(1+w)^{\theta-1}(a_{11}x_2 - a_{12}x_1^2 - sx_1 + a_{21}x_1 - a_{22}x_2^2 - d_1x_2u_1 - \beta x_2 + rx_3 \\
&\quad - \frac{r}{k_3}x_3^2 - d_2x_3u_2 - qEx_3 + \frac{r}{k_3}x_3u_1 - \frac{r}{k_3}u_1^2 + a_{12}x_2u_2 - a_{12}u_2^2) \\
&\quad + \frac{\sigma_1^2\theta(\theta-1)}{2}(1+w)^{\theta-2}(x_1^2+x_2^2) + \frac{\sigma_2^2\theta(\theta-1)}{2}(1+w)^{\theta-2}x_3^2 \\
&\le \theta(1+w)^{\theta-1}\bigg(- a_{12}x_1^2 + (a_{21}-s)x_1 - a_{22}x_2^2 + (a_{11}-\beta)x_2 - \frac{r}{k_3}x_3^2 \\
&\quad + (r-qE)x_3 + \frac{r}{2k_3}x_3^2 + \frac{r}{2k_3}u_1^2 - \frac{r}{k_3}u_1^2 + \frac{a_{12}}{2}x_2^2 + \frac{a_{12}}{2}u_2^2 - a_{12}u_2^2\bigg) \\
&\quad + \frac{\sigma_1^2\theta(\theta-1)}{2}(1+w)^{\theta-2}(x_1^2+x_2^2) + \frac{\sigma_2^2\theta(\theta-1)}{2}(1+w)^{\theta-2}x_3^2 \\
&= \theta(1+w)^{\theta-2}\bigg((1+w)(-a_{12}x_1^2 + (a_{21}-s)x_1 - \left(a_{22}+\frac{a_{12}}{2}\right)x_2^2 + (a_{11}-\beta)x_2 \\
&\quad - \frac{r}{2k_3}x_3^2 + (r-qE)x_3 - \frac{r}{2k_3}u_1^2 - \frac{a_{12}}{2}u_2^2) + \frac{\sigma_1^2\theta(\theta-1)}{2}(x_1^2+x_2^2) \\
&\quad + \frac{\sigma_2^2\theta(\theta-1)}{2}x_3^2\bigg) \\
&\le \theta(1+w)^{\theta-1}\bigg(- a_{12}x_1^2 + (a_{21}-s+\alpha_1)x_1 - \left(a_{22}+\frac{a_{12}}{2}\right)x_2^2 \\
&\quad + (a_{11}-\beta+\alpha_1)x_2 - \frac{r}{2k_3}x_3^2 + (r-qE+\alpha_1)x_3 - \alpha_1 w + \frac{a_{12}}{2}\left(\frac{\alpha_1}{\alpha_2}-1\right)u_2^2\bigg) \\
&\quad + \sigma_1^2\theta(\theta-1)(1+w)^{\theta-2}w^2 + \frac{\sigma_2^2\theta(\theta-1)}{2}(1+w)^{\theta-2}w^2 \\
&\le \theta(1+w)^{\theta-2}\bigg((1+w)(-a_{12}x_1^2 + (a_{21}-s+\alpha_2)x_1 - a_{22}x_2^2 + (a_{11}-\beta+\alpha_2)x_2 \\
&\quad - \frac{r}{2k_3}x_3^2 + (r-qE+\alpha_2)x_3 - \alpha_2 w) + \frac{(2\sigma_1^2+\sigma_2^2)}{2}(\theta-1)w^2\bigg) \\
&\le \theta(1+w)^{\theta-2}\left(-\big(\alpha_2 - \frac{(2\sigma_1^2+\sigma_2^2)}{2}(\theta-1)\big)w^2 + (M_1-\alpha_2)w + M_1\right),
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 = \sup_{x_1,x_2,x_3\in(0,+\infty)} \bigg\{ & - a_{12}x_1^2 + (a_{21}-s-\alpha_1)x_1 - a_{22}x_2^2 \\
& + (a_{11}-\beta+\alpha_1)x_2 - \frac{r}{2k_3}x_3^2 + (r-qE+\alpha_1)x_3\bigg\}.
\end{aligned}
$$

Choose $\theta \in \big(1, \frac{2\alpha_2}{2\sigma_1^2+\sigma_2^2}+1\big)$ such that $\lambda^* = \alpha_2 - \frac{2\sigma_1^2+\sigma_2^2}{2}(\theta-1) > 0$. Then

$$
\begin{aligned}
dV^* &\le \theta(1+w)^{\theta-2}\big(- \lambda^* w^2 + (M_1-\alpha_2)w + M_1\big)dt + \sigma_1(1+w)^{\theta-1}x_1dB_1(t) \\
&\quad + \sigma_1(1+w)^{\theta-1}x_2dB_1(t) + \sigma_2(1+w)^{\theta-1}x_3dB_2(t).
\end{aligned}
\tag{4.1}
$$

Hence, for $0 < \mu < \theta\lambda^*$, we have

$$
\begin{aligned}
d\big(e^{\mu t}V^*(w)\big) &\le L\big(e^{\mu t}V^*(w)\big)dt + \sigma_1\theta e^{\mu t}(1+w)^{\theta-1}x_1dB_1(t) + \sigma_1\theta e^{\mu t}(1+w)^{\theta-1}x_2dB_1(t) \\
&\quad + \sigma_2\theta e^{\mu t}(1+w)^{\theta-1}x_3dB_2(t),
\end{aligned}
\tag{4.2}
$$

where

$$\begin{aligned} L\big(e^{\mu t}V^*(w)\big) &\le \mu e^{\mu t}(1+w)^{\theta} + e^{\mu t}\theta(1+w)^{\theta-2}\big(-\lambda^* w^2 + (M_1-\alpha_2)w + M_1\big) \\ &= e^{\mu t}(1+w)^{\theta-2}\big(-(\theta\lambda^*-\mu)w^2 + (2\mu + M_1\theta - \alpha_2\theta)w + M_1\theta + \mu\big) \\ &\le e^{\mu t}M_2, \end{aligned}$$

where

$$M_2 = \sup_{w\in(0,+\infty)} (1+w)^{\theta-2}\big(-(\theta\lambda^*-\mu)w^2 + (2\mu + M_1\theta - \alpha_2\theta)w + M_1\theta + \mu\big).$$

Integrating from 0 to t and taking the expectation of two sides of (4.2) yields

$$\begin{aligned} E\big(e^{\mu t}V^*(w(t))\big) &= V^*\big(w(0)\big) + \int_0^t E\Big(L\big(e^{\mu\vartheta}V^*\big(w(\vartheta)\big)\big)\Big)d\vartheta \\ &\le (1+w(0))^{\theta} + \frac{M_2}{\mu}e^{\mu t}, \quad a.s. \end{aligned}$$

On account of the continuity of $V^*(w(t))$, there exists a constant $H>0$ such that

$$E\big((1+w(t))^{\theta}\big) \le H, \qquad t\ge 0, \ \ a.s. \tag{4.3}$$

From (4.1) and (4.3), for sufficiently small $\delta>0$, $n=1,2,\dots,$

$$E\Big(\sup_{n\delta\le t\le(n+1)\delta}(1+w(t))^{\theta}\Big) \le E\big((1+w(n\delta))^{\theta}\big) + I_1 + I_2, \tag{4.4}$$

where

$$\begin{aligned} I_1 &= \theta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big|\int_{n\delta}^t (1+w)^{\theta-2}\Big(-\lambda^* w^2 + (M_1-\alpha_2)w + M_1\Big)dt\Big|\Big) \\ I_2 &= \sigma_1\theta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big|\int_{n\delta}^t \Big(1+w(\vartheta)\Big)^{\theta-1} x_1(\vartheta)dB_1(\vartheta)\Big|\Big) \\ &\quad + \sigma_1\theta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big|\int_{n\delta}^t \Big(1+w(\vartheta)\Big)^{\theta-1} x_2(\vartheta)dB_1(\vartheta)\Big|\Big) \\ &\quad + \sigma_2\theta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big|\int_{n\delta}^t \Big(1+w(\vartheta)\Big)^{\theta-1} x_3(\vartheta)dB_2(\vartheta)\Big|\Big). \end{aligned}$$

Furthermore,

$$\begin{aligned} I_1 &\le \max\Big\{\lambda^*, \frac{1}{2}|M_1-\alpha_2|, M_1\Big\}\theta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big|\int_{n\delta}^t (1+w)^{\theta-2}(w^2+2w+1)dt\Big|\Big) \\ &\le C_1\delta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big(1+w(t)\Big)^{\theta}\Big), \end{aligned} \tag{4.5}$$

where $C_1 = \theta\max\{\lambda^*, \frac{1}{2}|M_1-\alpha_2|, M_1\}$. According to the Burkholder–Davis–Gundy inequality [1],

$$
\begin{aligned}
I_2 &\le \sqrt{32}\sigma_1\theta E\left(\left|\int_{n\delta}^{(n+1)\delta}\left(1+w(\vartheta)\right)^{2\theta-2}x_1^2(\vartheta)d\vartheta\right|^{\frac{1}{2}}\right)\\
&\quad+\sqrt{32}\sigma_1\theta E\left(\left|\int_{n\delta}^{(n+1)\delta}\left(1+w(\vartheta)\right)^{2\theta-2}x_2^2(\vartheta)d\vartheta\right|^{\frac{1}{2}}\right)\\
&\quad+\sqrt{32}\sigma_2\theta E\left(\left|\int_{n\delta}^{(n+1)\delta}\left(1+w(\vartheta)\right)^{2\theta-2}x_3^2(\vartheta)d\vartheta\right|^{\frac{1}{2}}\right)\\
&\le 2\sqrt{32}\sigma_1\theta E\left(\left|\int_{n\delta}^{(n+1)\delta}\left(1+w(\vartheta)\right)^{2\theta}d\vartheta\right|^{\frac{1}{2}}\right)+\sqrt{32}\sigma_2\theta E\left(\left|\int_{n\delta}^{(n+1)\delta}\left(1+w(\vartheta)\right)^{2\theta}d\vartheta\right|^{\frac{1}{2}}\right)\\
&\le 2\sqrt{32}\sigma_1\theta\sqrt{\delta}E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right)+\sqrt{32}\sigma_2\theta\sqrt{\delta}E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right)\\
&=(2\sigma_1+\sigma_2)\sqrt{32}\theta\sqrt{\delta}E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right).
\end{aligned}
\tag{4.6}
$$

By (4.4)-(4.6), we obtain that

$$(1-C_1\delta-(2\sigma_1+\sigma_2)\sqrt{32}\theta\sqrt{\delta})E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right)\le H$$

for a sufficiently small constant $\delta>0$ such that $C_1\delta+(2\sigma_1+\sigma_2)\sqrt{32}\theta\sqrt{\delta}\le\frac{1}{2}$. Then

$$E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right)\le 2H.$$

For arbitrary ϵ, according to Chebyshev's inequality,

$$P\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}>(n\delta)^{1+\epsilon}\right)\le\frac{E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right)}{(n\delta)^{1+\epsilon}}\le\frac{2H}{(n\delta)^{1+\epsilon}}.$$

From the Borel–Cantelli lemma [11], we have that $\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\le(n\delta)^{1+\epsilon}$, *a.s.* holds for all but finitely many n.

For $\epsilon\to 0$, we have $\limsup_{t\to+\infty}\frac{\ln(1+w(t))^{\theta}}{\ln t}\le 1$, *a.s.* Hence

$$\limsup_{t\to+\infty}\frac{\ln w(t)}{\ln t}\le\limsup_{t\to+\infty}\frac{\ln\left(1+w(t)\right)}{\ln t}\le\frac{1}{\theta}.$$

For $\epsilon_0<1$, there exists $T>0$ such that

$$\ln w(t)\le\left(\frac{1}{\theta}+\epsilon_0\right)\ln t,\quad\text{when } t\ge T.$$

Thus

$$\limsup_{t\to+\infty}\frac{w(t)}{t^2}\le\limsup_{t\to+\infty}t^{\frac{1}{\theta}+\epsilon_0-2}=0,$$

i.e.,

$$\limsup_{t\to+\infty} \frac{x_1(t)+x_2(t)+x_3(t)+\frac{a_{12}}{2\alpha_2}u_2^2(t)+\frac{r}{2k_3\alpha_1}u_1^2(t)}{t^2}=0,$$

which, together with the positivity of $x_1(t), x_2(t), x_3(t), u_1(t), u_2(t)$, gives

$$\lim_{t\to\infty}\frac{u_1(t)}{t}=0, \qquad \lim_{t\to\infty}\frac{u_2(t)}{t}=0, \quad a.s.$$

Indeed, integration of the system (1.3) from 0 to t yields

$$\begin{aligned}
\frac{u_1(t)-u_1(0)}{t} &= \alpha_1\langle x_3(t)\rangle - \alpha_1\langle u_1(t)\rangle\\
\frac{u_2(t)-u_2(0)}{t} &= \alpha_1\langle x_2(t)\rangle - \alpha_2\langle u_2(t)\rangle.
\end{aligned}$$

Thus

$$\langle u_1(t)\rangle = \langle x_3(t)\rangle - \frac{u_1(t)-u_1(0)}{\alpha_1 t}, \qquad \langle u_2(t)\rangle = \langle x_2(t)\rangle - \frac{u_2(t)-u_2(0)}{\alpha_2 t}. \qquad \square$$

Next, to obtain the optimal harvest strategy of system (1.3), we establish the following auxiliary systems:

$$\begin{cases}
dy_1(t) = y_1\big(a_{11}k_2 - a_{12}y_1(t) - s\big)dt + \sigma_1 y_1(t)dB_1(t),\\
dy_2(t) = y_2\big(a_{21}y_1 - a_{22}y_2(t) - \beta\big)dt + \sigma_1 y_2(t)dB_1(t),\\
dy_3(t) = \left(y_3(t)\left(r\left(1-\dfrac{y_3(t)}{k_3}\right) - d_2 v(t) - qE\right)\right)dt + \sigma_2 y_3(t)dB_2(t),\\
dv(t) = \alpha_2(y_2(t) - v(t)).
\end{cases} \tag{4.7}$$

On the basis of Lemma 4.4, we similarly obtain $\langle v(t)\rangle = \langle y_2(t)\rangle - \frac{v(t)-v(0)}{\alpha_2 t}$ and $\lim_{t\to\infty}\frac{v(t)}{t}=0$ a.s.

Theorem 4.5. *Under Assumption 1.1, if $a_{11}k_2 - b_1 > 0$, $a_{21}(a_{11}k_2-b_1) - a_{12}b_2 > 0$, then the solution $(y_1(t), y_2(t), y_3(t), v(t))$ of system (4.7) with initial value $(y_1(0), y_2(0), y_3(0), v(0))$ meets the conditions*

$$\lim_{t\to\infty}\langle y_1(t)\rangle = \frac{a_{11}k_2-b_1}{a_{12}} \quad a.s., \qquad \lim_{t\to\infty}\langle y_2(t)\rangle = \frac{a_{21}(a_{11}k_2-b_1)-a_{12}b_2}{a_{12}a_{22}} \quad a.s.$$

$$\begin{cases}
\lim\limits_{t\to\infty}\langle y_3(t)\rangle = 0 \ a.s. & \text{if } \Gamma_1 < a_{12}a_{22}qE,\\
\lim\limits_{t\to\infty}\langle y_3(t)\rangle = \dfrac{(\Gamma_1 - a_{12}a_{22}qE)k_3}{a_{12}a_{22}r} \ a.s. & \text{if } \Gamma_1 > a_{12}a_{22}qE.
\end{cases}$$

Proof. By Itô's formula, we have

$$\begin{aligned}
d\ln y_1(t) &= (a_{11}k_2 - a_{12}y_1(t) - b_1)dt + \sigma_1 dB_1(t),\\
d\ln y_2(t) &= (a_{21}y_1 - a_{22}y_2(t) - b_2)dt + \sigma_1 dB_1(t),\\
d\ln y_3(t) &= \left(-\frac{r}{k_3}y_3(t) - d_2 v(t) - qE + b_3\right)dt + \sigma_2 dB_2(t).
\end{aligned}$$

We integrate both sides of the above equation from 0 to t and divide by t to obtain

$$t^{-1}\ln\frac{y_1(t)}{y_1(0)} = -a_{12}\langle y_1(t)\rangle + a_{11}k_2 - b_1 + t^{-1}\sigma_1 B_1(t), \tag{4.8}$$

$$t^{-1}\ln\frac{y_2(t)}{y_2(0)} = a_{21}\langle y_1(t)\rangle - a_{22}\langle y_2(t)\rangle - b_2 + t^{-1}\sigma_1 B_1(t), \tag{4.9}$$

$$t^{-1}\ln\frac{y_3(t)}{y_3(0)} = -d_2\langle v(t)\rangle - \frac{r}{k_3}\langle y_3(t)\rangle - qE + b_3 + t^{-1}\sigma_2 B_2(t) \tag{4.10}$$

$$= -d_2\langle y_2(t)\rangle + d_2\frac{v(t)-v(0)}{\alpha_2 t} - \frac{r}{k_3}\langle y_3(t)\rangle - qE + b_3 + t^{-1}\sigma_2 B_2(t). \tag{4.11}$$

It is apparent that

$$\lim_{t\to\infty} t^{-1}\ln y_i(0) = 0, \qquad i = 1,2,3, \tag{4.12}$$

i.e., for any $\epsilon_1 > 0$, t is sufficiently large that

$$t^{-1}\ln y_1(t) \le -a_{12}\langle y_1(t)\rangle + a_{11}k_2 - b_1 + \epsilon_1 + t^{-1}\sigma_1 B_1(t),$$
$$t^{-1}\ln y_1(t) \ge -a_{12}\langle y_1(t)\rangle + a_{11}k_2 - b_1 - \epsilon_1 + t^{-1}\sigma_1 B_1(t).$$

Note that $a_{11}k_2 - b_1 > 0$. Let ϵ_1 be sufficiently small that $a_{11}k_2 - b_1 - \epsilon_1 > 0$. Then, by Lemma 4.1, we have

$$\lim_{t\to\infty}\langle y_1(t)\rangle \le \frac{a_{11}k_2 - b_1 + \epsilon_1}{a_{12}} \quad a.s, \qquad \lim_{t\to\infty}\langle y_1(t)\rangle \ge \frac{a_{11}k_2 - b_1 - \epsilon_1}{a_{12}} \quad a.s,$$

and by the arbitrariness of ϵ_1,

$$\lim_{t\to\infty}\langle y_1(t)\rangle = \frac{a_{11}k_2 - b_1}{a_{12}} \quad a.s. \tag{4.13}$$

Substitute (4.13) in (4.8) and note that $\lim\limits_{t\to\infty} t^{-1}\sigma_1 B_1(t) = 0$. Then, by Lemma 4.2,

$$\lim_{t\to\infty}\frac{\ln y_1(t)}{t} = 0. \tag{4.14}$$

Compute $a_{21}\times$(4.8)$+a_{12}\times$(4.9) to obtain

$$a_{21}t^{-1}\ln\frac{y_1(t)}{y_1(0)} + a_{12}t^{-1}\ln\frac{y_2(t)}{y_2(0)}$$
$$= -a_{12}a_{22}\langle y_2(t)\rangle + a_{21}(a_{11}k_2 - b_1) - a_{12}b_2 + (a_{21}+a_{12})t^{-1}\sigma_1 B_1(t), \tag{4.15}$$

and compute $a_{12}a_{22}\times$(4.10)$-d_2\times$(4.15) to obtain

$$\begin{aligned} a_{12}a_{22}t^{-1}\ln\frac{y_3(t)}{y_3(0)} &- a_{21}d_2t^{-1}\ln\frac{y_2(t)}{y_2(0)} - a_{12}d_2t^{-1}\ln\frac{y_1(t)}{y_1(0)} \\ &= \Gamma_1 - a_{12}a_{22}qE - \frac{r}{k_3}a_{12}a_{22}\langle y_3(t)\rangle + a_{12}a_{22}d_2\frac{v(t)-v(0)}{\alpha_2 t} \\ &\quad - d_2(a_{21}+a_{12})t^{-1}\sigma_1 B_1(t) + a_{12}a_{22}t^{-1}\sigma_2 B_2(t). \end{aligned} \tag{4.16}$$

Combining (4.12) with (4.14) yields that for any $0 < \epsilon_2 < a_{21}(a_{11}k_2 - b_1) - a_{12}b_2$, there exists $T_1 > 0$ such that

$$-\epsilon_2 < a_{21}t^{-1}\ln\frac{y_1(t)}{y_1(0)} + a_{12}t^{-1}\ln y_2(0) < \epsilon_2, \qquad t \ge T_1. \tag{4.17}$$

By (4.15) and (4.17), we can obtain that

$$a_{12}t^{-1}\ln y_2(t) \leq -a_{12}a_{22}\langle y_2(t)\rangle + a_{21}(a_{11}k_2 - b_1) - a_{12}b_2 + \epsilon_2 + (a_{21}+a_{12})t^{-1}\sigma_1 B_1(t),$$
$$a_{12}t^{-1}\ln y_2(t) \geq -a_{12}a_{22}\langle y_2(t)\rangle + a_{21}(a_{11}k_2 - b_1) - a_{12}b_2 - \epsilon_2 + (a_{21}+a_{12})t^{-1}\sigma_1 B_1(t).$$

It then follows from Lemma 4.1 that

$$\lim_{t\to\infty}\langle y_2(t)\rangle \leq \frac{a_{21}(a_{11}k_2 - b_1) - a_{12}b_2 + \epsilon_2}{a_{12}a_{22}} \quad a.s.,$$
$$\lim_{t\to\infty}\langle y_2(t)\rangle \geq \frac{a_{21}(a_{11}k_2 - b_1) - a_{12}b_2 - \epsilon_2}{a_{12}a_{22}} \quad a.s.$$

From the arbitrariness of ϵ_2, we can get that

$$\lim_{t\to\infty}\langle y_2(t)\rangle = \frac{a_{21}(a_{11}k_2 - b_1) - a_{12}b_2}{a_{12}a_{22}} \quad a.s. \tag{4.18}$$

From (4.14), (4.15), and (4.18), one can observe that

$$\lim_{t\to\infty}\frac{\ln y_2(t)}{t} = 0. \tag{4.19}$$

Analogously, applying Lemmas 4.1 and 4.4 and combining (4.12), (4.14), and (4.19) with (4.16), one can see that when $\Gamma_1 > a_{12}a_{22}qE$, we have

$$\lim_{t\to\infty}\langle y_3(t)\rangle = \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right) \quad a.s. \tag{4.20}$$

From (4.14), (4.19), (4.20), and (4.16), one can see that

$$\lim_{t\to\infty}\frac{\ln y_3(t)}{t} = 0, \tag{4.21}$$

and if $\Gamma_1 < a_{12}a_{22}qE$, then

$$\lim_{t\to\infty}\langle y_3(t)\rangle = 0. \tag{4.22}$$

The proof is completed. □

Then, for system (1.3), we have the following theorem.

Theorem 4.6. *Under Assumption 1.1 and when $\alpha_1 > \alpha_2$:*

(i) if $a_{11}k_2 < b_1$ and $b_3 < qE$, then all x_1, x_2, and x_3 go to extinction almost surely, i.e., $\lim_{t\to\infty} x_1(t) = 0$, $\lim_{t\to\infty} x_2(t) = 0$, $\lim_{t\to\infty} x_3(t) = 0$.

(ii) if $a_{11} > b_1k_1$, $a_{21} > b_2k_2$, and $\Gamma_1 < a_{12}a_{22}qE$, then x_1, x_2 are persistent in mean a.s., and x_3 goes to extinction a.s.

(iii) if $a_{11}k_2 < b_1$ and $b_3 > qE$, then both x_1 and x_2 go to extinction a.s., and x_3 is persistent in mean a.s.

(iv) if $a_{11} > b_1k_1$, $a_{12}a_{22}r(a_{21} - b_2k_2) > d_1k_3(\Gamma_1 - a_{12}a_{22}qE)$, and $\Gamma_1 > a_{12}a_{22}qE$, then x_1, x_2, x_3 are all persistent in mean a.s.

Proof. By the stochastic comparison theorem, we obtain

$$x_1(t) \le y_1(t), \qquad x_2(t) \le y_2(t), \qquad x_3(t) \le y_3(t). \tag{4.23}$$

So, it follows from (4.14), (4.19), and (4.21) that

$$\lim_{t\to\infty} \frac{\ln x_1(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\ln x_2(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\ln x_3(t)}{t} = 0. \tag{4.24}$$

Applying Itô's formula to system (1.3) yields

$$\begin{aligned}
d\ln x_1(t) &= \left(a_{11}\frac{x_2(t)}{x_1(t)} - a_{12}x_1(t) - b_1\right)dt + \sigma_1 dB_1(t),\\
d\ln x_2(t) &= \left(a_{21}\frac{x_1(t)}{x_2(t)} - a_{22}x_2(t) - d_1u_1(t) - b_2\right)dt + \sigma_1 dB_1(t),\\
d\ln x_3(t) &= \left(-\frac{r}{k_3}x_3(t) - d_2u_2(t) - qE + b_3\right)dt + \sigma_2 dB_2(t).
\end{aligned}$$

Integrate both sides of the above three equations from 0 to t, and divide by t to obtain

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)} = a_{11}t^{-1}\int_0^t \frac{x_2(v)}{x_1(v)}dv - a_{12}\langle x_1(t)\rangle - b_1 + t^{-1}\sigma_1 B_1(t), \tag{4.25}$$

$$\begin{aligned}
t^{-1}\ln\frac{x_2(t)}{x_2(0)} &= a_{21}t^{-1}\int_0^t \frac{x_1(v)}{x_2(v)}dv - a_{22}\langle x_2(t)\rangle - d_1\langle u_1(t)\rangle - b_2 + t^{-1}\sigma_1 B_1(t)\\
&= a_{21}t^{-1}\int_0^t \frac{x_1(v)}{x_2(v)}dv - a_{22}\langle x_2(t)\rangle - d_1\langle x_3(t)\rangle + d_1\frac{u_1(t)-u_1(0)}{\alpha_1 t}\\
&\quad - b_2 + t^{-1}\sigma_1 B_1(t),
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
t^{-1}\ln\frac{x_3(t)}{x_3(0)} &= -\frac{r}{k_3}\langle x_3(t)\rangle - d_2\langle u_2(t)\rangle - qE + b_3 + t^{-1}\sigma_2 B_2(t)\\
&= -\frac{r}{k_3}\langle x_3(t)\rangle - d_2\langle x_2(t)\rangle + d_2\frac{u_2(t)-u_2(0)}{\alpha_2 t} - qE + b_3 + t^{-1}\sigma_2 B_2(t).
\end{aligned} \tag{4.27}$$

Now, let us prove conclusion (i). We use Lemma 4.2 to obtain

$$\lim_{t\to\infty}\frac{\sigma_1 B_1(t)}{t} = 0, \qquad \lim_{t\to\infty}\frac{\sigma_2 B_2(t)}{t} = 0.$$

Then, for arbitrary $\epsilon_3 > 0$, there exists $T_2 > 0$ such that

$$\left|\frac{\sigma_1 B_1(t)}{t}\right| < \frac{\epsilon_3}{4}, \qquad \left|\frac{\sigma_2 B_2(t)}{t}\right| < \frac{\epsilon_3}{4}, \qquad \left|\frac{\ln x_i(0)}{t}\right| < \frac{\epsilon_3}{4}, \qquad i = 1,2,3.$$

Using the specific property of the limit superior in (4.25) gives

$$t^{-1}\ln x_1(t) \le a_{11}k_2 - b_1 - a_{12}\langle x_1(t)\rangle + \epsilon_3, \qquad t > T_2.$$

By the assumption $a_{11}k_2 < b_1$, we can let ϵ_3 be sufficiently small that $a_{11}k_2 < b_1 - \epsilon_3$, and by Lemma 4.1, $\lim_{t\to\infty} x_1(t) = 0$ and $\lim_{t\to\infty}\langle x_1(t)\rangle = 0$.

From Lemma (4.4), for the above ϵ_3, there exists $T_3 > 0$ such that

$$\left|\frac{u_1(t)-u_1(0)}{t}\right| < \frac{\alpha_1\epsilon_3}{2d_1}, \qquad t \ge T_3.$$

Using limit superior in (4.26) gives

$$t^{-1}\ln x_2(t) \le -b_2 + \epsilon_3 - a_{22}\langle x_2(t)\rangle - d_1\langle x_3(t)\rangle_*, \qquad t \ge T_3.$$

Let ϵ_3 be sufficiently small that $-b_2+\epsilon_3<0$. Then $\lim_{t\to\infty} x_2(t)=0$ by Lemma 4.1.

Similarly, from Lemma 4.4, there exists $T_4>0$ such that

$$\left|\frac{u_2(t)-u_1(0)}{t}\right| < \frac{\alpha_2\epsilon_3}{2d_2}, \qquad t\ge T_4.$$

Then

$$t^{-1}\ln x_3(t) \le b_3 - qE + \epsilon_3 - \frac{r}{k_3}\langle x_3(t)\rangle - d_2\langle x_2(t)\rangle_*, \qquad t\ge T_4.$$

Because $b_3<0$, ϵ_3 is sufficiently small that $b_3+\epsilon_5<0$, and we have $\lim_{t\to\infty} x_3(t)=0$ by Lemma 4.1.

Next, we prove (ii). Because $\lim_{t\to\infty} y_3(t)=0$ *a.s.* when $\Gamma_1 < a_{12}a_{22}qE$ from Theorem 4.5 and (4.23), we know $\lim_{t\to\infty} x_3(t)=0$ *a.s.* Hence system (1.3) can be simplified to a stage-structured single-population model,

$$\begin{cases} dx_1(t) = \big(a_{11}x_2(t) - a_{12}x_1^2(t) - sx_1(t)\big)dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = \big(a_{21}x_1(t) - a_{22}x_2^2(t) - \beta x_2(t)\big)dt + \sigma_1 x_2(t)dB_1(t). \end{cases}$$

Integrate both sides of the above equations from 0 to t and divide by t to obtain:

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)} \ge \frac{a_{11}}{k_1} - a_{12}\langle x_1(t)\rangle - b_1 + t^{-1}\sigma_1 B_1(t), \tag{4.28}$$

$$t^{-1}\ln\frac{x_2(t)}{x_2(0)} \ge \frac{a_{21}}{k_2} - a_{22}\langle x_2(t)\rangle - b_2 + t^{-1}\sigma_1 B_1(t). \tag{4.29}$$

According to Lemma 4.1 and the proof of Theorem 4.5, one can obtain that

$$\lim_{t\to\infty}\langle x_1(t)\rangle \ge \frac{a_{11}-b_1k_1}{a_{12}k_1} > 0,$$
$$\lim_{t\to\infty}\langle x_2(t)\rangle \ge \frac{a_{21}-b_2k_2}{a_{22}k_2} > 0,$$

and the proof of (ii) is completed.

Similar to (ii), we can see that $\lim_{t\to\infty} x_1(t)=0$, $\lim_{t\to\infty} x_2(t)=0$, from (i) under the condition $a_{11}k_2<b_1$, and system (1.3) can be simplified to a single-species model,

$$dx_3(t) = x_3(t)\left(r\left(1-\frac{x_3(t)}{k_3(t)}\right) - qE\right)dt + \sigma_2 x_3(t)dB_2(t).$$

Therefore,

$$t^{-1}\ln\frac{x_3(t)}{x_3(0)} = -\frac{r}{k_3}\langle x_3(t)\rangle + b_3 - qE + t^{-1}\sigma_2 B_2(t).$$

Applying Lemma 4.1 and similar proof with Theorem 4.5 to the above equation, we obtain:

$$\lim_{t\to\infty}\langle x_3(t)\rangle = \frac{(b_3-qE)k_3}{r} > 0.$$

Finally, we prove (iv). From (4.25)–(4.27), we obtain

$$
\begin{aligned}
t^{-1}\ln\frac{x_1(t)}{x_1(0)} &\geq \frac{a_{11}}{k_1} - a_{12}\langle x_1(t)\rangle - b_1 + t^{-1}\sigma_1 B_1(t),\\
t^{-1}\ln\frac{x_2(t)}{x_2(0)} &\geq \frac{a_{21}}{k_2} - a_{22}\langle x_2(t)\rangle - d_1\langle x_3(t)\rangle^* + \frac{d_1\big(u_1(t)-u_1(0)\big)}{\alpha_1 t} - b_2 + t^{-1}\sigma_1 B_1(t)\\
&\geq \frac{a_{21}}{k_2} - a_{22}\langle x_2(t)\rangle - d_1\frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right) + \frac{d_1\big(u_1(t)-u_1(0)\big)}{\alpha_1 t} - b_2\\
&\quad + t^{-1}\sigma_1 dB_1(t),\\
t^{-1}\ln\frac{x_3(t)}{x_3(0)} &\geq -\frac{r}{k_3}\langle x_3(t)\rangle - d_2\langle x_2(t)\rangle^* + \frac{d_2\big(u_2(t)-u_2(0)\big)}{\alpha_2 t} + b_3 - qE + t^{-1}\sigma_2 B_2(t)\\
&\geq -\frac{r}{k_3}\langle x_3(t)\rangle + \frac{d_2\big(u_2(t)-u_2(0)\big)}{\alpha_2 t} + \frac{\Gamma_1}{a_{12}a_{22}} - qE + t^{-1}\sigma_2 B_2(t).
\end{aligned}
$$

Simply, one can obtain that:

$$
\begin{aligned}
&\lim_{t\to\infty}\langle x_1(t)\rangle \geq \frac{a_{11}-k_1b_1}{a_{12}k_1} > 0,\\
&\lim_{t\to\infty}\langle x_2(t)\rangle \geq \frac{a_{21}-b_2k_2}{a_{22}k_2} - \frac{d_1k_3}{a_{22}r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right) > 0,\\
&\lim_{t\to\infty}\langle x_3(t)\rangle \geq \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right) > 0.
\end{aligned}
$$

□

Theorem 4.7. *If the conditions of Theorem 4.6 (iv) hold, then the optimal harvested efforts of species x_3 are*

$$E^\star = \frac{1}{2pq}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right),$$

and the maximum expectation of net economic revenue is

$$m(E^\star) = \frac{k_3}{4pqr}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right),$$

where p and $\frac{c}{x_3(t)}$ are respectively the unit price and unit cost of a commercially harvested population.

Proof. According to the conclusions of Theorems 4.5 and 4.6, we can obtain that

$$\lim_{t\to\infty}\langle x_3(t)\rangle \leq \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right), \qquad \lim_{t\to\infty}\langle x_3(t)\rangle \geq \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right).$$

Hence

$$\lim_{t\to\infty}\langle x_3(t)\rangle = \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right).$$

Then the net economic revenue is

$$
\begin{aligned}
m(E) &= \lim_{t\to\infty}\left(pE\frac{\int_0^t x_3(v)dv}{t} - \frac{c}{\frac{\int_0^t x_3(v)dv}{t}}\frac{\int_0^t x_3(v)dv}{t}E\right)\\
&= \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}}pE - pqE^2\right) - cE.
\end{aligned}
$$

Letting $\frac{dm(E)}{dE} = 0$, the optimal harvested efforts are

$$E^\star = \frac{1}{2pq}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right),$$

and the maximum expectation of net economic revenue is

$$m(E^\star) = \frac{k_3}{4pqr}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right). \qquad \square$$

5 Numerical analysis

We use some hypothetical parameter values to verify Theorems 4.6 and 4.7. We choose $k_1 = 50, k_2 = 50, k_3 = 100$, and initial values $x_1(0) = 5, x_2(0) = 5, x_3(0) = 8$. Assign different values to other parameters in Table 5.1, which satisfies Theorem 4.6 , to prove theoretical results. Fig. 5.1–Fig. 5.4 show the different survival states of the species, as demonstrated in Theorem 4.6.

Parameter	Fig. 5.1 values	Fig. 5.2 values	Fig. 5.3 values	Fig. 5.4–5.5 values
a_{11}	0.04	13.8	0.03	12.8
a_{12}	0.1	0.2	0.05	0.85
a_{21}	0.2	0.24	0.5	0.24
a_{22}	0.1	0.2	0.1	0.5
s	0.7	0.25	0.5	0.25
d_1	0.1	0.2	0.1	0.01
d_2	0.35	0.2	0.35	0.01
r	1	1.25	1.1	2
q	0.45	0.5	0.5	0.55
E	3	3	2	3.5
β	0.1	0.004	0.1	0.004
σ_1	1.62	0.05	1.42	0.1
σ_2	0.2	0.2	0.2	0.2
α_1	0.5	0.5	0.5	0.5
α_2	0.4	0.4	0.4	0.2

Table 5.1: Parameter values.

Regarding the optimal harvesting effort, we still select the same parameters with the Fig. 5.4. By Theorem 4.7, we obtain $E^* = 1.788$. Therefore, the optimal harvesting policy exists, we show it in Fig. 5.5. The maximum expectation of net economic revenue exists when $E^* = 1.788$.

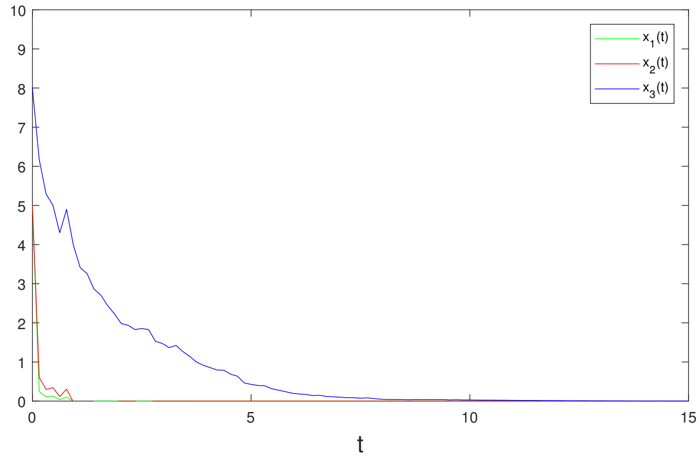

Figure 5.1: x_1, x_2 and x_3 all go to extinction.

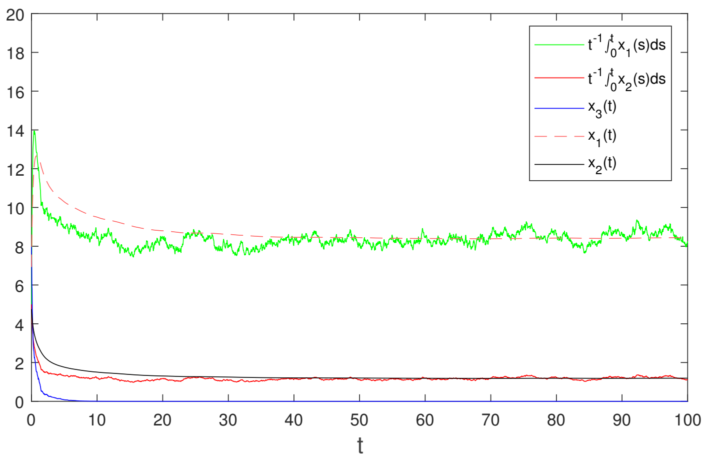

Figure 5.2: x_1 and x_2 are permanent, x_3 goes to extinction.

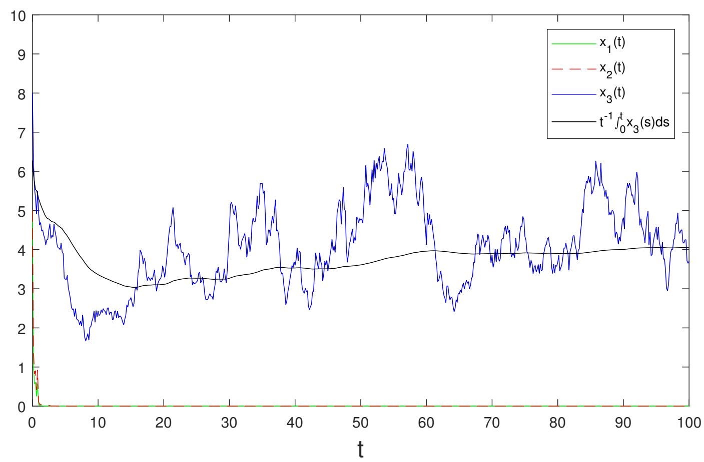

Figure 5.3: x_1,x_2 go to extinction, x_3 is permanent.

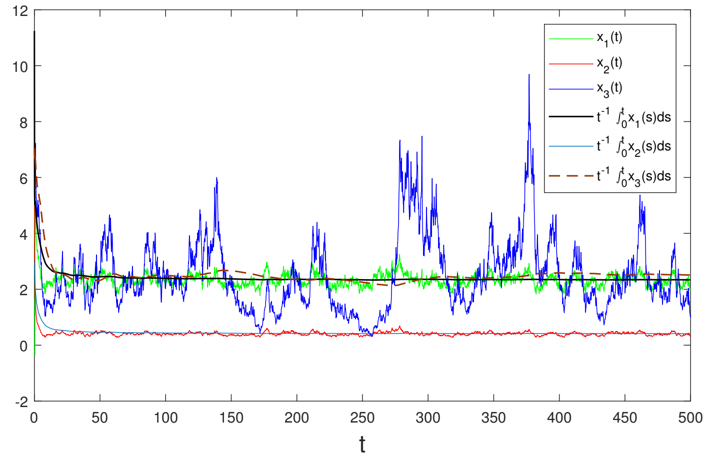

Figure 5.4: x_1, x_2 and x_3 are permanent.

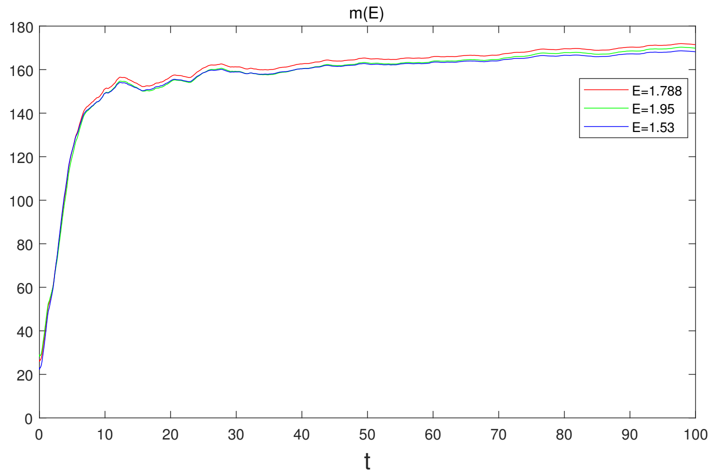

Figure 5.5: The optimal harvesting effort and the maximum of net economic revenue.

6 Conclusion

We investigated the dynamics of a stochastic stage-structured competitive system with distributed delay and harvesting. We took a weak kernel case as an example for convenience, and we similarly discuss the strong kernel case. Our objective was to study the optimal harvest strategy and the maximum net economic revenue. Some main results are as follows:

(i) The existence and uniqueness of the positive solution of system (1.3) was proved, using a Lyapunov function to ensure the rationality of the system and provide support for later results.

(ii) We showed that when $a_{12} > 2(a_{11} \vee a_{21})$, $a_{22} > \alpha_2 + (a_{11} \vee a_{21})$, $r > \alpha_1 k_3, \alpha_1 > d_1$, $\alpha_2 > d_2$, system (1.3) would be asymptotically stable in distribution.

(iii) The research of the optimal harvest and maximum expectation of net economic revenue of stochastic models has clear practical significance. Species extinction must be strictly prevented during fishing. First, sufficient conditions for persistence in mean and extinction were established. The optimal harvested efforts were

$$E^{\star} = \frac{1}{2pq}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right),$$

and the maximum expectation of net economic revenue was

$$m(E^{\star}) = \frac{k_3}{4pqr}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right).$$

We only considered the effect of white noise and delay on the dynamics of the stage-structured competitive system. It is also interesting to consider the effect of telephone noise, toxins, and Markovian switching, and these will be topics of our further research.

Acknowledgments

This research was supported by National Natural Science Foundation of China (61703083 and 61673100).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 26, 1–20; https://doi.org/10.14232/ejqtde.2021.1.26 www.math.u-szeged.hu/ejqtde/

Existence and multiplicity of positive solutions for a singular system via sub-supersolution method and Mountain Pass Theorem

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Received 3 April 2020, appeared 1 April 2021
Communicated by Gennaro Infante

Abstract: In this paper we show the existence and multiplicity of positive solutions using the sub-supersolution method and Mountain Pass Theorem in a general singular system which the operator is not homogeneous neither linear.

Keywords: $p\&q$-Laplacian operator, sub-supersolution method, singular system, Mountain Pass theorem.

2020 Mathematics Subject Classification: Primary 35J20, 35J50; Secondary 58E05.

1 Introduction

In this paper we treat the question of the existence and multiplicity of positive solutions for the following class of singular systems of nonlinear elliptic equation

$$\begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = h_1(x)u^{-\gamma_1} + F_u(x,u,v) \text{ in } \Omega, \\ -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = h_2(x)v^{-\gamma_2} + F_v(x,u,v) \text{ in } \Omega, \\ u,v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $N \geq 3$ and $2 \leq p_1, p_2 < N$. For $i = 1,2$, $\gamma_i > 0$ is a fixed constant, $a_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1-function and $h_i \geq 0$ is a nontrivial measurable function. More precisely, we suppose that the functions h_i and a_i satisfy the following assumptions:

(H) There exists $0 < \phi_0 \in C_0^1(\overline{\Omega})$ such that $h_i\phi_0^{-\gamma_i} \in L^\infty(\Omega)$.

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(A_1) There exist constants $k_1, k_2, k_3, k_4 > 0$ and $2 \le p_i \le q_i < N$ such that

$$k_1 t^{p_i} + k_2 t^{q_i} \le a_i(t^{p_i}) t^{p_i} \le k_3 t^{p_i} + k_4 t^{q_i}, \quad \text{for all } t \ge 0.$$

(A_2) The functions

$$t \longmapsto a_i(t^{p_i}) t^{p_i - 2} \quad \text{are increasing.}$$

(A_3) The functions

$$t \longmapsto A_i(t^{p_i}) \quad \text{are strictly convex,}$$

where $A_i(t) = \int_0^t a_i(s) ds$.

(A_4) There exist constants μ_i, $\frac{1}{q_1^*} < \theta_s < \frac{1}{q_1}$ and $\frac{1}{q_2^*} < \theta_t < \frac{1}{q_2}$ such that

$$\frac{1}{\mu_i} a_i(t) t \le A_i(t), \quad \text{for all } t \ge 0,$$

with $\frac{q_1}{p_1} \le \mu_1 < \frac{1}{\theta_s p_1}$ and $\frac{q_2}{p_2} \le \mu_2 < \frac{1}{\theta_t p_2}$.

Notice that the functions a_i satisfy suitable monotonicity conditions which allow to consider a larger class of $p\&q$ type problems. In order to illustrate the degree of generality of the kind of problems studied here, in the following we present some examples of functions a_i which are interesting from the mathematical point of view and have a wide range of applications in physics and related sciences.

Example 1.1. If $a_i \equiv 1$, for each $i = 1, 2$, our operator is the p-Laplacian and so problem (1.1) becomes

$$\begin{cases} -\Delta_{p_1} u = h_1(x) u^{-\gamma_1} + F_u(x, u, v) \text{ in } \Omega, \\ -\Delta_{p_2} v = h_2(x) v^{-\gamma_2} + F_v(x, u, v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

with $q_i = p_i$, $k_1 + k_2 = 1$ and $k_3 + k_4 = 1$.

Example 1.2. If $a_i(t) = 1 + t^{\frac{q_i - p_i}{p_i}}$, for each $i = 1, 2$, we obtain

$$\begin{cases} -\Delta_{p_1} u - \Delta_{q_1} u = h_1(x) u^{-\gamma_1} + F_u(x, u, v) \text{ in } \Omega, \\ -\Delta_{p_2} v - \Delta_{q_2} v = h_2(x) v^{-\gamma_2} + F_v(x, u, v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

with $k_1 = k_2 = k_3 = k_4 = 1$.

Example 1.3. Taking $a_i(t) = 1 + \frac{1}{(1+t)^{\frac{p_i - 2}{p_i}}}$, for each $i = 1, 2$, we get

$$\begin{cases} -\text{div}\left(|\nabla u|^{p_1 - 2} \nabla u + \dfrac{|\nabla u|^{p_1 - 2} \nabla u}{(1 + |\nabla u|^{p_1})^{\frac{p_1 - 2}{p_1}}} \right) = h_1(x) u^{-\gamma_1} + F_u(x, u, v) \text{ in } \Omega, \\ -\text{div}\left(|\nabla v|^{p_2 - 2} \nabla v + \dfrac{|\nabla v|^{p_2 - 2} \nabla v}{(1 + |\nabla v|^{p_2})^{\frac{p_2 - 2}{p_2}}} \right) = h_2(x) v^{-\gamma_2} + F_v(x, u, v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

with $q_i = p_i$, $k_1 + k_2 = 1$ and $k_3 + k_4 = 2$.

Example 1.4. If we consider $a_i(t) = 1 + t^{\frac{q_i - p_i}{p_i}} + \frac{1}{(1+t)^{\frac{p_i-2}{p_i}}}$, for each $i = 1, 2$, we obtain

$$\begin{cases} -\Delta_{p_1} u - \Delta_{q_1} u - \operatorname{div}\left(\dfrac{|\nabla u|^{p_1-2}\nabla u}{(1+|\nabla u|^{p_1})^{\frac{p_1-2}{p_1}}} \right) = h_1(x) u^{-\gamma_1} + F_u(x,u,v) \text{ in } \Omega, \\ -\Delta_{p_2} v - \Delta_{q_2} v - \operatorname{div}\left(\dfrac{|\nabla v|^{p_2-2}\nabla v}{(1+|\nabla v|^{p})^{\frac{p_2-2}{p_2}}} \right) = h_2(x) v^{-\gamma_2} + F_v(x,u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

where $k_1 = k_2 = k_4 = 1$ and $k_3 = 2$.

Remark 1.5. Note that by hypothesis (H) we have $h_i \in L^\infty(\Omega)$ because

$$|h_i| = |h_i \phi_0^{-\gamma_i} \phi_0^{\gamma_i}| \leq \|h_i \phi_0^{-\gamma_i}\|_\infty \phi_0^{\gamma_i}.$$

Here F is a function on $\overline{\Omega} \times \mathbb{R}^2$ of class C^1 satisfying

(F_1) There exists $0 < \delta < \frac{1}{2}$ such that

$$-h_1(x) \leq F_s(x,s,t) \leq 0 \quad \text{a.e. in } \Omega, \text{ for all } 0 \leq s \leq \delta$$

and

$$-h_2(x) \leq F_t(x,s,t) \leq 0 \quad \text{a.e. in } \Omega, \text{ for all } 0 \leq t \leq \delta.$$

It is worthwhile to point out that, since $p_i < q_i$ and by the boundedness of Ω, $W_0^{1,p_i}(\Omega) \cap W_0^{1,q_i}(\Omega) = W_0^{1,q_i}(\Omega)$. Thus, in order to show the existence and multiplicity of solutions to system (1.1), we define the Sobolev space $X = W_0^{1,q_1}(\Omega) \times W_0^{1,q_2}(\Omega)$ endowed with the norm

$$\|(u,v)\| = \|u\|_{1,q_1} + \|v\|_{1,q_2},$$

where

$$\|u\|_{1,q_i} = \left(\int_\Omega |\nabla u|^{q_i} dx \right)^{\frac{1}{q_i}}.$$

Moreover, we say that a pair $(u,v) \in X$ is a positive weak solution of system (1.1) if $u, v > 0$ in Ω and it verifies

$$\int_\Omega a_1(|\nabla u|^{p_1}) |\nabla u|^{p_1-2} \nabla u \nabla \phi \, dx = \int_\Omega h_1(x) u^{-\gamma_1} \phi \, dx + \int_\Omega F_u(x,u,v) \phi \, dx$$

and

$$\int_\Omega a_2(|\nabla v|^{p_2}) |\nabla v|^{p_2-2} \nabla v \nabla \varphi \, dx = \int_\Omega h_2(x) v^{-\gamma_2} \varphi \, dx + \int_\Omega F_v(x,u,v) \varphi \, dx,$$

for all $(\phi, \varphi) \in X$.

In our first theorem we apply the sub-supersolution method to establish the existence of a weak solution for system (1.1).

Theorem 1.6. *Suppose that (H), (F_1) and (A_1)–(A_3) are satisfied. Then system (1.1) has a positive weak solution if $\|h_i\|_\infty$ is sufficiently small, for $i = 1, 2$.*

Furthermore, we assume the conditions below to prove the existence of two solutions for problem (1.1).

(F_2) For $i = 1, 2$, there exists $1 < r < q_i^* = \frac{Nq_i}{(N-q_i)}$ ($q_i^* = \infty$ if $q_i \geq N$) such that

$$F_s(x, s, t) \leq h_1(x)(1 + s^{r-1} + t^{r-1}) \quad \text{a.e. in } \Omega, \text{ for all } s \geq 0$$

and

$$F_t(x, s, t) \leq h_2(x)(1 + s^{r-1} + t^{r-1}) \quad \text{a.e. in } \Omega, \text{ for all } t \geq 0.$$

(F_3) There exist $s_0, t_0 > 0$ such that

$$0 < F(x, s, t) \leq \theta_s s F_s(x, s, t) + \theta_t t F_t(x, s, t) \quad \text{a.e. in } \Omega, \text{ for all } s \geq s_0 \text{ and } t \geq t_0,$$

where θ_s and θ_t appeared in (A_4).

Theorem 1.7. *Suppose that (H), (F_1)–(F_3) and (A_1)–(A_4) are satisfied. Then system (1.1) has two positive weak solutions if $\|h_i\|_\infty$ is sufficiently small, for $i = 1, 2$.*

Singular problems has been much studied in last years. We are going to cite some authors in last ten years. System (1.1) with Laplacian operator in both equations was studied in [9], where it was investigated the questions of existence, non-existence and uniqueness for solutions. The results in [9] were complemented in [16]. The general operator as we consider in this paper was studied in [5] using continuous unbounded of solutions. The cases with Laplacian operator involving weights were studied in [7] and [11].

In this paper we complement the results that can be found in [5], [7], [9], [11] and [16] because we consider a general problem with singularity without restrictions in the exponents. Moreover, we are considering the sub-supersolution method for a system that involves a non-linear and nonhomegeneous operator. The reader can see the generality of the operator in [5].

We would like to highlight that our theorems can be applied for the model nonlinearity

$$F(x, s, t) = h_1(x)\left(\frac{s^r}{r} - s\delta^{r-1}\right) + h_2(x)\left(\frac{t^r}{r} - t\delta^{r-1}\right).$$

This paper is organized in the following way. Section 2 is devoted to some preliminary results in order to prove the main results. The first theorem is proved in the Section 3 and the second theorem in the Section 4.

2 Preliminary results

The next lemma provides the uniqueness of solution to the linear problem. The proof can be found in [5, Lemma 1]. However, for the convenience of the reader, we also prove it here.

Lemma 2.1. *Assume that the conditions (A_1) and (A_2) hold. Then, there exists an unique solution $u_i \in W_0^{1,q_i}(\Omega)$ of the linear problem*

$$\begin{cases} -\operatorname{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) = h_i(x) \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

where $h_i \in (W_0^{1,q_i}(\Omega))'$, for all $i = 1, 2$ and $2 \leq p_i \leq q_i < N$.

Proof. Consider the operator $T_i : W_0^{1,q_i}(\Omega) \longrightarrow (W_0^{1,q_i}(\Omega))'$ given by

$$\langle T_i u_i, \phi_i \rangle = \int_\Omega a_i(|\nabla u_i|^{p_i})\, |\nabla u_i|^{p_i-2} \nabla u_i \nabla \phi_i \, dx.$$

In virtue of hypothesis (A_1), we can show that the operator T_i is well defined and it is continuous. Furthermore, by considering the hypothesis (A_2), we argument as [8, Lemma 2.4] to obtain the following inequality

$$C_i |u_i - v_i|^{p_i} \le \langle a_i(|u_i|^{p_i})|u_i|^{p_i-2}u_i - a_i(|v_i|^{p_i})|v_i|^{p_i-2}v_i, u_i - v_i \rangle,$$

for some $C_i > 0$ and for all $i = 1, 2$. Therefore,

$$\langle T_i u_i - T_i v_i, u_i - v_i \rangle > 0, \text{ for all } u_i, v_i \in W_0^{1,q_i}(\Omega) \text{ with } u_i \ne v_i,$$

which implies that T_i is monotone. Moreover, using (A_1) again we get

$$\frac{\langle T_i u_i, u_i \rangle}{\|u_i\|_{1,q_i}} \ge k_2 \|u_i\|_{1,q_i}^{q_i-1}$$

and hence

$$\lim_{\|u_i\|_{1,q_i} \to \infty} \frac{\langle T_i u_i, u_i \rangle}{\|u_i\|_{1,q_i}} = +\infty,$$

which shows that T_i is coercive. Thus, applying the Minty–Browder Theorem [2, Theorem 5.15] there exists an unique $u_i \in W_0^{1,q_i}(\Omega)$ such that $T_i u_i = h_i(x)$. □

Our approach in the study of system (1.1) rests heavily on the following Weak Comparison Principle for the $p\&q$-Laplacian operator. The proof of the result below for the scalar case can be found in [6, Lemma 2.1].

Lemma 2.2. *Let Ω a bounded domain and consider $u_i, v_i \in W_0^{1,q_i}(\Omega)$ satisfying*

$$\begin{cases} -\mathrm{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) \le -\mathrm{div}(a_i(|\nabla v_i|^{p_i})|\nabla v_i|^{p_i-2}\nabla v_i) \text{ in } \Omega, \\ u_i \le v_i \text{ on } \partial\Omega, \end{cases}$$

then $u_i \le v_i$ a.e. in Ω, for all $i = 1, 2$ and $2 \le p_i \le q_i < N$.

Proof. Using the test function $\phi_i = (u_i - v_i)^+ := \max\{u_i - v_i, 0\} \in W_0^{1,q_i}(\Omega)$, we get

$$\int_{\Omega \cap \{u_i > v_i\}} \langle a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i - a_i(|\nabla v_i|^{p_i})|\nabla v_i|^{p_i-2}\nabla v_i, \nabla u_i - \nabla v_i \rangle dx \le 0.$$

From Lemma 2.1, $\|(u_i - v_i)^+\| \le 0$, which implies that $u_i \le v_i$ a.e. in Ω. □

Now, using Lemma 2.2, it is possible to repeat the same arguments of [13, Hopf's Lemma] to obtain the next result

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $i = 1, 2$. If $u_i \in C^1(\overline{\Omega}) \cap W_0^{1,q_i}(\Omega)$, with $2 \le p_i \le q_i < N$, and*

$$\begin{cases} -\mathrm{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) \ge 0 \text{ in } \Omega, \\ u_i > 0 \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, $\frac{\partial u_i}{\partial \eta} < 0$ on $\partial\Omega$, where η is the outwards normal to $\partial\Omega$.

We enunciate an iteration lemma due to Stampacchia that we will use to prove the L^∞-regularity of the solutions for this class of $p\&q$ type problems.

Lemma 2.4 (See [14]). *Assume that $\phi : [0,\infty) \to [0,\infty)$ is a nonincreasing function such that if $h > k > k_0$, for some $\alpha > 0$, $\beta > 1$, $\phi(h) \leq \frac{C(\phi(k))^\beta}{(h-k)^\alpha}$. Then, $\phi(k_0 + d) = 0$, where $d^\alpha = C2^{\frac{\alpha\beta}{\beta-1}}\phi(k_0)^{\beta-1}$ and C is positive constant.*

Lemma 2.5. *Let $u_i \in W_0^{1,q_i}(\Omega)$ be solution to problem*

$$\begin{cases} -\operatorname{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) = f_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

such that $f_i \in L^{r_i}(\Omega)$ with $r_i > \frac{q_i^*}{q_i^*-q_i}$. Then, $u_i \in L^\infty$. In particular, if $\|f_i\|_{r_i}$ is small, then also $\|u_i\|_\infty$ is small, for all $i = 1,2$ and $2 \leq p_i \leq q_i < N$.

Proof. Since u_i is the weak solution to (2.1) we can write

$$\int_\Omega a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i \nabla \phi_i \, dx = \int_\Omega f_i \phi_i \, dx, \ \forall \phi_i \in W_0^{1,q_i}(\Omega).$$

For $k > 0$, we define the test function

$$v_i = sign(u_i)(|u_i| - k) = \begin{cases} u - k, \text{ if } u > k, \\ 0, \text{ if } u = k, \\ u + k, \text{ if } u < k. \end{cases}$$

Then, $u_i = v_i + k\,\text{sign}(u_i)$ and $\frac{\partial u_i}{\partial x_j} = \frac{\partial v_i}{\partial x_j}$ in the set $A(k) = \{x \in \Omega; |u(x)| > k\}$, $v_i = 0$ in $\Omega - A(k)$ and $v_i \in W_0^{1,q_i}(\Omega)$. By considering the test function v_i and using the Hölder inequality, we get

$$\int_{A(k)} a_i(|\nabla v_i|^{p_i})\, |\nabla v_i|^{p_i} dx = \int_\Omega f_i v_i \, dx \leq \left(\int_{A(k)} |v_i|^{q_i^*} dx \right)^{\frac{1}{q_i^*}} \left(\int_{A(k)} |f_i|^{r_i} dx \right)^{\frac{1}{r_i}} |A(k)|^{1-\left(\frac{1}{q_i^*}+\frac{1}{r_i}\right)},$$

where $|A(k)|$ denotes the Lebesgue measure of $A(k)$. Moreover, applying (A_1) and Sobolev inequality we obtain

$$k_2 S \left(\int_{A(k)} |v_i|^{q_i^*} dx \right)^{\frac{q_i-1}{q_i^*}} \leq \left(\int_{A(k)} |f_i|^{r_i} dx \right)^{\frac{1}{r_i}} |A(k)|^{1-\left(\frac{1}{q_i^*}+\frac{1}{r_i}\right)}, \tag{2.2}$$

where S is the best constant in the Sobolev inclusion.

Note that if $0 < k < h$, then $A(h) \subset A(k)$ and

$$|A(k)|^{\frac{1}{q_i^*}} (h-k) = \left(\int_{A(h)} (h-k)^{q_i^*} dx \right)^{\frac{1}{q_i^*}} \leq \left(\int_{A(h)} |v_i|^{q_i^*} dx \right)^{\frac{1}{q_i^*}} \leq \left(\int_{A(k)} |v_i|^{q_i^*} dx \right)^{\frac{1}{q_i^*}}. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$|A(k)| \leq \frac{1}{(h-k)^{q_i^*}} \frac{1}{(k_2 S)^{\frac{q_i^*}{q_i-1}}} \|f_i\|_{r_i}^{\frac{q_i^*}{q_i-1}} |A(k)|^{\frac{q_i^*}{q_i-1}\left[1-\left(\frac{1}{q_i^*}+\frac{1}{r_i}\right)\right]}.$$

Since $r_i > \frac{q_i^*}{q_i^*-q_i}$ we have $\beta := \frac{q_i^*}{q_i-1}\left[1-\left(\frac{1}{q_i^*}+\frac{1}{r_i}\right)\right] > 1$. Therefore, if we define

$$\phi(h) = |A(h)|, \qquad \alpha = q_i^*, \qquad \beta := \frac{q_i^*}{q_i-1}\left[1-\left(\frac{1}{q_i^*}+\frac{1}{r_i}\right)\right], \qquad k_0 = 0,$$

we obtain that ϕ is a nonincreasing function and

$$\phi(h) \leq \frac{C(\phi(k))^\beta}{(h-k)^\alpha}, \quad \text{for all } h > k > 0.$$

By Lemma 2.4, we conclude that $\phi(d) = 0$ for $d = C\frac{\|f_i\|_{r_i}^{\frac{1}{q_i-1}}}{(k_2 S)^{\frac{1}{q_i-1}}}|\Omega|^{\frac{\beta-1}{\alpha}}$ and hence,

$$\|u_i\|_\infty \leq C\frac{\|f_i\|_{r_i}^{\frac{1}{q_i-1}}}{(k_2 S)^{\frac{1}{q_i-1}}}|\Omega|^{\frac{\beta-1}{\alpha}},$$

where β, α, S and C are constants that do not depend on f_i and u_i. □

Regarding the regularity of the solution of (2.1) the next result hold and the proof can be done repeating the same arguments of [10, Theorem 1].

Lemma 2.6. *Fix $h_i \in L^\infty(\Omega)$, for all $i = 1, 2$, and consider $u_i \in W_0^{1,q_i}(\Omega) \cap L^\infty(\Omega)$, with $2 \leq p_i \leq q_i < N$, satisfying the problem*

$$\begin{cases} -\operatorname{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) = h_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

Then, $u_i \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$.

The following result can be found in [12, Lemma 2.6]. The proof is presented for the completeness of the paper.

Lemma 2.7. *Let $\phi, \omega > 0$ be any functions on $C_0^1(\overline{\Omega})$. If $\frac{\partial\phi}{\partial\nu} > 0$ in $\partial\Omega$, where ν is the inwards normal to $\partial\Omega$, then there exists $C > 0$ such that*

$$\frac{\phi(x)}{\omega(x)} \geq C > 0, \quad \textit{for all } x \in \Omega.$$

Proof. For $\delta > 0$ sufficiently small, we consider the following set

$$\Omega_\delta = \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

Since $\phi, \omega > 0$ in Ω and $\Omega \setminus \Omega_\delta$ is compact, there exists $m > 0$ such that

$$\frac{\phi(x)}{\omega(x)} \geq m, \quad \text{for all } x \in \Omega \setminus \Omega_\delta. \tag{2.4}$$

It follows from $\frac{\partial\phi}{\partial\nu} > 0$ in $\partial\Omega$ that $\frac{\partial\phi}{\partial\eta} < 0$, where η is the outwards normal to $\partial\Omega$. Furthermore, since $\Omega \subset \mathbb{R}^n$ is bounded domain, then $\partial\Omega$ is a compact set and consequently, there exists $C_1 < 0$ satisfying

$$\frac{\partial\phi(x)}{\partial\eta} \leq C_1, \quad \text{for all } x \in \overline{\Omega}_\delta.$$

Since $\omega \in C_0^1(\overline{\Omega})$, there exists $C_2 > 0$ such that

$$\left|\frac{\partial\omega(x)}{\partial\eta}\right| \leq C_2, \quad \text{for all } x \in \overline{\Omega}_\delta.$$

Consider $K_0 = \inf_{\overline{\Omega}_\delta} \frac{\partial\omega}{\partial\eta} < 0$ and define the function $H(x) = \alpha\omega(x) - \phi(x)$, for all $x \in \overline{\Omega}_\delta$ and $\alpha \in \mathbb{R}$ to be chosen later. Since $0 < \alpha < \frac{C_1}{K_0}$ we obtain

$$\frac{\partial H(x)}{\partial\eta} = \alpha\frac{\partial\omega(x)}{\partial\eta} - \frac{\partial\phi(x)}{\partial\eta} \geq \alpha K_0 - C_1 > 0, \quad \text{for all } x \in \overline{\Omega}_\delta.$$

Now, fix $x \in \overline{\Omega}_\delta$ and consider the function

$$f(x) = H(x + s\eta), \quad \text{for all } s \in \mathbb{R}.$$

For every $x \in \overline{\Omega}_\delta$, we choose an unique $\widetilde{x} \in \overline{\Omega}_\delta$ so that there exists $\widehat{s} > 0$ such that $x + \widehat{s}\eta = \widetilde{x} \in \partial\Omega$. Hence, since $H(\partial\Omega) \equiv 0$ we have

$$f(\widehat{s}) = H(x + \widehat{s}\eta) = H(\widetilde{x}) = 0.$$

Applying the Mean Value Theorem, there exists $\xi \in (0, \widehat{s})$ such that

$$f(\widehat{s}) - f(0) = f'(\xi)(\widehat{s} - 0),$$

which implies that

$$-H(x) = \frac{\partial H}{\partial\eta}(x + \xi\eta)\widehat{s} > 0 \quad \text{in } \overline{\Omega}_\delta.$$

Therefore, $H(x) \leq 0$ for all $x \in \overline{\Omega}_\delta$ and hence,

$$\alpha\omega(x) - \phi(x) \leq 0, \quad \text{for all } x \in \overline{\Omega}_\delta,$$

which result in

$$\alpha\omega(x) \leq \phi(x), \quad \text{for all } x \in \overline{\Omega}_\delta.$$

Thus,

$$\frac{\phi(x)}{\omega(x)} \geq \alpha > 0, \quad \text{for all } x \in \overline{\Omega}_\delta. \tag{2.5}$$

By virtue of (2.4) and (2.5), we conclude that there exists $C > 0$ so that

$$\frac{\phi(x)}{\omega(x)} \geq C, \quad \text{for all } x \in \Omega. \qquad \square$$

3 Proof of Theorem 1.6

In the proof of Theorem 1.6, we combine the sub-supersolution method with minimization arguments. Before this, we need of the following definition.

We say that $(\underline{u},\underline{v}),(\overline{u},\overline{v}) \in X$ form a pair of sub and supersolution for system (1.1) if $\underline{u},\overline{u},\underline{v},\overline{v} \in L^\infty(\Omega)$ with

(a) $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$ in Ω and $\underline{u} = 0 \leq \overline{u}$, $\underline{v} = 0 \leq \overline{v}$ on $\partial\Omega$,

(b) Given $(\phi,\varphi) \in X$, with $\phi,\varphi \geq 0$, we have

$$\begin{cases} \displaystyle\int_\Omega a_1(|\nabla \underline{u}|^{p_1})|\nabla \underline{u}|^{p_1-2}\nabla \underline{u}\nabla\phi\, dx \leq \int_\Omega h_1(x)\underline{u}^{-\gamma_1}\phi\, dx + \int_\Omega F_u(x,\underline{u},w)\phi\, dx, & \text{for all } w \in [\underline{v},\overline{v}], \\ \displaystyle\int_\Omega a_2(|\nabla \underline{v}|^{p_2})|\nabla \underline{v}|^{p_2-2}\nabla \underline{v}\nabla\varphi\, dx \leq \int_\Omega h_2(x)\underline{v}^{-\gamma_2}\varphi\, dx + \int_\Omega F_v(x,w,\underline{v})\varphi\, dx, & \text{for all } w \in [\underline{u},\overline{u}] \end{cases}$$

and

$$\begin{cases} \displaystyle\int_\Omega a_1(|\nabla \overline{u}|^{p_1})|\nabla \overline{u}|^{p_1-2}\nabla \overline{u}\ \nabla\phi\, dx \geq \int_\Omega h_1(x)\overline{u}^{-\gamma_1}\phi\, dx + \int_\Omega F_u(x,\overline{u},w)\phi\, dx, & \text{for all } w \in [\underline{v},\overline{v}] \\ \displaystyle\int_\Omega a_2(|\nabla \overline{v}|^{p_2})|\nabla \overline{v}|^{p_2-2}\nabla \overline{v}\ \nabla\varphi\, dx \geq \int_\Omega h_2(x)\overline{v}^{-\gamma_2}\varphi,\, dx + \int_\Omega F_v(x,w,\overline{v})\varphi\, dx, & \text{for all } w \in [\underline{u},\overline{u}]. \end{cases}$$

The next result is essential to provide the existence of a subsolution and a supersolution for system (1.1) whenever we fix the value of $\|h_i\|_\infty$ with $i = 1,2$.

Lemma 3.1. *Suppose that (H), (F_1) and (A_1)–(A_2) are satisfied. If $\|h_i\|_\infty$ is small, for $i = 1,2$, then there exist $\underline{u},\overline{u},\underline{v},\overline{v} \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, such that*

i) $h_1\underline{u}^{-\gamma_1}, h_2\underline{v}^{-\gamma_2} \in L^\infty(\Omega)$, $\|\underline{u}\|_\infty \leq \delta$ *and* $\|\underline{v}\|_\infty \leq \delta$, *where δ is the constant that appeared in the hypothesis (F_1);*

ii) $\|\overline{u}\|_\infty \leq \delta$ *and* $\|\overline{v}\|_\infty \leq \delta$, *where δ is the constant that appeared in the hypothesis (F_1);*

iii) $0 < \underline{u}(x) \leq \overline{u}(x)$ *a.e. in Ω and* $0 < \underline{v}(x) \leq \overline{v}(x)$ *a.e. in Ω;*

iv) $(\underline{u},\underline{v})$ *is a subsolution and* $(\overline{u},\overline{v})$ *is a supersolution for system (1.1).*

Proof. From Lemma 2.1 and maximum principle, there exists an unique positive solution $0 < \underline{u} \in W_0^{1,q_1}(\Omega)$ satisfying the problem below

$$\begin{cases} -\mathrm{div}(a_1(|\nabla \underline{u}|^{p_1})|\nabla \underline{u}|^{p_1-2}\nabla \underline{u}) = h_1(x) \text{ in } \Omega, \\ \underline{u} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

Similary, there exists an unique positive solution $0 < \underline{v} \in W_0^{1,q_2}(\Omega)$ satisfying

$$\begin{cases} -\mathrm{div}(a_2(|\nabla \underline{v}|^{p_2})|\nabla \underline{v}|^{p_2-2}\nabla \underline{v}) = h_2(x) \text{ in } \Omega, \\ \underline{v} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.2}$$

Since $h_1, h_2 \in L^\infty(\Omega)$, it follows from Lemma 2.5 that $\underline{u}, \underline{v} \in L^\infty(\Omega)$ and there exist $C_1, C_2 > 0$ such that

$$\|\underline{u}\|_\infty \leq C_1 \|h_1\|_\infty^{\frac{1}{p_1-1}} \quad \text{and} \quad \|\underline{v}\|_\infty \leq C_2 \|h_2\|_\infty^{\frac{1}{p_2-1}},$$

where C_1 and C_2 are constants that does not depend on h_i, $\underline{u}$ and $\underline{v}$. Therefore, we may choose $\|h_i\|_\infty$ sufficiently small, with $i = 1, 2$, so that

$$\|\underline{u}\|_\infty \leq \delta < \frac{1}{2} \quad \text{and} \quad \|\underline{v}\|_\infty \leq \delta < \frac{1}{2}.$$

Moreover, from Lemma 2.6 we have $\underline{u}, \underline{v} \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$. Thus, by virtue of Lemmas 2.3 and 2.7, there exist $C_3, C_4 > 0$ such that

$$\frac{\underline{u}(x)^{-\gamma_1}}{\phi_0(x)^{-\gamma_1}} \leq C_3^{-\gamma_1} \quad \text{and} \quad \frac{\underline{v}(x)^{-\gamma_2}}{\phi_0(x)^{-\gamma_2}} \leq C_4^{-\gamma_2}, \quad \text{for all } x \in \Omega.$$

Therefore, by (H) we get

$$|h_1 \underline{u}^{-\gamma_1}| \leq C_3^{-\gamma_1} \|h_1 \phi_0^{-\gamma_1}\|_\infty \quad \text{and} \quad |h_2 \underline{v}^{-\gamma_2}| \leq C_4^{-\gamma_2} \|h_2 \phi_0^{-\gamma_2}\|_\infty, \tag{3.3}$$

implying that $h_1 \underline{u}^{-\gamma_1}, h_2 \underline{v}^{-\gamma_2} \in L^\infty(\Omega)$, which ends the proof of condition (i).

In order to prove (ii), we invoke Lemma 2.1 and maximum principle once again to claim that there exists an unique positive solution $0 < \overline{u} \in W_0^{1,q_1}(\Omega)$ satisfying

$$\begin{cases} -\text{div}(a_1(|\nabla \overline{u}|^{p_1})|\nabla \overline{u}|^{p_1-2}\nabla \overline{u}) = h_1(x)\underline{u}^{-\gamma_1} \text{ in } \Omega, \\ \overline{u} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.4}$$

and there exists an unique positive solution $0 < \overline{v} \in W_0^{1,q_2}(\Omega)$ satisfying

$$\begin{cases} -\text{div}(a_2(|\nabla \overline{v}|^{p_2})|\nabla \overline{v}|^{p_2-2}\nabla \overline{v}) = h_2(x)\underline{v}^{-\gamma_2} \text{ in } \Omega, \\ \overline{v} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.5}$$

Since $h_1 \underline{u}^{-\gamma_1}, h_2 \underline{v}^{-\gamma_2} \in L^\infty(\Omega)$, we use Lemma 2.5 to obtain $\overline{u}, \overline{v} \in L^\infty(\Omega)$ and hence, from Lemma 2.6 we obtain $\overline{u}, \overline{v} \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$. Furthermore, note that using (3.3) we have

$$\|\overline{u}\|_\infty \leq C_1^* \|h_1 \underline{u}^{-\gamma_1}\|_\infty^{\frac{1}{p_1-1}} \leq C_1^* \|h_1\|_\infty^{\frac{1}{p_1-1}} C_3^{-\gamma_1\left(\frac{1}{p_1-1}\right)} \|\phi_0\|_\infty^{-\gamma_1\left(\frac{1}{p_1-1}\right)}$$

and

$$\|\overline{v}\|_\infty \leq C_2^* \|h_2 \underline{v}^{-\gamma_2}\|_\infty^{\frac{1}{p_2-1}} \leq C_2^* \|h_2\|_\infty^{\frac{1}{p_2-1}} C_4^{-\gamma_2\left(\frac{1}{p_2-1}\right)} \|\phi_0\|_\infty^{-\gamma_1\left(\frac{1}{p_1-1}\right)}.$$

So, choosing $\|h_i\|_\infty$ sufficiently small, with $i = 1, 2$, we obtain

$$\|\overline{u}\|_\infty \leq \delta < \frac{1}{2} \quad \text{and} \quad \|\overline{v}\|_\infty \leq \delta < \frac{1}{2}.$$

Now, since $\|\underline{u}\|_\infty$ and $\|\underline{v}\|_\infty$ are small it follows from (3.1), (3.2), (3.4) and (3.5) that

$$\begin{aligned} -\text{div}(a_1(|\nabla \overline{u}|^{p_1})|\nabla \overline{u}|^{p_1-2}\nabla \overline{u}) &= h_1(x)\underline{u}^{-\gamma_1} \geq h_1(x)\|\underline{u}\|_\infty^{-\gamma_1} \geq h_1(x) \\ &= -\text{div}(a_1(|\nabla \underline{u}|^{p_1})|\nabla \underline{u}|^{p_1-2}\nabla \underline{u}) \end{aligned}$$

and

$$-\operatorname{div}(a_2(|\nabla\overline{v}|^{p_2})|\nabla\overline{v}|^{p_2-2}\nabla\overline{v}) = h_2(x)\underline{v}^{-\gamma_2} \geq h_2(x)\|\underline{v}\|_\infty^{-\gamma_2} \geq h_2(x)$$
$$= -\operatorname{div}(a_2(|\nabla\underline{v}|^{p_2})|\nabla\underline{v}|^{p_2-2}\nabla\underline{v}).$$

Therefore, applying the Weak Comparison Principle for the $p\&q$-Laplacian operator we conclude that

$$0 < \underline{u}(x) \leq \overline{u}(x) \quad \text{a.e. in } \Omega \quad \text{and} \quad 0 < \underline{v}(x) \leq \overline{v}(x) \quad \text{a.e. in } \Omega,$$

which proves (iii).

Our final task is to check that the condition (iv) holds. Indeed, we invoke (F_1), (i), (3.1) and (3.2) to obtain

$$-\operatorname{div}(a_1(|\nabla\underline{u}|^{p_1})|\nabla\underline{u}|^{p_1-2}\nabla\underline{u}) - h_1(x)\underline{u}^{-\gamma_1} - F_{\underline{u}}(x,\underline{u},v)$$
$$\leq 2h_1(x) - h_1(x)\underline{u}^{-\gamma_1} \leq h_1(x)(2 - \|\underline{u}\|_\infty^{-\gamma_1}) \leq 0$$

and

$$-\operatorname{div}(a_2(|\nabla\underline{v}|^{p_2})|\nabla\underline{v}|^{p_2-2}\nabla\underline{v}) - h_2(x)\underline{v}^{-\gamma_2} - F_{\underline{v}}(x,u,\underline{v}),$$
$$\leq 2h_2(x) - h_2(x)\underline{v}^{-\gamma_2} \leq h_2(x)(2 - \|\underline{v}\|_\infty^{-\gamma_2}) \leq 0,$$

which implies that $(\underline{u},\underline{v})$ is a subsolution for system (1.1). Finally, we use (F_1), (ii), (iii), (3.4) and (3.5) to get

$$-\operatorname{div}(a_1(|\nabla\overline{u}|^{p_1})|\nabla\overline{u}|^{p_1-2}\nabla\overline{u}) - h_1(x)\overline{u}^{-\gamma_1} - F_{\overline{u}}(x,\overline{u},v) \geq h_1(x)(\underline{u}^{-\gamma_1} - \overline{u}^{-\gamma_1}) \geq 0$$

and

$$-\operatorname{div}(a_2(|\nabla\overline{v}|^{p_2})|\nabla\overline{v}|^{p_2-2}\nabla\overline{v} - h_2(x)\overline{v}^{-\gamma_2} - F_{\overline{v}}(x,u,\overline{v}) \geq h_2(x)(\underline{v}^{-\gamma_2} - \overline{v}^{-\gamma_2}) \geq 0,$$

which shows that $(\overline{u},\overline{v})$ is a supersolution for system (1.1). □

Following the same idea in [4] (see also [3]), we introduce the truncation operators $T : W_0^{1,q_1}(\Omega) \to L^\infty(\Omega)$ and $S : W_0^{1,q_2}(\Omega) \to L^\infty(\Omega)$ given by

$$Tu(x) = \begin{cases} \overline{u}(x), & \text{if } u(x) > \overline{u}(x) \\ u(x), & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x) \\ \underline{u}(x), & \text{if } u(x) < \underline{u}(x) \end{cases} \tag{3.6}$$

and

$$Sv(x) = \begin{cases} \overline{v}(x), & \text{if } v(x) > \overline{v}(x) \\ v(x), & \text{if } \underline{v}(x) \leq v(x) \leq \overline{v}(x) \\ \underline{v}(x), & \text{if } v(x) < \underline{v}(x). \end{cases} \tag{3.7}$$

It is well that the truncation operators T and S are continuous and bounded. Now, we consider the following functions

$$G_u(x,u,v) = h_1(x)(Tu)^{-\gamma_1} + F_u(x,Tu,Sv) \tag{3.8}$$

and

$$G_v(x,u,v) = h_2(x)(Sv)^{-\gamma_2} + F_v(x,Tu,Sv) \tag{3.9}$$

and the auxiliary problem

$$\begin{cases} -\text{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = G_u(x,u,v) \text{ in } \Omega, \\ -\text{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = G_v(x,u,v) \text{ in } \Omega, \\ u,v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.10}$$

Define the energy functional $\Phi : X \to \mathbb{R}$ associated with problem (3.10) by

$$\Phi(u,v) = \frac{1}{p_1}\int_\Omega A_1(|\nabla u|^{p_1})dx + \frac{1}{p_2}\int_\Omega A_2(|\nabla v|^{p_2})dx - \int_\Omega G(x,u,v)dx, \qquad \forall (u,v) \in X,$$

where $G(x,s,t) = \int_0^s G_\xi(x,\xi,t)d\xi + \int_0^t G_\xi(x,s,\xi)d\xi$.

It follows from Lemma 3.1 (i)–(iii), (3.8), (3.9) and (F_1) that

$$|G_u(x,u,v)| \le K_1 \quad \text{a.e. in } \Omega, \text{ for some } K_1 > 0, \forall (u,v) \in X. \tag{3.11}$$

Similarly,

$$|G_v(x,u,v)| \le K_2 \quad \text{a.e. in } \Omega, \text{ for some } K_2 > 0, \forall (u,v) \in X. \tag{3.12}$$

Consequently, we use (A_1) to show that the functional Φ is well defined and it is of class C^1 on Sobolev space X with

$$\begin{aligned} \Phi'(u,v)(\phi,\varphi) = &\int_\Omega \left[a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u\nabla\phi + a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v\nabla\varphi\right] dx \\ &- \int_\Omega G_u(x,u,v)\phi\, dx - \int_\Omega G_v(x,u,v)\varphi\, dx, \forall (u,v),(\phi,\varphi) \in X. \end{aligned}$$

Next, consider

$$M = \{(u,v) \in X; \underline{u} \le u \le \overline{u} \text{ a.e. in } \Omega \text{ and } \underline{v} \le v \le \overline{v} \text{ a.e. in } \Omega\}.$$

We claim that Φ is bounded from below in M. Indeed, for all $(u,v) \in X$, we use (A_1), (3.11), (3.12) and continuous embedding $W_0^{1,q_i}(\Omega) \hookrightarrow L^{1,q_i}(\Omega)$, for $i = 1,2$, to obtain that Φ is coercive in M. Moreover, since (A_3) holds and $G_u, G_v \in L^\infty(\Omega)$ we have that Φ is weak lower semi-continuous on M. Thus, as M is closed and convex in X, we use [15, Theorem 1.2] to conclude that Φ is bounded from below in M and attains it is infimum at a point $(u,v) \in M$.

Using the same arguments as in the proof of [15, Theorem 2.4], we see that this minimum point (u,v) is a weak solution of problem (3.10). Indeed, for all $\phi, \varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$, let the functions $u_\varepsilon, v_\varepsilon \in M$ be given by

$$u_\varepsilon(x) = \begin{cases} \overline{u}(x), & u(x) + \varepsilon\phi(x) > \overline{u}(x) \\ u(x) + \varepsilon\phi(x), & \underline{u}(x) \le u(x) + \varepsilon\phi(x) \le \overline{u}(x) \\ \underline{u}(x), & u(x) + \varepsilon\phi(x) < \underline{u}(x) \end{cases}$$

and

$$v_\varepsilon(x) = \begin{cases} \overline{v}(x), & v(x) + \varepsilon\varphi(x) > \overline{v}(x) \\ v(x) + \varepsilon\varphi(x), & \underline{v}(x) \le v(x) + \varepsilon\varphi(x) \le \overline{v}(x) \\ \underline{v}(x), & v(x) + \varepsilon\varphi(x) < \underline{v}(x). \end{cases}$$

The functions u_ε and v_ε can be written as

$$u_\varepsilon = (u + \varepsilon\phi) - (\overline{\phi}_\varepsilon - \underline{\phi}_\varepsilon) \in M \quad \text{and} \quad v_\varepsilon = (v + \varepsilon\varphi) - (\overline{\varphi}_\varepsilon - \underline{\varphi}_\varepsilon) \in M,$$

where $\overline{\phi}_\varepsilon = \max\{0, u + \varepsilon\phi - \overline{u}\} \geq 0$, $\underline{\phi}_\varepsilon = -\min\{0, u + \varepsilon\phi - \underline{u}\} \geq 0$, $\overline{\varphi}_\varepsilon = \max\{0, v + \varepsilon\varphi - \overline{v}\} \geq 0$ and $\underline{\varphi}_\varepsilon = -\min\{0, v + \varepsilon\varphi - \underline{v}\} \geq 0$.

Note that $\overline{\phi}_\varepsilon, \underline{\phi}_\varepsilon \in W_0^{1,q_1}(\Omega) \cap L^\infty(\Omega)$, $\overline{\varphi}_\varepsilon, \underline{\varphi}_\varepsilon \in W_0^{1,q_2}(\Omega) \cap L^\infty(\Omega)$ and Φ is differentiable in direction $(u_\varepsilon - u, v_\varepsilon - v)$. Since $(u, v) \in M$ minimizes the functional Φ in M, then

$$0 \leq \Phi'(u, v)(u_\varepsilon - u, v_\varepsilon - v) = \varepsilon\Phi'(u, v)(\phi, \varphi) - \Phi'(u, v)(\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) + \Phi'(u, v)(\underline{\phi}_\varepsilon, \underline{\varphi}_\varepsilon).$$

Thus,

$$\Phi'(u, v)(\phi, \varphi) \geq \frac{1}{\varepsilon}\left[\Phi'(u, v)(\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) - \Phi'(u, v)(\underline{\phi}_\varepsilon, \underline{\varphi}_\varepsilon)\right]. \tag{3.13}$$

Now, since $(\overline{u}, \overline{v})$ is a supersolution to system (1.1), we obtain

$$\begin{aligned}
\Phi'(u, v)(\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) &= \Phi'(\overline{u}, \overline{v})(\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) + \left[\Phi'(u, v) - \Phi'(\overline{u}, \overline{v})\right](\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) \\
&\geq \left[\Phi'(u, v) - \Phi'(\overline{u}, \overline{v})\right](\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) \\
&= \int\limits_{\Omega_\varepsilon} \left[a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u - a_1(|\nabla \overline{u}|^{p_1})|\nabla \overline{u}|^{p_1-2}\nabla \overline{u}\right] \nabla(u + \varepsilon\phi - \overline{u})dx \\
&\quad + \int\limits_{\Omega_\varepsilon} \left[a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v - a_2(|\nabla \overline{v}|^{p_2})|\nabla \overline{v}|^{p_2-2}\nabla \overline{v}\right] \nabla(v + \varepsilon\varphi - \overline{v})dx \\
&\quad - \int\limits_{\Omega_\varepsilon} \left[G_u(x, u, v) - G_{\overline{u}}(x, \overline{u}, \overline{v})\right](u + \varepsilon\phi - \overline{u})dx \\
&\quad - \int\limits_{\Omega_\varepsilon} \left[G_v(x, u, v) - G_{\overline{v}}(x, \overline{u}, \overline{v})\right](v + \varepsilon\varphi - \overline{v})dx \\
&\geq \varepsilon \int\limits_{\Omega_\varepsilon} \left[a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u - a_1(|\nabla \overline{u}|^{p_1})|\nabla \overline{u}|^{p_1-2}\nabla \overline{u}\right] \nabla\phi dx \\
&\quad + \varepsilon \int\limits_{\Omega_\varepsilon} \left[a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v - a_2(|\nabla \overline{v}|^{p_2})|\nabla \overline{v}|^{p_2-2}\nabla \overline{v}\right] \nabla\varphi dx \\
&\quad - \varepsilon \int\limits_{\Omega_\varepsilon} |G_u(x, u, v) - F_{\overline{u}}(x, \overline{u}, \overline{v})|\,|\phi|dx - \varepsilon \int\limits_{\Omega_\varepsilon} |G_v(x, u, v) - G_{\overline{v}}(x, \overline{u}, \overline{v})|\,|\varphi|dx,
\end{aligned}$$

where $\Omega_\varepsilon = \{x \in \Omega; u(x) + \varepsilon\phi(x) > \overline{u}(x) \geq u(x) \text{ and } v(x) + \varepsilon\varphi(x) > \overline{v}(x) \geq v(x)\}$. Note that $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$. Then, $\Phi'(u, v)(\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) \geq o(\varepsilon)$, where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. Similarly, we obtain $\Phi'(u, v)(\underline{\phi}_\varepsilon, \underline{\varphi}_\varepsilon) \leq o(\varepsilon)$ and consequently, by (3.13) we conclude that $\Phi'(u, v)(\phi, \varphi) \geq 0$, for all $\phi, \varphi \in C_0^\infty(\Omega)$. Repeating the above arguments for $(-\phi, -\varphi)$ we have $\Phi'(u, v)(\phi, \varphi) \leq 0$, for all $\phi, \varphi \in C_0^\infty(\Omega)$ and hence, $\Phi'(u, v)(\phi, \varphi) = 0$. Therefore, since $C_0^\infty(\Omega)$ is dense in $W_0^{q_i}$, $\forall i = 1, 2$, we prove that $\Phi'(u, v) = 0$, which implies that (u, v) weakly solves (3.10).

Since $(u, v) \in M$ it follows from $G_u(x, u, v) = h_1(x)u^{-\gamma_1} + F_u(x, u, v)$ and $G_v(x, u, v) = h_2(x)v^{-\gamma_2} + F_v(x, u, v)$, for $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, that $(u, v) \in X$ is precisely a positive weak solution for system (1.1).

4 Proof of Theorem 1.7

Let $(\underline{u}, \underline{v}) \in L^{\infty}(\Omega) \times L^{\infty}(\Omega)$ be the subsolution of system (1.1). Consider $T : W_0^{1,q_1}(\Omega) \to L^{\infty}(\Omega)$ and $S : W_0^{1,q_2}(\Omega) \to L^{\infty}(\Omega)$ the truncation operators given by

$$\widehat{T}u(x) = \begin{cases} u(x), & \text{if } u(x) > \underline{u}(x) \\ \underline{u}(x), & \text{if } u(x) \leq \underline{u}(x), \end{cases} \tag{4.1}$$

$$\widehat{S}v(x) = \begin{cases} v(x), & \text{if } v(x) > \underline{v}(x) \\ \underline{v}(x), & \text{if } v(x) \leq \underline{v}(x). \end{cases} \tag{4.2}$$

and the following functions

$$\widehat{G}_u(x,u,v) = h_1(x)(\widehat{T}u)^{-\gamma_1} + F_u(x, \widehat{T}u, \widehat{S}v) \tag{4.3}$$

and

$$\widehat{G}_v(x,u,v) = h_2(x)(\widehat{S}v)^{-\gamma_2} + F_v(x, \widehat{T}u, \widehat{S}v) \tag{4.4}$$

Next, consider the auxiliary problem

$$\begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = \widehat{G}_u(x,u,v) \text{ in } \Omega, \\ -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = \widehat{G}_v(x,u,v) \text{ in } \Omega, \\ u, v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.5}$$

and define the functional $\widehat{\Phi} : X \to \mathbb{R}$ associated with problem (4.5) by

$$\widehat{\Phi}(u,v) = \frac{1}{p_1}\int_{\Omega} A_1(|\nabla u|^{p_1})dx + \frac{1}{p_2}\int_{\Omega} A_2(|\nabla v|^{p_2})dx - \int_{\Omega} \widehat{G}(x,u,v)dx,$$

where $\widehat{G}(x,s,t) = \int_0^s \widehat{G}_{\xi}(x,\xi,t)d\xi + \int_0^t \widehat{G}_{\xi}(x,s,\xi)d\xi$.

Note that, applying (4.3), (4.4), (F_1) and (F_2) we obtain

$$\widehat{G}_u(x,u,v) \leq h_1(x)\underline{u}^{-\gamma_1} + h_1(x)(1 + |u|^{r-1} + |\widehat{S}v|^{r-1}) \quad \text{a.e. in } \Omega, \forall u, v \geq 0 \tag{4.6}$$

Similarly,

$$\widehat{G}_v(x,u,v) \leq h_2(x)\underline{v}^{-\gamma_2} + h_2(x)(1 + |\widehat{T}u|^{r-1} + |v|^{r-1}) \quad \text{a.e. in } \Omega, \forall u, v \geq 0. \tag{4.7}$$

Again, using (A_1) its possible to prove that the functional $\widehat{\Phi} \in C^1(X, \mathbb{R})$ with the following Fréchet derivative

$$\begin{aligned} \widehat{\Phi}'(u,v)(\phi,\varphi) = & \int_{\Omega} \left[a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u \nabla \phi + a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v \nabla \varphi\right] dx \\ & - \int_{\Omega} \widehat{G}_u(x,u,v)\phi\, dx - \int_{\Omega} \widehat{G}_v(x,u,v)\varphi\, dx, \end{aligned}$$

for all $(u,v), (\phi,\varphi) \in X$. Furthermore, any critical point of $\widehat{\Phi}$ is a weak solution for auxiliary system (4.5).

In our next result we prove that the functional $\widehat{\Phi}$ satisfies the two geometries of the Mountain Pass Theorem [1].

Lemma 4.1. *Suppose that (H), (F_1)–(F_3) and (A_1)–(A_4) are satisfied. Then, for $\|h_i\|_\infty$ small, $\forall i = 1,2$, $\widehat{\Phi}$ satisfies*

($\widehat{\Phi}_1$) *There exist R, α, β with $R > \|(\underline{u}, \underline{v})\|$ and $\alpha < \beta$ such that*

$$\widehat{\Phi}(\underline{u}, \underline{v}) \leq \alpha < \beta \leq \inf_{\partial B_R(0)} \widehat{\Phi}.$$

($\widehat{\Phi}_2$) *There exists $e \in X \setminus \overline{B_R(0)}$ such that $\widehat{\Phi}(e) < \beta$.*

Proof. Since $(\underline{u}, \underline{v})$ is a subsolution of (1.1) it follows from Lemma 3.1(i), (F_1), (4.3) and (4.4) that

$$\widehat{G}(x, \underline{u}, \underline{v}) \geq \left[h_1(x)\underline{u}^{-\gamma_1} - h_1(x)\right] \underline{u} + \left[h_2(x)\underline{v}^{-\gamma_2} - h_2(x)\right] \underline{v} \quad \text{a.e. in } \Omega$$

and hence, in view of Lemma 3.1(i) again we obtain $0 < \alpha \in \mathbb{R}$ such that

$$\widehat{\Phi}(\underline{u}, \underline{v}) \leq \frac{1}{p_1} \int_\Omega A_1(|\nabla \underline{u}|^{p_1})\, dx + \frac{1}{p_2} \int_\Omega A_2(|\nabla \underline{v}|^{p_2})\, dx \equiv \alpha. \tag{4.8}$$

Now, without loss of generality, we can consider $q_1 \leq q_2$. So, using (H), (A_1), (4.6), (4.7), Lemma 3.1 and Sobolev embedding there exist positive constants such that

$$\begin{aligned}
\widehat{\Phi}(u, v) \geq{}& \frac{K}{2^{q_1}} \|(u, v)\|^{q_1} - c_1 \|h_1 \underline{u}^{-\gamma_1}\|_\infty \|(u, v)\| - c_2 \|h_1\|_\infty \; \|(u, v)\| \\
& - c_3 \|h_1\|_\infty \|(u, v)\|^r - \|h_1\|_\infty \int_\Omega |\widehat{S}v|^{r-1} |u| dx - c_4 \|h_2 \underline{v}^{-\gamma_2}\|_\infty \|(u, v)\| \\
& - c_5 \|h_2\|_\infty \|(u, v)\| - c_6 \|h_2\|_\infty \|(u, v)\|^r - \|h_2\|_\infty \int_\Omega |\widehat{T}u|^{r-1} |v| dx,
\end{aligned} \tag{4.9}$$

where $K = \min\left\{\frac{\tilde{k_2}}{q_1}, \frac{\tilde{k_2}}{q_2}\right\}$. Note that, invoking Young's inequality and Sobolev embedding we get

$$\begin{aligned}
\|h_1\|_\infty \int_\Omega |\widehat{S}v|^{r-1} |u| dx &= \|h_1\|_\infty \int_{v \leq \underline{v}} |\underline{v}|^{r-1} |u| dx + \|h_1\|_\infty \int_{v > \underline{v}} |v|^{r-1} |u| dx \\
&\leq c_7 \|h_1\|_\infty \|\underline{v}\|_\infty^{r-1} \|(u, v)\| + c_8 \|h_1\|_\infty \|(u, v)\|^r + c_9 \|h_1\|_\infty \|(u, v)\|^r
\end{aligned}$$

and

$$\begin{aligned}
\|h_2\|_\infty \int_\Omega |\widehat{T}u|^{r-1} |v| dx &= \|h_2\|_\infty \int_{u \leq \underline{u}} |\underline{u}|^{r-1} |v| dx + \|h_2\|_\infty \int_{u > \underline{u}} |u|^{r-1} |v| dx \\
&\leq c_{10} \|h_2\|_\infty \|\underline{u}\|_\infty^{r-1} \|(u, v)\| + c_{11} \|h_2\|_\infty \|(u, v)\|^r + c_{12} \|h_2\|_\infty \|(u, v)\|^r.
\end{aligned}$$

Thus, taking $\|(u, v)\| = R$ with $R > \max\{1, \|(\underline{u}, \underline{v})\|\}$ and $\|h_i\|_\infty$ sufficiently small, for $i = 1,2$, there exists $0 < \beta \in \mathbb{R}$, with $\beta > \alpha$, such that $\widehat{\Phi}(u, v) \geq \beta$, for all $(u, v) \in \partial B_R(0)$. Hence, the choices of α, β, R and $\|h_i\|_\infty$ combined with inequalities (4.8) and (4.9) result in

$$\widehat{\Phi}(\underline{u}, \underline{v}) \leq \alpha < \beta \leq \inf_{(u,v) \in \partial B_R(0)} \widehat{\Phi},$$

which shows the condition $\widehat{\Phi}_1$.

Now, by definition (4.3) we have

$$\widehat{G}_{s\underline{u}}(x, s\underline{u}, 0) \geq F_{s\underline{u}}(x, s\underline{u}, 0), \quad \text{for all } s \geq 1, \text{ a.e. in } \Omega$$

and invoking (A_1) we obtain

$$\widehat{\Phi}(s\underline{u}, 0) \leq \frac{k_3}{p_1} s^{p_1} \|\underline{u}\|_{1,p_1}^{p_1} + \frac{k_4}{q_1} s^{q_1} \|\underline{u}\|_{1,q_1}^{q_1} - \int_\Omega F(x, s\underline{u}, 0) dx.$$

The hypothesis (F_3) provides $d_1, d_2 > 0$ such that $F(x,s,0) \geq d_1 s^{\frac{1}{\theta_s}} - d_2$, for all $s \geq \max\{1, s_0\}$, where s_0 is the constant that appeared in (F_3). Then, by Sobolev embedding there exist positive constants $d_3, d_4 > 0$ such that

$$\widehat{\Phi}(s\underline{u}, 0) \leq \frac{k_3}{p_1} s^{p_1} \|\underline{u}\|_{1,p_1}^{p_1} + \frac{k_4}{q_1} s^{q_1} \|\underline{u}\|_{1,q_1}^{q_1} - d_3 s^{\frac{1}{\theta_s}} \|\underline{u}\|^{\frac{1}{\theta_s}} + d_4.$$

Since $2 \leq p_1 \leq q_1 < \frac{1}{\theta_s} < q_1^*$, we conclude that $\widehat{\Phi}(s\underline{u}, 0) \to -\infty$ as $s \to +\infty$. So, we may find $s^* > 0$ with $e = s^*(\underline{u}, 0) \in X$ such that $\|e\| > R$ and $\widehat{\Phi}(e) < \beta$, which satisfies the condition $\widehat{\Phi}_2$. □

Lemma 4.2. *The functional $\widehat{\Phi}$ satisfies the Palais–Smale condition for all $c \in \mathbb{R}$.*

Proof. Consider $(u_n, v_n) \subset X$ a Palais–Smale sequence, i.e.,

$$\widehat{\Phi}(u_n, v_n) \to c \quad \text{and} \quad \widehat{\Phi}'(u_n, v_n) \to 0. \tag{4.10}$$

Thus, for all $n \in \mathbb{N}$ sufficiently large, there exists $C > 0$ such that

$$\widehat{\Phi}(u_n, v_n) - \left[\theta_{u_n} \hat{\Phi}'(u_n, v_n)(u_n, 0) + \theta_{v_n} \widehat{\Phi}'(u_n, v_n)(0, v_n)\right] \leq C(1 + \|(u_n, v_n)\|).$$

On the other hand, we use (A_1) and (A_4) to obtain

$$\begin{aligned}
&\widehat{\Phi}(u_n, v_n) - \left[\theta_{u_n} \hat{\Phi}'(u_n, v_n)(u_n, 0) + \theta_{v_n} \widehat{\Phi}'(u_n, v_n)(0, v_n)\right] \\
&\geq \left(\frac{1}{p_1 \mu_1} - \theta_{u_n}\right) \tilde{k}_2 \|u_n\|_{1,q_1}^{q_1} + \left(\frac{1}{p_2 \mu_2} - \theta_{v_n}\right) \tilde{k}_2 \|v_n\|_{1,q_2}^{q_2} \\
&\quad + \int_\Omega \left[\theta_{u_n} \widehat{G}_{u_n}(x, u_n, v_n) u_n + \theta_{v_n} \widehat{G}_{v_n}(x, u_n, v_n) v_n - \widehat{G}(x, u_n, v_n)\right].
\end{aligned}$$

Therefore, since $\theta_{u_n} < \frac{1}{\mu_1 p_1}$, $\theta_{v_n} < \frac{1}{\mu_2 p_2}$ and $q_1 \leq q_2$, without loss of generality, we have

$$\begin{aligned}
&C + (1 + \|(u_n, v_n)\|) \\
&\quad \geq \frac{\overline{K}}{2^{q_1}} \|(u_n, v_n)\|^{q_1} + \int_\Omega \left[\theta_{u_n} \widehat{G}_{u_n}(x, u_n, v_n) u_n + \theta_{v_n} \widehat{G}_{v_n}(x, u_n, v_n) v_n - \widehat{G}(x, u_n, v_n)\right],
\end{aligned} \tag{4.11}$$

where $\overline{K} = \min\left\{\tilde{k}_2\left(\frac{1}{p_1 \mu_1} - \theta_{u_n}\right), \tilde{k}_2\left(\frac{1}{p_2 \mu_2} - \theta_{v_n}\right)\right\}$.

Considering s_0 and t_0 given in (F_3), it follows from (4.3), (4.4) and (F_3) that there exists $\widehat{C} > 0$ such that

$$\int_\Omega \left[\theta_{u_n}\widehat{G}_{u_n}(x,u_n,v_n)u_n + \theta_{v_n}\widehat{G}_{v_n}(x,u_n,v_n)v_n - \widehat{G}(x,u_n,v_n)\right]$$

$$\geq \int_\Omega \left[\theta_{u_n}h_1(x)(\widehat{T}_{u_n})^{-\gamma_1}u_n + \theta_{v_n}h_2(x)(\widehat{S}_{v_n})^{-\gamma_2}v_n - \int_0^{u_n}h_1(x)(\widehat{T}_{u_n})^{-\gamma_1} - \int_0^{v_n}h_2(x)(\widehat{S}_{v_n})^{-\gamma_2}\right] - \widehat{C}$$

$$= \int_{\{u_n\leq \underline{u}\}\cup\{v_n\leq\underline{v}\}} \left[(\theta_{u_n}-1)\,h_1(x)\underline{u}^{1-\gamma_1} + (\theta_{v_n}-1)\,h_2(x)\underline{v}^{1-\gamma_2}\right]$$

$$+ \int_{\{u_n>\underline{u}\}\cup\{v_n>\underline{v}\}} \left[\left(\theta_{u_n} - \frac{1}{1-\gamma_1}\right)h_1(x)u_n^{1-\gamma_1} + \left(\theta_{v_n} - \frac{1}{1-\gamma_2}\right)h_2(x)u_n^{1-\gamma_2}\right] - \widehat{C}. \quad (4.12)$$

Now, using (A_4), Lemma 3.1(i), (4.11) and (4.12) we consider the following cases below:
Case 1: If $\gamma_1,\gamma_2 > 1$, then there exists $M > 0$ such that

$$M + C\|(u_n,v_n)\| \geq \frac{\overline{K}}{2^{q_1}}\|(u_n,v_n)\|^{q_1}.$$

Case 2: If $0 < \gamma_1,\gamma_2 < 1$, we apply Hölder's inequality in (4.12) to obtain

$$M{+}C\|(u_n,v_n)\|{+}\left(\frac{1}{1{-}\gamma_1} - \theta_{u_n}\right)\|h_1\|_{1,q_1}^{q_1+(\gamma_1-1)}\|u_n\|_{1,q_1}^{1-\gamma_1}$$
$$+\left(\frac{1}{1{-}\gamma_2} - \theta_{v_n}\right)\|h_2\|_{1,q_2}^{q_2+(\gamma_2-1)}\|v_n\|_{1,q_2}^{1-\gamma_2} \geq \frac{\overline{K}}{2^{q_1}}\|(u_n,v_n)\|^{q_1},$$

Case 3: If $\gamma_1 > 1$ and $0 < \gamma_2 < 1$, we get

$$M{+}C\|(u_n,v_n)\|{+}\left(\frac{1}{1{-}\gamma_2} - \theta_{v_n}\right)\|h_2\|_{1,q_2}^{q_2+(\gamma_2-1)}\|v_n\|_{1,q_2}^{1-\gamma_2} \geq \frac{\overline{K}}{2^{q_1}}\|(u_n,v_n)\|^{q_1}.$$

Case 4: If $\gamma_2 > 1$ and $0 < \gamma_1 < 1$, then

$$M{+}C\|(u_n,v_n)\|{+}\left(\frac{1}{1{-}\gamma_1} - \theta_{u_n}\right)\|h_1\|_{1,q_1}^{q_1+(\gamma_1-1)}\|u_n\|_{1,q_1}^{1-\gamma_1} \geq \frac{\overline{K}}{2^{q_1}}\|(u_n,v_n)\|^{q_1}.$$

Case 5: Making $\gamma_1,\gamma_2 = 1$ in (4.3) and (4.4) we have

$$M + C\|(u_n,v_n)\|{+}\|h_1\|_\infty\|u_n\|{+}\|h_2\|_\infty\|v_n\| \geq \frac{\overline{K}}{2^{q_1}}\|(u_n,v_n)\|^{q_1}.$$

So, analyzing all cases above, we conclude that (u_n,v_n) is a bounded sequence in X. Thus, up to subsequence, there exists $(u,v) \in X$ such that

$$\begin{cases} u_n \rightharpoonup u_2 \text{ in } W_0^{1,q_1}(\Omega), \\ u_n \to u_2 \text{ in } L^s(\Omega),\ 1 \leq s < q_1^*, \\ u_n(x) \to u_2(x) \text{ a.e. in } \Omega \end{cases} \quad (4.13)$$

and

$$\begin{cases} v_n \rightharpoonup v_2 \text{ in } W_0^{1,q_1}(\Omega), \\ v_n \to v_2 \text{ in } L^t(\Omega),\ 1 \leq t < q_2^*, \\ v_n(x) \to v_2(x) \text{ a.e. in } \Omega. \end{cases} \quad (4.14)$$

Using (A_2), Lemma 4.1(i), (4.10), (4.13) and (4.14) we can argue as in [5, Lemma 1] to obtain

$$C_{q_1}\|u_n - u\|_{1,q_1}^{q_1} + C_{q_2}\|v_n - v\|_{1,q_2}^{q_2} \leq \int_\Omega \left[\widehat{G}_{u_n}(x,u_n,v_n)(u_n - u) + \widehat{G}_{v_n}(x,u_n,v_n)(v_n - v)\right] dx. \quad (4.15)$$

Moreover, we invoke (4.6), (4.7), (4.13), (4.14) and Lebesgue's Dominated Convergence Theorem to get

$$\int_\Omega \left[\widehat{G}_{u_n}(x,u_n,v_n)(u_n - u) + \widehat{G}_{v_n}(x,u_n,v_n)(v_n - v)\right] dx \to 0, \quad \text{as } n \to +\infty. \quad (4.16)$$

Note that, without loss of generality, we can consider $q_1 \geq q_2$. It follows from (4.15) and (4.16) that $(u_n, v_n) \to (u, v)$ in X. □

Next, let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ be the subsolution and the supersolution, respectively, of system (1.1) given in Lemma 3.1 and (u_1, v_1) a weak solution of system (1.1) obtained in Theorem 1.6. Invoking Lemmas 4.1 and 4.4, it follows from Mountain Pass Theorem that there exists $(u_2, v_2) \in X$ such that

$$\beta < \widehat{\Phi}(u_2, v_2) = c,$$

where c is the minimax value of $\widehat{\Phi}$. Furthermore, since $G_u(x,u,v) = \widehat{G}_u(x,u,v)$ and $G_v(x,u,v) = \widehat{G}_v(x,u,v)$, for all $(u,v) \in [0,\overline{u}] \times [0,\overline{v}]$, then $\Phi(u,v) = \widehat{\Phi}(u,v)$, for all $(u,v) \in [0,\overline{u}] \times [0,\overline{v}]$. Thus, $\widehat{\Phi}(u_1,v_1) = \inf_M \Phi$, where $(u_1,v_1) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$ and M is given in the proof of Theorem 1.6. Thus, auxiliary system (4.5) has two positive weak solutions $(u_1,v_1), (u_2,v_2) \in X$ such that

$$\widehat{\Phi}(u_1,v_1) \leq \widehat{\Phi}(\underline{u},\underline{v}) \leq \alpha < \beta \leq \widehat{\Phi}(u_2,v_2) = c.$$

Finally, let's show that $\underline{u} \leq u_2$ and $\underline{v} \leq v_2$. Indeed, taking $((\underline{u} - u_2)^+, (\underline{v} - v_2)^+)$ as test function and defining $\{(u_2,v_2) < (\underline{u},\underline{v})\} := \{x \in \Omega; u_2(x) < \underline{u}(x) \text{ and } v_2(x) < \underline{v}(x)\}$, we have

$$\begin{aligned}
&\int_\Omega a_1(|\nabla u_2|^{p_1})|\nabla u_2|^{p_1-2}\nabla u_2 \nabla(\underline{u} - u_2)^+ dx + \int_\Omega a_2(|\nabla v_2|^{p_2})|\nabla v_2|^{p_2-2}\nabla v_2 \nabla(\underline{v} - v_2)^+ dx \\
&\quad = \int_{\{u_2<\underline{u}\}} \left[h_1(x)\underline{u}^{-\gamma_1} + F_{u_2}(x,\underline{u},\widehat{S}v_2)\right] (\underline{u} - u_2)^+ dx \\
&\qquad + \int_{\{v_2<\underline{v}\}} \left[h_2(x)\underline{v}^{-\gamma_2} + F_{v_2}(x,\widehat{T}u_2,\underline{v})\right] (\underline{v} - v_2)^+ dx.
\end{aligned}$$

Since $(\underline{u}, \underline{v})$ is subsolution for system (1.1), then

$$\int_\Omega a_1(|\nabla \underline{u}|^{p_1})|\nabla \underline{u}|^{p_1-2}\nabla \underline{u} \nabla(\underline{u} - u_2)^+ dx - \int_\Omega a_1(|\nabla u_2|^{p_1})|\nabla u_2|^{p_1-2}\nabla u_2 \nabla(\underline{u} - u_2)^+ dx \leq 0$$

and

$$\int_\Omega a_2(|\nabla \underline{v}|^{p_2})|\nabla \underline{v}|^{p_2-2}\nabla \underline{v} \nabla(\underline{v} - v_2)^+ dx - \int_\Omega a_2(|\nabla v_2|^{p_2})|\nabla v_2|^{p_2-2}\nabla v_2 \nabla(\underline{v} - u_2)^+ dx \leq 0,$$

which implies that $(\underline{u} - u_2)^+ = 0$ and $(\underline{v} - v_2)^+ = 0$. Therefore, we conclude that $0 < \underline{u} \leq u_2$ a.e. in Ω and $0 < \underline{v} \leq v_2$ a.e. in Ω, as claimed. It follows from (4.3) and (4.4) that

$$\widehat{G}_{u_2}(x, u_2, v_2) = h(x)u_2^{-\gamma_1} + F_{u_2}(x, u_2, v_2) \quad \text{in } \Omega$$

and

$$\widehat{G}_{v_2}(x, u_2, v_2) = h(x)v_2^{-\gamma_2} + F_{v_2}(x, u_2, v_2) \quad \text{in } \Omega.$$

Then, (u_1, v_1) and (u_2, v_2) are two positive weak solutions for system (1.1).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 27, 1–19; https://doi.org/10.14232/ejqtde.2021.1.27 www.math.u-szeged.hu/ejqtde/

Periodic solutions of second order Hamiltonian systems with nonlinearity of general linear growth

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Received 31 October 2020, appeared 2 April 2021
Communicated by Gabriele Bonanno

Abstract. In this paper we consider a class of second order Hamiltonian system with the nonlinearity of linear growth. Compared with the existing results, we do not assume an asymptotic of the nonlinearity at infinity to exist. Moreover, we allow the system to be resonant at zero. Under some general conditions, we will establish the existence and multiplicity of nontrivial periodic solutions by using the Morse theory and two critical point theorems.

Keywords: second order Hamiltonian systems, periodic solutions, Morse theory, critical groups.

2020 Mathematics Subject Classification: 34C25, 37B30, 37J45.

1 Introduction

Consider the following second order Hamiltonian systems

$$-\ddot{x} = V_x(t,x), \tag{1.1}$$

where $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ with $V(t+T,x) = V(t,x)$ for some $T > 0$. During the past forty years, the existence and multiplicity of periodic solutions for second order Hamiltonian systems have been extensively studied by variational methods. There has been a lot of results under various suitable solvability conditions, such as the sublinear conditions (see [14, 18, 22, 23, 27, 28] and references therein), the superlinear conditions (see [3, 8, 9, 16, 17, 21, 24, 29] and references therein), and the asymptotically linear conditions (see [2, 6, 10, 15, 19, 20, 30] and references therein).

In this paper, we shall study the existence and multiplicity of nontrivial periodic solutions for (1.1) when the nonlinearity $V_x(t,x)$ has linear growth. Compared with the existing results, we do not make any assumptions at infinity on the asymptotic behaviors of the nonlinearity $V_x(t,x)$. Specifically, we do not require the system to be asymptotically linear at infinity. Instead, we assume that there exists a T-periodic symmetric matrix function $A_\infty(t)$ such that for some $K > 0$,

$$V_{xx}(t,x) \geq A_\infty(t) \quad (\text{or } V_{xx}(t,x) \leq A_\infty(t)), \qquad \forall t \in [0,T],\ |x| \geq K,$$

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where for two symmetric matrices A and B, $A \leq B$ means that $B - A$ is semi-positively definite. Under this general linear growth condition, we will construct a sequence of approximate systems and use the Morse theory and two critical point theorems to establish the existence and multiplicity of nontrivial periodic solutions for the system. The idea of our proof is closely related to the work of Liu, Su and Wang [13], where they dealt with the existence of nontrivial solutions of elliptic problems. Note that in [13] the authors assumed that the elliptic problem was nonresonant at zero. By contrast, here we allow system (1.1) to be resonant at zero. On the other hand, system (1.1) with periodic boundary condition is rather different from the elliptic problems with Dirichlet boundary condition. These lead us to need some new technique.

Now let us say some words about the idea of the proof. We first construct a sequence of approximate systems which are asymptotically linear and non-resonant at infinity. Then in a crucial step we establish the L^∞ bound to the solutions of the approximate systems whose Morse index is controlled by the Morse index at infinity. Finally, we use the Morse theory and two critical point theorems to obtain the nontrivial periodic solutions with the controlled Morse index for the approximate systems, therefore using the previous L^∞ estimate they are also the nontrivial periodic solutions of the original system.

We make the following assumptions:

(H_1) $V(t,x) \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ with $V(t,0) = 0$ and $V(t+T,x) = V(t,x)$;

(H_2) There exist $C_1 > 0$ and $C_2 > 0$ such that

$$|V_x(t,x)| \leq C_1(1+|x|), \qquad |V_{xx}(t,x)| \leq C_2, \qquad t \in [0,T],\ x \in \mathbb{R}^N;$$

(H_3) $V_x(t,x) = A_0(t)x + (G_0)_x(t,x)$, where $A_0(t)$ is a T-periodic continuous symmetric matrix function and $(G_0)_x(t,x) = o(|x|)$ as $|x| \to 0$;

($H_4^{\pm}$) There exists $\delta > 0$ such that

$$\pm G_0(t,x) > 0, \qquad \forall t \in [0,T],\ 0 < |x| < \delta;$$

($H_5^{\pm}$) There exists a T-periodic continuous symmetric matrix function $A_\infty(t)$ such that for some $K > 0$,

$$\pm V_{xx}(t,x) \geq \pm A_\infty(t), \qquad \forall t \in [0,T],\ |x| \geq K.$$

Let $E := H_T^1(\mathbb{R}, \mathbb{R}^N)$, the Hilbert space of T-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the inner product

$$\langle x, y \rangle = \int_0^T (\dot{x} \cdot \dot{y} + x \cdot y)dt, \qquad \forall x, y \in E,$$

and norm $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$. We define the functional I on E by

$$I(x) = \frac{1}{2}\int_0^T |\dot{x}(t)|^2 dt - \int_0^T V(t,x)dt. \tag{1.2}$$

By (H_1) and (H_2), $I \in C^2(E, \mathbb{R})$ and the critical points of I in E are T-periodic solutions of (1.1).

Clearly, the set $\sigma = \{(\frac{2k\pi}{T})^2 \mid k \in \mathbb{Z}^+\}$ is the set of the eigenvalues of

$$-\ddot{x} = \lambda x \tag{1.3}$$

with T-periodic boundary condition. Consider the eigenvalue problem of the following system

$$-\ddot{x} - A_\infty x = \lambda x \tag{1.4}$$

with T-periodic boundary condition. Without loss of generality, in $(H_5^\pm)$ by considering $A_\infty(t) \mp \epsilon I_N$ instead of $A_\infty(t)$ for ϵ small if necessary we may assume that 0 is not the eigenvalue of (1.4). Let $\lambda_1 < \lambda_2 < \cdots < \lambda_l < 0 < \lambda_{l+1} < \lambda_{l+2} < \cdots$ be distinct eigenvalues of (1.4). Clearly, $\lambda_i \to \infty$ as $i \to \infty$. Let $E(\lambda_i)$ be the eigenspace of (1.4) corresponding to λ_i, $i \in \mathbb{Z}^+$.

We define the linear operator $\tilde{L}$ on E by

$$\langle \tilde{L}x, y\rangle := \int_0^T \dot{x} \cdot \dot{y} dt, \qquad \forall x, y \in E.$$

Then $\tilde{L}$ is a bounded self-adjoint operator. Define the linear operators B_0 and B_∞ on E by

$$\langle B_0 x, y\rangle := \int_0^T A_0(t)x \cdot y dt, \qquad \forall x, y \in E$$

and

$$\langle B_\infty x, y\rangle := \int_0^T A_\infty(t)x \cdot y dt, \qquad \forall x, y \in E.$$

Then B_0 and B_∞ are bounded self-adjoint compact operators on E. Let $L_0 := \tilde{L} - B_0$ and $L_\infty := \tilde{L} - B_\infty$. Since 0 is not an eigenvalue of (1.4), we have that L_∞ is a non-degenerate operator on E. Denote by E_0^+, E_0^-, E_∞^+ and E_∞^- the positive and negative spectral subspaces of L_0 and L_∞ respectively, and let $E_0^0 = \ker L_0$. Then there exists a constant $c_0 > 0$ such that for any $x \in E_0^+$ and $y \in E_0^-$,

$$\langle L_0 x, x\rangle \geq c_0\|x\|^2, \qquad \langle L_0 y, y\rangle \leq -c_0\|y\|^2. \tag{1.5}$$

Clearly,

$$E_\infty^- = \bigoplus_{i=1}^{l} E(\lambda_i), \qquad E_\infty^+ = \bigoplus_{i=l+1}^{\infty} E(\lambda_i),$$
$$E = E_0^+ \bigoplus E_0^0 \bigoplus E_0^- = E_\infty^+ \bigoplus E_\infty^-.$$

Set

$$i_0^0 = \dim E_0^0, \qquad i_0^- = \dim E_0^-, \qquad i_\infty^- = \dim E_\infty^-.$$

By (H_3), we see that $x = 0$ is a periodic solutions of (1.1) which is called trivial periodic solution. Our aim is to find nontrivial periodic solutions of (1.1). Now we give our main results as follows.

Theorem 1.1. *Assume that (H_1), (H_2), (H_3) hold. Then (1.1) has at least one nontrivial periodic solution in each of the following cases:*

(1) (H_4^+), (H_5^+) and $i_0^- + i_0^0 < i_\infty^- - 1$;

(2) (H_4^-), (H_5^+) and $i_0^- < i_\infty^- - 1$;

(3) (H_4^+), (H_5^-) and $i_0^- + i_0^0 > i_\infty^- + 1$;

(4) (H_4^-), (H_5^-) and $i_0^- > i_\infty^- + 1$.

Theorem 1.2. *Assume that* (H_1), (H_2), (H_3) *hold, and* $V(t,-x) = V(t,x)$ *for any* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

(1) *If* (H_4^+), (H_5^+) *hold and* $i_0^- + i_0^0 < i_\infty^- - 1$, *then* (1.1) *has at least* $i_\infty^- - i_0^- - i_0^0 - 1$ *pairs of nontrivial periodic solutions;*

(2) *If* (H_4^-), (H_5^+) *hold and* $i_0^- < i_\infty^- - 1$, *then* (1.1) *has at least* $i_\infty^- - i_0^- - 1$ *pairs of nontrivial periodic solutions;*

(3) *If* (H_4^+), (H_5^-) *hold and* $i_0^- + i_0^0 > i_\infty^- + 1$, *then* (1.1) *has at least* $i_0^- + i_0^0 - i_\infty^- - 1$ *pairs of nontrivial periodic solutions;*

(4) *If* (H_4^-), (H_5^-) *hold and* $i_0^- > i_\infty^- + 1$, *then* (1.1) *has at least* $i_0^- - i_\infty^- - 1$ *pairs of nontrivial periodic solutions.*

Remark 1.3. In what follows, we assume that $x = 0$ is an isolated critical point of I in E. In fact, if $x = 0$ is not an isolated critical point of I, then I has infinitely many critical points and therefore (1.1) has infinitely many periodic solutions. Therefore Theorem 1.1 and 1.2 hold naturally.

The paper is organized as follows. In Section 2, we construct a sequence of approximate systems and establish the L^∞ bound to the solutions of these approximate systems with appropriate Morse indexes. In Section 3, we will give the proof of Theorem 1.1 by using Morse theory and previous estimate. In Section 4, we will prove Theorem 1.2 by using two critical point theorems for even functional and previous estimate.

2 Preliminaries

In this section we give some important preliminary lemmas. Let H be a real Hilbert space and $J \in C^2(H,\mathbb{R})$. Denote $K(J) = \{u \in H \mid J'(u) = 0\}$. For $u \in K(J)$, we denote the Morse index of u by $m^-(J''(u))$ which is the dimension of the negative spectral subspace of $J''(u)$. The augmented Morse index of u is defined by

$$m^*(J''(u)) = m^-(J''(u)) + \dim\ker(J''(u)),$$

where $\ker(J''(u))$ is the kernel of $J''(u)$.

To construct a sequence of approximate systems of (1.1), we first construct a sequence of approximate functions $V_k(t,x)$. The following result is from [13].

Lemma 2.1. *Assume that* (H_1), (H_2) *and* (H_5^+) *(resp.* (H_5^-)*) hold. Then there exists a sequence functions* $V_k(t,x) \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ *satisfying the following properties:*

(a) $V_k(t+T,x) = V_k(t,x)$ *and there exists an increasing sequence of real numbers* $R_k \to \infty$ $(k \to \infty)$ *such that*
$$V_k(t,x) = V(t,x), \qquad \forall |x| \le R_k, t \in [0,T];$$

(b) *there exist* $C_1' > 0$ *and* $C_2' > 0$ *independent of* k *such that*
$$|(V_k)_x(t,x)| \le C_1'(1+|x|), \qquad |(V_k)_{xx}(t,x)| \le C_2';$$

(c) *for each* $k \in \mathbb{Z}^+$, $(V_k)_{xx}(t,x) \ge A_\infty(t)$ *(resp.* $(V_k)_{xx}(t,x) \le A_\infty(t)$*) for all* $t \in [0,T], |x| \ge K$;

(d) *there is* $\gamma > 0$ *independent of* k *such that* $\left(\frac{2p\pi}{T}\right)^2 < \gamma < \left(\frac{2(p+1)\pi}{T}\right)^2$ *for some* $p \in \mathbb{Z}^+$, *and for each* $k \in \mathbb{Z}^+$ *fixed,*

$$V_k(t,x) = \frac{\gamma}{2}|x|^2 + o(|x|^2), \qquad (V_k)_x(t,x) = \gamma x + o(|x|), \qquad (V_k)_{xx}(t,x) = \gamma I_N + o(1)$$

as $|x| \to \infty$;

(e) *if* $V(t,-x) = V(t,x)$, $\forall t \in [0,T], x \in \mathbb{R}^N$, *then for every* $k \in \mathbb{Z}^+$, $V_k(t,-x) = V_k(t,x)$, $\forall t \in [0,T], x \in \mathbb{R}^N$.

Let

$$I_k(x) := \frac{1}{2}\int_0^T |\dot{x}|^2 dt - \psi_k(x), \qquad x \in E, \tag{2.1}$$

where

$$\psi_k(x) := \int_0^T V_k(t,x)dt.$$

Clearly, $I_k(x) \in C^2(E,\mathbb{R})$ and the critical points of I_k correspond to the periodic solutions of the following system

$$-\ddot{x} = (V_k)_x(t,x). \tag{2.2}$$

By Lemma 2.1 (a) and Remark 1.3, $x = 0$ is also an isolated critical point of I_k for every $k \in \mathbb{Z}^+$. Define the linear operator $B_\gamma : E \to E$ by

$$\langle B_\gamma x, y\rangle := \int_0^T \gamma x \cdot y dt, \qquad \forall x, y \in E.$$

Let $L_\gamma := \widetilde{L} - B_\gamma$, then by Lemma 2.1, L_γ is a non-degenerate bounded linear self-adjoint operator on E. We have the decomposition $E = E_\gamma^- \oplus E_\gamma^+$, where E_γ^- and E_γ^+ are the negative and positive spectral subspaces of L_γ. Then there exists a constant $c_\gamma > 0$ such that for any $x \in E_\gamma^+$ and $y \in E_\gamma^-$,

$$\langle L_\gamma x, x\rangle \geq c_\gamma \|x^2\|, \qquad \langle L_\gamma y, y\rangle \leq -c_\gamma \|y^2\|. \tag{2.3}$$

Denote

$$j_\infty^- = \dim E_\gamma^-.$$

By Lemma 2.1 (c), (d), if (H_5^+) holds, then $\gamma I_N \geq A_\infty(t)$, which implies that

$$E_\infty^- \subset E_\gamma^- \quad \text{and} \quad j_\infty^- \geq i_\infty^-. \tag{2.4}$$

If (H_5^-) holds, then $\gamma I_N \leq A_\infty(t)$, which implies that

$$E_\gamma^- \subset E_\infty^- \quad \text{and} \quad j_\infty^- \leq i_\infty^-. \tag{2.5}$$

Let

$$G_k(t,x) = V_k(t,x) - \frac{\gamma}{2}|x|^2, \qquad G_{0k}(t,x) = V_k(t,x) - \frac{1}{2}A_0(t)x \cdot x$$

and

$$\varphi_k(x) = \int_0^T G_k(t,x)dt, \qquad \varphi_{0k}(x) = \int_0^T G_{0k}(t,x)dt.$$

By (H_3), Lemma 2.1 (a), (d), we see that $(G_k)_x(t,x) = o(|x|)$ as $|x| \to \infty$ and $(G_{0k})_x(t,x) = o(|x|)$ as $|x| \to 0$. Then we have

$$\varphi_k'(x) = o(\|x\|) \quad \text{as } \|x\| \to \infty \quad \text{and} \quad \varphi_{0k}'(x) = o(\|x\|) \quad \text{as } \|x\| \to 0. \tag{2.6}$$

And we can rewrite the functional I_k by

$$I_k(x) = \frac{1}{2}\langle L_\gamma x, x\rangle - \varphi_k(x) = \frac{1}{2}\langle L_0 x, x\rangle - \varphi_{0k}(x), \qquad x \in E. \tag{2.7}$$

Lemma 2.2. *Assume that* (H_1), (H_2), (H_3) *and* (H_5^+) *(resp.* (H_5^-)*) hold. For every* $k \in \mathbb{Z}^+$, *if* x_k *is a critical point of* I_k *with* $m^-(I_k''(x_k)) \le i_\infty^- - 1$ *(resp.* $m^*(I_k''(x_k)) \ge i_\infty^- + 1$*), then there exists a constant* $\beta > 0$ *independent of* k *such that* $\|x_k\|_{L^\infty} \le \beta$.

Proof. We use an indirect argument. Assume that $\|x_k\|_{L^\infty} \to \infty$ as $k \to \infty$. By the Sobolev inequality $\|x\|_{L^\infty([0,T])} \le C\|x\|$, we have that $\|x_k\| \to \infty$ as $k \to \infty$.

Let

$$\bar{x}_k = \frac{x_k}{\|x_k\|}.$$

Then $\bar{x}_k$ satisfies

$$-\ddot{\bar{x}}_k = \frac{(V_k)_x(t, x_k)}{\|x_k\|}. \tag{2.8}$$

Up to a subsequence, we have that for some $\bar{x} \in E$, $\bar{x}_k \rightharpoonup \bar{x}$ in E, $\bar{x}_k \to \bar{x}$ in $L^2([0,T])$. And it follows from Proposition 1.2 in [20] that $\bar{x}_k$ converges uniformly to $\bar{x}$ on $[0,T]$. By (H_2), (H_3) and Lemma 2.1, there exists $C_1' > 0$ such that $|(V_k)_x(t,x_k)| \le C_1'|x_k|$. Thus for every k,

$$\left|\frac{(V_k)_x(t, x_k)}{\|x_k\|}\right| \le C_1'|\bar{x}_k|. \tag{2.9}$$

Multiplying (2.8) by $\bar{x}_k$, one has

$$1 = \|\bar{x}_k\|^2 \le (C_1' + 1)\|\bar{x}_k\|^2_{L^2([0,T])}.$$

Letting $k \to \infty$, we get

$$\|\bar{x}\|^2_{L^2([0,T])} \ge \frac{1}{C_1' + 1} > 0. \tag{2.10}$$

Now we show that up to a subsequence $\dot{\bar{x}}_k$ converges uniformly to $\dot{\bar{x}}$ on $[0,T]$. For any $t \in [0,T]$, by (2.8), (2.9) and Hölder inequality, we have

$$\begin{aligned}
|\dot{\bar{x}}_k(0)| &= \left|\dot{\bar{x}}_k(t) + \int_0^t \frac{(V_k)_x(s,x_k)}{\|x_k\|} ds\right| \\
&\le \left|\dot{\bar{x}}_k(t)\right| + \left|\int_0^t C_1'|\bar{x}_k(s)|\, ds\right| \\
&\le |\dot{\bar{x}}_k(t)| + C_1'\sqrt{T}\|\bar{x}_k\|_{L^2} \\
&\le |\dot{\bar{x}}_k(t)| + C_1'\sqrt{T},
\end{aligned}$$

thus

$$\begin{aligned}
\int_0^T |\dot{\bar{x}}_k(0)| dt &\le \int_0^T |\dot{\bar{x}}_k(t)| dt + \int_0^T C_1'\sqrt{T} dt \\
&\le \sqrt{T}\|\dot{\bar{x}}_k\|_{L^2} + C_1'\sqrt{T}T \\
&\le \sqrt{T} + C_1'\sqrt{T}T.
\end{aligned}$$

Hence

$$|\dot{\bar{x}}_k(0)| \le C_2,$$

where $C_2 = \frac{\sqrt{T}}{T} + C_1'\sqrt{T}$. Then for any $t \in [0, T]$,

$$\begin{aligned} |\dot{\bar{x}}_k(t)| &= \left| \dot{\bar{x}}_k(0) + \int_0^t -\frac{(V_k)_x(s, x_k)}{\|x_k\|} ds \right| \\ &\le |\dot{\bar{x}}_k(0)| + \left| \int_0^t C_1' |\bar{x}_k(s)| ds \right| \\ &\le C_2 + C_1'\sqrt{T}\|\bar{x}_k\|_{L^2} \\ &\le C_2 + C_1'\sqrt{T}, \end{aligned}$$

which implies that for every $k \in \mathbb{Z}^+$,

$$\|\dot{\bar{x}}_k(t)\|_{C^0} \le C_2 + C_1'\sqrt{T}. \tag{2.11}$$

For any $\Delta t \in \mathbb{R}$, by (2.8) and (2.9) we have

$$\begin{aligned} |\dot{\bar{x}}_k(t + \Delta t) - \dot{\bar{x}}_k(t)| &= \left| \int_t^{t+\Delta t} \ddot{\bar{x}}_k(s) ds \right| \\ &= \left| \int_t^{t+\Delta t} -\frac{(V_k)_x(t, x_k)}{\|x_k\|} ds \right| \\ &\le \left| \int_t^{t+\Delta t} C_1' |\bar{x}_k| ds \right| \\ &\le C_1' |\Delta t|^{\frac{1}{2}} \|\bar{x}_k\|_{L^2} \le C_1' |\Delta t|^{\frac{1}{2}}. \end{aligned} \tag{2.12}$$

Thus by (2.11) and (2.12), we have

$$\|\dot{\bar{x}}_k(t)\|_{C^{\frac{1}{2}}} \le C.$$

Then by the Arzelà–Ascoli theorem, $\dot{\bar{x}}_k$ converges uniformly to $\dot{\bar{x}}$ on $[0, T]$.

We claim that $\bar{x}(t) \neq 0$ a.e. in $[0, T]$. In fact, conversely, if $\bar{x}(t) = 0$ in a positive measure subset of $[0, T]$, then there exists a point $t_0 \in [0, T]$ such that $\bar{x}(t_0) = 0$ and $\dot{\bar{x}}(t_0) = 0$. Recall that $\bar{x}_k$ and $\dot{\bar{x}}_k$ converge uniformly to $\bar{x}$ and $\dot{\bar{x}}$ respectively on $[0, T]$, we have

$$\bar{x}_k(t_0) \to 0 \quad \text{and} \quad \dot{\bar{x}}_k(t_0) \to 0 \tag{2.13}$$

as $k \to \infty$. Let $\bar{y}_k := \dot{\bar{x}}_k$, then $(\bar{x}_k, \bar{y}_k)$ satisfies the following system

$$\begin{cases} \dot{\bar{x}}_k = \bar{y}_k, \\ \dot{\bar{y}}_k = -\frac{(V_k)_x(t, x_k)}{\|x_k\|}. \end{cases} \tag{2.14}$$

For any $t \in [0, T]$,

$$\begin{aligned} |(\bar{x}_k(t), \bar{y}_k(t))| &= \left| (\bar{x}_k(t_0), \bar{y}_k(t_0)) + \int_{t_0}^t \left(\bar{y}_k(s), -\frac{(V_k)_x(s, x_k)}{\|x_k\|} \right) ds \right| \\ &\le |(\bar{x}_k(t_0), \bar{y}_k(t_0))| + \left| \int_{t_0}^t \left| \left(\bar{y}_k(s), -\frac{(V_k)_x(s, x_k)}{\|x_k\|} \right) \right| ds \right| \\ &\le |(\bar{x}_k(t_0), \bar{y}_k(t_0))| + \left| \int_{t_0}^t \sqrt{1 + C_1'^2}\, |(\bar{x}_k(s), \bar{y}_k(s))|\, ds \right|. \end{aligned}$$

Thus by Gronwall's inequality, we have

$$|(\bar{x}_k(t), \bar{y}_k(t))| \le |(\bar{x}_k(t_0), \bar{y}_k(t_0))| e^{|\int_{t_0}^t \sqrt{1+C_1'^2} ds|} \le C |(\bar{x}_k(t_0), \bar{y}_k(t_0))|, \tag{2.15}$$

where $C = e^{\sqrt{1+C_1'^2}T}$. Then letting $k \to \infty$ in (2.15), we get $\bar{x}(t) = 0$ and $\bar{y}(t) = 0$ for any $t \in [0, T]$, which is contrary to (2.10). Hence the claim is proved. Note that $\|x_k\| \to \infty$, then by this claim one has

$$|x_k| \to \infty \quad \text{a.e. in } [0, T] \tag{2.16}$$

as $k \to \infty$.

If (H_5^+) holds, then by (2.16), Lemma 2.1 (b), (c) and Fatou's Lemma, for any fixed $x \in E_\infty^- \setminus \{0\}$,

$$\begin{aligned}
\limsup_{k\to\infty} \langle I_k''(x_k)x, x\rangle &= \langle \tilde{L}x, x\rangle - \liminf_{k\to\infty} \int_0^T (V_k)_{xx}(t, x_k)x \cdot x dt \\
&\le \langle \tilde{L}x, x\rangle - \int_0^T \liminf_{k\to\infty} (V_k)_{xx}(t, x_k)x \cdot x dt \\
&\le \langle \tilde{L}x, x\rangle - \int_0^T A_\infty(t)x \cdot x dt \\
&= \langle L_\infty x, x\rangle < 0,
\end{aligned}$$

which implies that there exists $k(x) \in \mathbb{Z}^+$ such that $\langle I_k''(x_k)x, x\rangle < 0$ when $k \ge k(x)$. Note that E_∞^- is finite dimensional, there must exist $k_0 \in \mathbb{Z}^+$ independent of $x \in E_\infty^- \setminus \{0\}$ such that

$$\langle I_k''(x_k)x, x\rangle < 0$$

for all $x \in E_\infty^- \setminus \{0\}$ and $k \ge k_0$. This means that $m^-(I_k''(x_k)) \ge i_\infty^-$ for $k \ge k_0$, which leads to a contradiction.

If (H_5^-) holds, since E_∞^+ is infinite dimensional, the above argument cannot be used directly. To overcome this difficulty, we will split E_∞^+ into two parts. Let

$$M = \max_{t\in[0,T]} |A_\infty(t)|.$$

Since $\lambda_i \to \infty$ as $i \to \infty$, then there exists $i_0 \in \mathbb{Z}^+$ such that $\lambda_{i_0} \ge 2(M + C_2')$ where C_2' is the constant as in Lemma 2.1 (b). Let

$$E_1 = \bigoplus_{i=l+1}^{i_0-1} E(\lambda_i), \qquad E_2 = \bigoplus_{i=i_0}^{\infty} E(\lambda_i).$$

Then $E_\infty^+ = E_1 \oplus E_2$ and E_1 is finite dimensional. For any $y_1 \in E_2 \setminus \{0\}$, note that

$$\int_0^T (|\dot{y}_1|^2 - A_\infty y_1 \cdot y_1) dt \ge \lambda_{i_0} \int_0^T |y_1|^2 dt,$$

then

$$\begin{aligned}
\langle I_k''(x_k)y_1, y_1\rangle &= \int_0^T |\dot{y}_1|^2 dt - \int_0^T (V_k)_{xx}(t, x_k)y_1 \cdot y_1 dt \\
&\ge \lambda_{i_0} \int_0^T |y_1|^2 dt + \int_0^T A_\infty y_1 \cdot y_1 dt - \int_0^T (V_k)_{xx}(t, x_k)y_1 \cdot y_1 dt \\
&\ge \lambda_{i_0} \int_0^T |y_1|^2 dt - \int_0^T M|y_1|^2 dt - \int_0^T C_2'|y_1|^2 dt \\
&\ge \frac{\lambda_{i_0}}{2} \int_0^T |y_1|^2 dt > 0.
\end{aligned} \tag{2.17}$$

For any $y_2 \in E_1 \setminus \{0\}$, by (2.16), Lemma 2.1 (b), (c) and Fatou's Lemma,

$$\begin{aligned}\liminf_{k\to\infty}\langle I_k''(x_k)y_2, y_2\rangle &= \int_0^T |\dot{y}_2|^2 dt - \limsup_{k\to\infty}\int_0^T (V_k)_{xx}(t,x_k)y_2\cdot y_2 dt \\ &\geq \int_0^T |\dot{y}_2|^2 dt - \int_0^T \limsup_{k\to\infty}(V_k)_{xx}(t,x_k)y_2\cdot y_2 dt \\ &\geq \int_0^T |\dot{y}_2|^2 dt - \int_0^T A_\infty(t)y_2\cdot y_2 dt \\ &= \langle L_\infty y_2, y_2\rangle > 0,\end{aligned}$$

which implies that there exists $k(y_2) \in \mathbb{Z}^+$ such that $\langle I_k''(x_k)y_2, y_2\rangle > 0$ for $k \geq k(y_2)$. Note that E_1 is finite dimensional, there must exist $k_1 \in \mathbb{Z}^+$ independent of $y_2 \in E_1 \setminus \{0\}$ such that

$$\langle I_k''(x_k)y_2, y_2\rangle > 0 \tag{2.18}$$

for all $y_2 \in E_1 \setminus \{0\}$ and $k \geq k_1$. Hence by (2.17) and (2.18), for any $y \in E_\infty^+ \setminus \{0\}$ and every $k \geq k_1$,

$$\langle I_k''(x_k)y, y\rangle > 0.$$

This implies that $m^*(I_k''(x_k)) \leq i_\infty^-$ for $k \geq k_1$, which leads to a contradiction.

Therefore the lemma is proved. □

3 Proof of Theorem 1.1

In this section, we will use Morse theory to prove the existence of nontrivial periodic solution for (1.1). Let H be a real Hilbert space and $J \in C^2(H, \mathbb{R})$ be a functional satisfying the (PS) condition, i.e., any sequence $\{u_n\} \subset H$ for which $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence. Denote by $H_q(A, B)$ the q-th singular relative homology group of the topological pair (A, B) with coefficients in a field $\mathcal{F}$. Let u be an isolated critical point of J with $J(u) = c$. The groups

$$C_q(J, u) := H_q(J^c, J^c \setminus \{u\}), \qquad q \in \mathbb{Z}$$

are called the critical groups of J at u, where $J^c = \{u \in H \mid J(u) \leq c\}$. Denote $K = K(J) = \{u \in H \mid J'(u) = 0\}$. Suppose that $J(K)$ is bounded from below by $a \in \mathbb{R}$. The critical groups of J at infinity are defined by

$$C_q(J, \infty) := H_q(H, J^a), \qquad q \in \mathbb{Z}.$$

We say that J has a local linking structure at 0 with respect to the direct sum decomposition $H = H^- \oplus H^+$ if there exists $r > 0$ such that

$$J(u) > 0 \quad \text{for } u \in H^+ \text{ with } 0 < \|u\| \leq r, \qquad J(u) \leq 0 \quad \text{for } u \in H^- \text{ with } \|u\| \leq r.$$

The following results can be found in [1], [26] and [4].

Proposition 3.1 (See [1]). *Suppose J satisfies (PS) condition. If $K = \emptyset$, then $C_q(J, \infty) \cong 0, q \in \mathbb{Z}$. If $K = \{u_0\}$, then $C_q(J, \infty) \cong C_q(J, u_0), q \in \mathbb{Z}$.*

Proposition 3.2 (See [26]). *Let 0 be an isolated critical point of $J \in C^2(H,\mathbb{R})$ with Morse index μ_0 and nullity ν_0. Assume that J has a local linking structure at 0 with respect to the direct sum decomposition $H = H^- \oplus H^+$ and $k = \dim H^- < \infty$. If $k = \mu_0$ or $k = \mu_0 + \nu_0$, then*

$$C_q(J,u) = \delta_{q,k}\mathcal{F}, \qquad q \in \mathbb{Z}.$$

Let A be a nondegenerate bounded self-adjoint operator defined on H. According to its spectral decomposition, $H = H^+ \oplus H^-$, where H^+, H^- are invariant subspaces corresponding to the positive and negative spectrum of A respectively. Let

$$J(x) = \frac{1}{2}\langle Ax, x\rangle + g(x),$$

and the following assumptions are given:

(A_1) $A_\pm := A\,|_{H^\pm}$ has a bounded inverse on $H^\pm$;

(A_2) $\kappa := \dim H^- < \infty$;

(A_3) $g \in C^1(H,\mathbb{R}^1)$ has a compact derivative $g'(x)$ and $\|g'(x)\| = o(\|x\|)$ as $\|x\| \to \infty$.

Proposition 3.3 (See [4]). *Under the assumptions (A_1), (A_2) and (A_3), we have that J satisfies (PS) condition and $C_q(J,\infty) = \delta_{q,\kappa}\mathcal{F}$.*

Proposition 3.4 (See [4]). *Suppose that $J \in C^2(H,\mathbb{R})$ satisfies (PS) condition, and $K = \{u_1, \dots, u_k\}$, then*

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1+t)Q(t),$$

where $Q(t)$ is a formal series with nonnegative coefficients, $M_q = \sum_{i=0}^{k} \operatorname{rank} C_q(J,u_k)$ and $\beta_q = \operatorname{rank} C_q(J,\infty)$, $q = 0, 1, 2, \dots$

Now we compute the critical groups of I_k at zero and at infinity.

Lemma 3.5. *Assume that (H$_1$)–(H$_3$) hold. Then for every $k \in \mathbb{Z}^+$,*

(1) if (H_4^+) holds,

$$C_q(I_k, 0) = \delta_{q,i_0^- + i_0^0}\mathcal{F}, \qquad q \in \mathbb{Z}.$$

(2) if (H_4^-) holds,

$$C_q(I_k, 0) = \delta_{q,i_0^-}\mathcal{F}, \qquad q \in \mathbb{Z}.$$

Proof. (1) We first show that I_k has a local linking structure at 0 with respect to $E = E^- \oplus E^+$, where $E^- = E_0^- \oplus E_0^0$ and $E^+ = E_0^+$. For $x \in E_0^+$, by (1.5) and (2.6) we have

$$\begin{aligned} I_k(x) &= \frac{1}{2}\langle L_0 x, x\rangle - \varphi_{0k}(x) \\ &\geq \frac{c_0}{2}\|x\|^2 - o(\|x\|^2) \end{aligned} \tag{3.1}$$

as $\|x\| \to 0$. This means that there exists small $r > 0$ such that

$$I_k(x) > 0, \quad \text{for } x \in E_0^+ \text{ with } 0 < \|x\| \leq r. \tag{3.2}$$

For $x \in E_0^- \oplus E_0^0$, we write $x = x^- + x^0$ with $x^- \in E_0^-$ and $x^0 \in E_0^0$. Then

$$\begin{aligned} I_k(x) &= \frac{1}{2}\langle L_0 x^-, x^- \rangle - \int_0^T G_{0k}(t,x)dt \\ &\leq -\frac{c_0}{2}\|x^-\|^2 - \int_0^T G_{0k}(t,x)dt. \end{aligned} \tag{3.3}$$

By (H_4^+) and Lemma 2.1 (a),

$$\int_{|x|\leq\delta} G_{0k}(t,x)dt \geq 0. \tag{3.4}$$

If $|x| > \delta$, since E_0^0 is finite dimensional, we have

$$|x^-| \geq |x| - |x^0| \geq |x| - \|x^0\|_{L^\infty} \geq |x| - C\|x^0\| \geq |x| - C\|x\|,$$

thus let $0 < r < \frac{\delta}{3C}$, for $\|x\| \leq r$, we have

$$|x^-| \geq |x| - \frac{\delta}{3} \geq |x| - \frac{1}{3}|x| = \frac{2}{3}|x|. \tag{3.5}$$

By Lemma 2.1 (b), (d), there exists $C_\delta > 0$ such that for $|x| > \delta$,

$$|G_{0k}(t,x)| \leq C_\delta |x|^3. \tag{3.6}$$

Hence, by (3.3)–(3.6), for $x \in E_0^- \oplus E_0^0$ with $\|x\| \leq r$, we have

$$\begin{aligned} I_k(x) &\leq -\frac{c_0}{2}\|x^-\|^2 - \int_0^T G_{0k}(t,x)dt \\ &\leq -\frac{c_0}{2}\|x^-\|^2 - \int_{|x|\leq\delta} G_{0k}(t,x)dt - \int_{|x|>\delta} G_{0k}(t,x)dt \\ &\leq -\frac{c_0}{2}\|x^-\|^2 + \int_{|x|>\delta} C_\delta |x|^3 dt \\ &\leq -\frac{c_0}{2}\|x^-\|^2 + C_\delta \int_{|x|>\delta} \left(\frac{3}{2}\right)^3 |x^-|^3 dt \\ &\leq -\frac{c_0}{2}\|x^-\|^2 + C'_\delta \|x^-\|^3. \end{aligned} \tag{3.7}$$

This implies that there exists $r > 0$ small enough such that

$$I_k(x) < 0, \quad \text{for } x \in E_0^- \oplus E_0^0 \text{ with } \|x\| \leq r \text{ and } \|x^-\| > 0. \tag{3.8}$$

On the other hand, for $x^0 \in E_0^0$, we can choose $r > 0$ small enough such that

$$0 < \|x^0\|_{L^\infty} < \delta, \quad \text{when } 0 < \|x^0\| \leq r.$$

Then for $x^0 \in E_0^0$ with $0 < \|x^0\| \leq r$, since $x^0 \in C^2([0,T], \mathbb{R}^N)$, there must exist $0 < t_1 < t_2 < T$ such that

$$0 < |x^0(t)| < \delta, \qquad \forall t \in [t_1, t_2].$$

Then by (H_4^+) and Lemma 2.1 (a), for $x^0 \in E_0^0$ with $0 < \|x^0\| \leq r$,

$$I_k(x^0) = -\int_0^T G_{0k}(t,x^0)dt = -\int_0^T G_0(t,x^0)dt \leq -\int_{t_1}^{t_2} G_0(t,x^0)dt < 0. \tag{3.9}$$

Hence, by (3.8) and (3.9), there exists $r > 0$ such that

$$I_k(x) < 0, \quad \text{for } x \in E_0^- \oplus E_0^0 \text{ with } 0 < \|x\| \le r. \tag{3.10}$$

Therefore, it follows from (3.2) and (3.10) that I_k has a local linking structure at 0 with respect to $E = E^- \oplus E^+$, where $E^- = E_0^- \oplus E_0^0$ and $E^+ = E_0^+$. Then by Proposition 3.2, we have

$$C_q(I_k, 0) = \delta_{q, i_0^- + i_0^0}\mathcal{F}, \qquad q \in \mathbb{Z}.$$

(2) By a similar argument as (1), we can prove that I_k has a local linking structure at 0 with respect to $E = E^- \oplus E^+$, where $E^- = E_0^-$ and $E^+ = E_0^+ \oplus E_0^0$. Then by Proposition 3.2, we have

$$C_q(I_k, 0) = \delta_{q, i_0^-}\mathcal{F}, \qquad q \in \mathbb{Z}. \qquad \square$$

Lemma 3.6. *Assume that (H_1)-(H_3), (H_5^+)(or (H_5^-)) hold. Then for every $k \in \mathbb{Z}^+$, I_k satisfies (PS) condition and the critical groups of I_k at infinity are*

$$C_q(I_k, \infty) = \delta_{q, j_\infty^-}\mathcal{F}, \qquad q \in \mathbb{Z}.$$

Proof. Note that

$$I_k(x) = \frac{1}{2}\langle L_\gamma x, x\rangle - \varphi_k(x)$$

Since L_γ is a nondegenerate operator on E, then $L_\gamma|_{E_\gamma^\pm}$ has a bounded inverse on $E_\gamma^\pm$. Recall that $\dim E_\gamma^- = j_\infty^- < \infty$. Thus the assumptions (A_1) and (A_2) in Proposition 3.3 are satisfied. On the other hand, note that $\varphi_k(x) \in C^2(E, \mathbb{R})$ has compact derivative $\varphi_k'(x)$ and $\varphi_k'(x) = o(\|x\|)$ as $\|x\| \to \infty$, then the assumption (A_3) in Proposition 3.3 is also satisfied. Hence, by Proposition 3.3, we have

$$C_q(I_k, \infty) = \delta_{q, j_\infty^-}\mathcal{F}, \qquad q \in \mathbb{Z}. \qquad \square$$

Remark 3.7. Since $I_k'(x) = L_\gamma x + \varphi_k'(x) = L_\gamma x + o(\|x\|)$ as $\|x\| \to \infty$ and L_γ is invertible, it is easy to see that the critical point set $K(I_k)$ is bounded for every $k \in \mathbb{Z}^+$. Then since I_k satisfies (PS) condition by Lemma 3.6, we conclude that $K(I_k)$ is a compact set for every $k \in \mathbb{Z}^+$.

Proof of Theorem 1.1. We only prove the result for the case (1), the proofs for the cases (2), (3) and (4) are similar.

For every $k \in \mathbb{Z}^+$, since $x = 0$ is an isolated critical point of I_k, there exists $\sigma > 0$ such that $I_k(x)$ has no nontrivial critical points in $B_\sigma(0) := \{x \mid \|x\| \le \sigma\}$. Since $i_0^- + i_0^0 < i_\infty^- - 1$, then by (2.4), Lemma 3.5 (1) and Lemma 3.6 we have

$$C_q(I_k, \infty) \ne C_q(I_k, 0)$$

for some $q \in \mathbb{Z}$. So by Proposition 3.1 and Remark 3.7, the set $K(I_k) \setminus \{0\}$ is not empty and compact. Denote $\mathcal{K}_k = K(I_k) \setminus \{0\}$.

Now we show that for every $k \in \mathbb{Z}^+$ there exists a nontrivial critical point $x_k \in \mathcal{K}_k$ such that

$$m^-(I_k''(x_k)) \le i_\infty^- - 1. \tag{3.11}$$

We use an indirect argument. Suppose that for any $x_k \in \mathcal{K}_k$,

$$m^-(I_k''(x_k)) > i_\infty^- - 1. \tag{3.12}$$

For $A \subset E$ and $a > 0$, set

$$N_a(A) := \{x \in E \mid \mathrm{dist}(x, A) < a\}.$$

Using the Marino–Prodi perturbation technique from [25], for any $\epsilon > 0$ and $0 < \tau < \min\{\frac{\sigma}{3}, 1\}$, we can obtain a C^2 functional J_k such that:

(i) $\|I_k - J_k\|_{C^2} < \epsilon$;

(ii) $I_k(x) = J_k(x)$, $x \in E \setminus N_{2\tau}(\mathcal{K}_k)$;

(iii) $I_k''(x) = J_k''(x)$ for any $x \in N_\tau(K(I_k))$, $K(J_k) \setminus \{0\} \subset N_\tau(\mathcal{K}_k)$, and the nontrivial critical points of J_k are all non-degenerate.

By (iii), $J_k''(0) = I_k''(0)$, thus by Proposition 3.2 and Lemma 3.5, we have

$$C_q(J_k, 0) = C_q(I_k, 0) = \delta_{q, i_0^- + i_0^0}\mathcal{F}. \tag{3.13}$$

By (ii), $I_k(x) = J_k(x)$ for $x \in E \setminus N_{2\tau}(\mathcal{K}_k)$, then by Lemma 3.6, J_k also satisfies (PS) condition and

$$C_q(J_k, \infty) = C_q(I_k, \infty) = \delta_{q, j_\infty^-}\mathcal{F}. \tag{3.14}$$

Since $K(J_k) \subset N_\tau(K(I_k))$ and $K(I_k)$ is compact, $K(J_k)$ is also a compact set. Moreover, note that the notrivial critical points of J_k are all non-degenerate, we have that $K(J_k)$ is a finite set. Suppose that

$$K(J_k) \setminus \{0\} = \{x_{k1}, x_{k2}, x_{k3}, \ldots, x_{kn}\}.$$

By (iii) and (3.12), we can choose τ small enough such that for all $1 \le i \le n$,

$$m^-(J_k''(x_{ki})) > i_\infty^- - 1. \tag{3.15}$$

By (3.13), (3.14), and Proposition 3.4 we have

$$t^{i_0^- + i_0^0} + \sum_{i=1}^{n} t^{m^-(J_k''(x_{ki}))} = t^{j_\infty^-} + (1+t)Q(t). \tag{3.16}$$

Note that $i_0^- + i_0^0 < i_\infty^- - 1$ and $i_\infty^- \le j_\infty^-$, it follows from (3.16) that $(1+t)Q(t)$ has a nonzero term with exponent $i_0^- + i_0^0$. Then this means that the left hand side of (3.16) has a nonzero term with exponent $i_0^- + i_0^0 - 1$ or $i_0^- + i_0^0 + 1$. Thus there exists a $1 \le i \le n$ such that

$$m^-(J_k''(x_{ki})) = i_0^- + i_0^0 - 1 \quad \text{or} \quad m^-(J_k''(x_{ki})) = i_0^- + i_0^0 + 1.$$

Since $i_0^- + i_0^0 < i_\infty^- - 1$, we have that $m^-(J_k''(x_{ki})) \le i_\infty^- - 1$ for some $1 \le i \le n$. This is contrary to (3.15), thus (3.11) is proved.

By Lemma 2.2 and (3.11), for every $k \in \mathbb{Z}^+$ the functional I_k has a nontrivial critical point x_k such that $\|x_k\|_{L^\infty} \le \beta$. By Lemma 2.1, for k large enough such that $R_k > \beta$, x_k is also a nontrivial critical point of I, and thus x_k is a nontrivial periodic solution of (1.1). □

4 Proof of Theorem 1.2

We introduce two critical point theorems which will be used in proving Theorem 1.2. Let H be a Hilbert space. Assume that $J \in C^2(H, \mathbb{R})$ is an even functional, satisfies the (PS) condition, $J(0) = 0$ and $K(J)$ is a compact set. Let $B_a = \{y \in H \mid \|y\| \le a\}$ and $S_a = \partial B_a = \{y \in H \mid \|y\| = a\}$. The following two critical point theorems follow from Ghoussoub [7] and Chang [4] (see also [13]).

Proposition 4.1 (See [7]). *Assume $H = Y \bigoplus Z$, and let X be a subspace of H, satisfying* $\dim X = j > k = \dim Y$. *If there exist $R > r > 0$ and $\alpha > 0$ such that*

$$\inf J(S_r \cap Z) \geq \alpha, \qquad \sup J(S_R \cap X) \leq 0,$$

then J has $j-k$ pairs of nontrivial critical points $\{\pm u_1, \pm u_2, \dots, \pm u_{j-k}\}$ so that $m^-(J''(u_i)) \leq k+i$ for $i = 1, 2, \dots, j-k$.

Proposition 4.2 (See [4]). *Assume $H = Y \bigoplus Z$, and let X be a subspace of H, satisfying* $\dim X = j > k = \dim Y$. *If there exist $r > 0$ and $\alpha > 0$ such that*

$$\inf J(Z) > -\infty, \qquad \sup J(S_r \cap X) \leq -\alpha,$$

then J has at least $j-k$ pairs of nontrivial critical points $\{\pm u_1, \pm u_2, \dots, \pm u_{j-k}\}$ so that $m^*(J''(u_i)) \geq k+i-1$ for $i = 1, 2, \dots, j-k$.

For every $k \in \mathbb{Z}^+$, by Lemma 2.1 (e), we see that $I_k(x)$ is an even functional on E. From Lemma 3.6 and Remark 3.7, I_k satisfies (PS) condition and $K(I_k)$ is compact. Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. (1) We will use Proposition 4.1 to prove this case. Let $Y = E_0^- \oplus E_0^0$, $Z = E_0^+$ and $X = E_\infty^-$. Then $E = Y \oplus Z$ and $\dim X = i_\infty^- > i_0^- + i_0^0 = \dim Y$.

For $x \in E_0^+$, by (1.5) and (2.6) we have

$$I_k(x) = \frac{1}{2}\langle L_0 x, x\rangle - \varphi_{0k}(x) \geq \frac{c_0}{2}\|x\|^2 + o(\|x\|^2) \tag{4.1}$$

as $\|x\| \to 0$. Then there exists $\alpha > 0$ and sufficiently small $r > 0$ such that $I_k(x) \geq \alpha$ for any $x \in S_r \cap E_0^+$, that is

$$\inf I_k(S_r \cap E_0^+) \geq \alpha. \tag{4.2}$$

On the other hand, recall that $E_\infty^- \subset E_\gamma^-$ in this case, then by (2.3) for $x \in E_\infty^-$ we have

$$I_k(x) = \frac{1}{2}\langle L_\gamma x, x\rangle - \varphi_k(x) \leq -\frac{c_\gamma}{2}\|x\|^2 + o(\|x\|^2) \tag{4.3}$$

as $\|x\| \to \infty$. Thus there exists $R > r$ such that $I_k(x) \leq 0$ for any $x \in S_R \cap E_\infty^-$, that is

$$\sup I_k(S_R \cap E_\infty^-) \leq 0. \tag{4.4}$$

For every $k \in \mathbb{Z}^+$, by (4.2), (4.4) and using Proposition 4.1, we have that $I_k(x)$ has $i_\infty^- - i_0^- - i_0^0$ pairs of nontrivial critical points $\{\pm x_k^1, \pm x_k^2, \dots, \pm x_k^{i_\infty^- - i_0^- - i_0^0}\}$ with $m^-(I_k''(x_k^i)) \leq i_0^- + i_0^0 + i$ for $i = 1, 2, \dots, i_\infty^- - i_0^- - i_0^0$. By Lemma 2.2, $\|x_k^i\|_{L^\infty} \leq \beta$ for $i = 1, 2, \dots, i_\infty^- - i_0^- - i_0^0 - 1$. Then for k large enough such that $R_k > \beta$, $\{\pm x_k^1, \pm x_k^2, \dots, \pm x_k^{i_\infty^- - i_0^- - i_0^0 - 1}\}$ are also nontrivial critical points of I, and therefore are nontrivial periodic solutions of (1.1).

(2) We will also use Proposition 4.1 to prove this case. Let $Y = E_0^-$, $Z = E_0^+ \oplus E_0^0$ and $X = E_\infty^-$. Then $E = Y \oplus Z$ and $\dim X = i_\infty^- > i_0^- = \dim Y$.

For $x \in E_0^+ \oplus E_0^0$, we write $x = x^+ + x^0$ where $x^+ \in E_0^+$ and $x^0 \in E_0^0$. For $x \in (E_0^+ \cap S_r) \oplus (E_0^0 \cap B_r)$, by (1.5) we have

$$\begin{aligned} I_k(x) &= \frac{1}{2}\langle L_0 x^+, x^+\rangle - \varphi_{0k}(x^+ + x^0) \\ &\geq \frac{c_0}{2}\|x^+\|^2 - o(\|x^+ + x^0\|^2) \\ &\geq \frac{c_0}{4} r^2 \end{aligned} \tag{4.5}$$

provided r is small enough.

Now we consider I_k on $(E_0^+ \cap B_r) \oplus (E_0^0 \cap S_r)$. For $x \in (E_0^+ \cap B_r) \oplus (E_0^0 \cap S_r)$, by (1.5) we have that

$$\begin{aligned} I_k(x) &= \frac{1}{2}\langle L_0 x^+, x^+\rangle - \varphi_{0k}(x^+ + x^0) \\ &\geq -\varphi_{0k}(x^+ + x^0) \\ &\geq -\frac{1}{4}r^2 \end{aligned} \tag{4.6}$$

provided r is small enough. Inspired by [12], we define a function $g : E_0^0 \cap S_r \to \mathbb{R}$ by

$$g(x^0) = \inf\{I_k(x^+ + x^0) \mid x^+ \in E_0^+ \cap B_r\}.$$

Then by (4.6), g is well defined and continuous. For any fixed $x^0 \in E_0^0 \cap S_r$, by a standard minimization method, we see that $g(x^0)$ is attained at some $\bar{x}^+ \in E_0^+ \cap B_r$, i.e.,

$$g(x^0) = I_k(\bar{x}^+ + x^0).$$

By the Sobolev inequality $\|x\|_{L^\infty} \leq C\|x\|$, we can choose r small enough such that

$$\|\bar{x}^+ + x^0\|_{L^\infty} < \delta.$$

Thus by (H_4^-),

$$G_{0k}(t, (\bar{x}^+ + x^0)(t)) < 0$$

for any t satisfying $(\bar{x}^+ + x^0)(t) \neq 0$. Since $x^0 \in E_0^0 \cap S_r$, then $\bar{x}^+ + x^0$ is not identically equal to zero. This implies that

$$\int_0^T G_{0k}(t, (\bar{x}^+ + x^0)(t))dt < 0$$

and

$$g(x^0) = I_k(\bar{x}^+ + x^0) = \frac{1}{2}\langle L_0\bar{x}^+, \bar{x}^+\rangle - \int_0^T G_{0k}(t, (\bar{x}^+ + x^0)(t))dt > 0.$$

Since E_0^0 is finite dimensional, $E_0^0 \cap S_r$ is a compact set. Then there exists $\alpha_0 > 0$ such that

$$g(x^0) \geq \alpha_0, \qquad \forall x^0 \in E_0^0 \cap S_r.$$

Hence, by the definition of g we have

$$I_k(x^+ + x^0) \geq g(x^0) \geq \alpha_0, \qquad \forall x^+ + x^0 \in (E_0^+ \cap B_r) \oplus (E_0^0 \cap S_r). \tag{4.7}$$

Let $\alpha = \min\{\alpha_0, \frac{c_0}{4}r^2\}$. Notice that

$$\partial[(E_0^+ \cap B_r) \oplus (E_0^0 \cap B_r)] = [(E_0^+ \cap S_r) \oplus (E_0^0 \cap B_r)] \cup [(E_0^+ \cap B_r) \oplus (E_0^0 \cap S_r)],$$

then by (4.5) and (4.7) we have

$$I_k(x^+ + x^0) \geq \alpha, \qquad \forall x^+ + x^0 \in \partial[(E_0^+ \cap B_r) \oplus (E_0^0 \cap B_r)]. \tag{4.8}$$

Taking $\alpha > 0$ smaller if necessary, we obtain

$$I_k(x^+ + x^0) \geq \alpha, \qquad \forall x^+ + x^0 \in (E_0^+ \oplus E_0^0) \cap S_r, \tag{4.9}$$

that is

$$\inf I_k((E_0^+ \oplus E_0^0) \cap S_r) \geq \alpha. \tag{4.10}$$

By (H_5^+), it is easy to see that (4.4) also holds in this case. Then by (4.4) and (4.10), using Proposition 4.1 and a similar argument as the case (1), we can prove that system (1.1) has at least $i_\infty^- - i_0^- - 1$ pairs of nontrivial periodic solutions;.

(3) We will use Proposition 4.2 to prove this case. Let $Y = E_\infty^-$, $Z = E_\infty^+$ and $X = E_0^- \oplus E_0^0$. Then $E = Y \oplus Z$ and $\dim X = i_0^- + i_0^0 > i_\infty^- = \dim Y$.

For $x \in E_\infty^+$, note that $E_\infty^+ \subset E_\gamma^+$ by (2.5) in this case, we have

$$I_k(x) = \frac{1}{2}\langle L_\gamma x, x\rangle - \varphi_k(x) \geq \frac{c_\gamma}{2}\|x\|^2 - o(\|x\|^2) \tag{4.11}$$

as $\|x\| \to \infty$. Then there exists $M_k > 0$ such that

$$I_k(x) \geq 0, \qquad \forall x \in E_\infty^+ \text{ with } \|x\| \geq M_k. \tag{4.12}$$

On the other hand, by Lemma 2.1, there exists a constant $C_1' > 0$ such that

$$|V_k(t,x)| \leq C_1'|x|^2.$$

Thus for $x \in E_\infty^+$ with $\|x\| \leq M_k$, we have

$$I_k(x) = \frac{1}{2}\int_0^T |\dot{x}|^2 dt - \int_0^T V_k(t,x)dt \geq -C_1' \int_0^T |x|^2 dt \geq -C_1' M_k^2. \tag{4.13}$$

By (4.12) and (4.13), we have

$$\inf I_k(E_\infty^+) > -\infty. \tag{4.14}$$

For $x \in E_0^- \oplus E_0^0$, by using a similar argument as in obtaining (4.9), we have that there exist $r > 0$ and $\alpha > 0$ such that

$$I_k(x) \leq -\alpha, \qquad \forall x \in (E_0^- \oplus E_0^0) \cap S_r,$$

that is

$$\sup I_k((E_0^- \oplus E_0^0) \cap S_r) \leq -\alpha. \tag{4.15}$$

Then by (4.14) and (4.15), using Proposition 4.2 and a similar argument as the case (1), we can prove that the system (1.1) has at least $i_0^- + i_0^0 - i_\infty^- - 1$ pairs of nontrivial periodic solutions.

(4) We will also use Proposition 4.2 to prove this case. Let $Y = E_\infty^-$, $Z = E_\infty^+$ and $X = E_0^-$. Then $E = Y \oplus Z$ and $\dim X = i_0^- > i_\infty^- = \dim Y$.

It is easy to see that (4.14) also holds in this case. For $x \in E_0^-$,

$$I_k(x) = \frac{1}{2}\langle L_0 x, x\rangle - \varphi_{0k}(x) \leq -\frac{c_0}{2}\|x\|^2 + o(\|x\|^2) \tag{4.16}$$

as $\|x\| \to 0$. By (4.16), there exist $r > 0$ and $\alpha > 0$ such that

$$\sup I_k(E_0^- \cap S_r) \leq -\alpha. \tag{4.17}$$

By (4.14) and (4.17), using Proposition 4.2 and a similar argument as the case (1), we can prove that system (1.1) has at least $i_0^- - i_\infty^- - 1$ pairs of nontrivial periodic solutions. $\square$

Acknowledgements

The author wishes to thank the referee for careful reading of the manuscript and helpful suggestions which improved this paper. This work is supported by Shandong Provincial Natural Science Foundation (Grant No. ZR2019BA019) and National Natural Science Foundation of China (Grant No. 11901270).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **28**, 1–12; https://doi.org/10.14232/ejqtde.2021.1.28 www.math.u-szeged.hu/ejqtde/

A time-nonlocal inverse problem for a hyperbolic equation with an integral overdetermination condition

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Received 12 January 2021, appeared 2 April 2021
Communicated by Alberto Cabada

Abstract. This article is concerned with the study of the unique solvability of a time-nonlocal inverse boundary value problem for second-order hyperbolic equation with an integral overdetermination condition. To study the solvability of the inverse problem, we first reduce the considered problem to an auxiliary system with trivial data and prove its equivalence (in a certain sense) to the original problem. Then using the Banach fixed point principle, the existence and uniqueness of a solution to this system is shown. Further, on the basis of the equivalency of these problems the existence and uniqueness theorem for the classical solution of the inverse coefficient problem is proved for the smaller value of time.

Keywords: inverse problem, hyperbolic equation, overdetermination condition, classical solution, existence, uniqueness.

2020 Mathematics Subject Classification: Primary 35R30, 35L10; Secondary 35A01, 35A02, 35A09.

1 Introduction

In practice, it is often required to recover the coefficients in an ordinary or partial differential equation from the final overspecified data. Problems of these types are called inverse problems of mathematical physics and are one of the most complicated and practically important problems. The theory of inverse problems is widely used to solve practical problems in almost all fields of science, in particular, in physics, medicine, ecology, and economics. Such problems include the locating groundwater, investigating locations for landfills, acoustics, oil and gas exploration, electromagnetic, X-ray tomography, laser tomography, elasticity, fluid dynamics, and so on.

In the modern mathematical literature, the theory of inverse boundary-value problems for equations of hyperbolic type of the second-order is stated rather satisfactory. In particular, the solvability of the inverse problems in various formulations with different overdetermination

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conditions for partial differential equations is extensively studied in many monographs and papers (see for example, [2,4,5,7,8,10,11,13,14,16,17,20,21,26], and the references therein).

Recently, problems with nonlocal conditions for partial differential equations have been of great interest to applied sciences. In the literature, the term "nonlocal boundary value problems" refers to problems that contain conditions relating the values of the solution and/or its derivatives either at different points of the boundary or at boundary points and some interior points [19]. It is well known that direct nonlocal boundary value problems with integral conditions (with respect to spatial variable) [3,6,9,15] are widely used for thermo-elasticity, chemical engineering, heat conduction, and plasma physics. As well as the direct nonlocal boundary value problems for hyperbolic equations with integral conditions (with respect to time variable) are considered in the papers [12,22] and the references therein. Moreover, In [23–25] the authors present a regularity result for solutions of partial differential equations in the framework of mixed Morrey spaces.

It should also be noted that the statement of the problem and the proof technique used in this paper differ from those of the above articles, and the conditions in the theorems are significantly different from those in them. A distinctive feature of this article is the consideration the inverse boundary value problem for a hyperbolic equation with both spatial and time nonlocal conditions.

2 Mathematical formulation

In the region defined by $D: 0 < x < 1,\ 0 < t < T,\ \ D_T = \overline{D}$, we consider the problem of determining the unknown functions $u(x,t) \in C^1(D_T) \cap C^2(D)$ and $a(t) \in C[0,T]$ such that the pair $\{u(x,t), a(t)\}$ satisfies a one-dimensional hyperbolic equation

$$u_{tt}(x,t) - u_{xx}(x,t) = a(t)u(x,t) + f(x,t), \qquad (x,t) \in D, \tag{2.1}$$

with the nonlocal initial conditions

$$u(x,0) + \delta_1 u(x,T) = \varphi(x), \qquad u_t(x,0) + \delta_2 u_t(x,T) = \psi(x), \qquad 0 \le x \le 1, \tag{2.2}$$

the boundary conditions

$$u_x(0,t) = u(1,t) = 0, \qquad 0 \le t \le T, \tag{2.3}$$

and integral overdetermination condition of the first kind

$$\int_0^1 w(x)u(x,t)dx = H(t), \qquad 0 \le t \le T, \tag{2.4}$$

where $\delta_1, \delta_2 \ge 0$, and $0 < T < +\infty$ are given numbers, and $f(x,t), \varphi(x), \psi(x), w(x),\ H(t)$ are known functions.

To study problem (2.1)–(2.4), we consider the equation

$$y''(t) = \gamma(t)y(t), \qquad 0 < t < T, \tag{2.5}$$

with the boundary conditions

$$y(0) + \delta_1 y(T) = 0, \qquad y'(0) + \delta_2 y'(T) = 0, \tag{2.6}$$

where $\delta_1, \delta_2 \geq 0$ are fixed numbers, $\gamma(t) \in C[0,T]$ is given function, and $y = y(t)$ is desired function.

Clearly, the problem

$$y''(t) = 0, \qquad y(0) + \delta_1 y(T) = 0, \qquad y'(0) + \delta_2 y'(T) = 0 \tag{2.7}$$

has unique trivial solution, for all nonnegative values of δ_1 and δ_2.

It is known [18] that boundary value problem (2.7) has a Green's function of the form

$$G(t,\tau) = \begin{cases} -\frac{\delta_2 t + \delta_1 (T-\tau) + \delta_1\delta_2 (t-\tau)}{(1+\delta_1)(1+\delta_2)}, & \in [0,\tau], \\ -\frac{\delta_2 t + \delta_1 (T-\tau) - (1+\delta_1+\delta_2)(t-\tau)}{(1+\delta_1)(1+\delta_2)}, & t \in [\tau, T]. \end{cases} \tag{2.8}$$

Lemma 2.1. *Suppose that the function $\gamma(t)$ is continuous on the interval $[0,T]$. If $\delta_1, \delta_2 \geq 0$ and*

$$\frac{1+2\delta_1+3\delta_2+\delta_1\delta_2}{2(1+\delta_1)(1+\delta_2)} \|\gamma(t)\|_{C[0,T]} T^2 < 1, \tag{2.9}$$

then problem (2.5), (2.6) has only a trivial solution.

Proof. Since problem (2.7) has a unique Green function defined by formula (2.8), then it could be argued [18] that boundary-value problem (2.5), (2.6) is equivalent to the integral equation

$$y(t) = \int_0^T G(t,\tau)\gamma(\tau) y(\tau) d\tau. \tag{2.10}$$

Let us introduce the notation

$$A(y(t)) = \int_0^T G(t,\tau)\gamma(\tau) y(\tau) d\tau. \tag{2.11}$$

Then the equation (2.10) can be rewritten as

$$y(t) = A(y(t)). \tag{2.12}$$

Obviously, the operator A is continuous in the space $C[0,T]$.

Now we prove that A is a contraction operator in the space $C[0,T]$. It is easy to see that the inequality

$$\|A(y_1(t)) - A(y_2(t))\|_{C[0,T]} \leq \frac{1+2\delta_1+3\delta_2+\delta_1\delta_2}{2(1+\delta_1)(1+\delta_2)} T^2 \|\gamma(t)\|_{C[0,T]} \|y_1(t) - y_2(t)\|_{C[0,T]} \tag{2.13}$$

holds for any functions $y_1(t), y_2(t) \in C[0,T]$.

In view of (2.9) and (2.13) it is clear that the operator A is contractive in $C[0,T]$. Therefore, the operator A has a unique fixed point $y(t)$ in the space $C[0,T]$ which is a solution of equation (2.12). Thus, the integral equation (2.10) has a unique solution in $C[0,T]$. Consequently, problem (2.5), (2.6) also has a unique solution in the indicated space. Since $y(t) = 0$ is a solution to problem (2.5), (2.6), it follows that this problem has a unique trivial solution. □

Now, to study problem (2.1)–(2.4), we consider the following auxiliary inverse boundary value problem: it is required to find a pair of functions $u(x,t) \in C^1(D_T) \cap C^2(D)$, $a(t) \in C[0,T]$ from (2.1)–(2.3) and

$$H''(t) - \int_0^1 w(x) u_{xx}(x,t) dx = H(t)a(t) + \int_0^1 w(x) f(x,t) dx, \qquad 0 < t < T. \tag{2.14}$$

Theorem 2.2. *Assume that* $\varphi(x),\ \psi(x) \in C[0,1],\ H(t) \in C^1[0,T] \cap C^2(0,T),\ H(t) \neq 0,\ 0 \le t \le T,\ f(x,t) \in C(D_T)$, *and that the following compatibility conditions are fulfilled*

$$\int_0^1 w(x)\varphi(x)dx = H(0) + \delta_1 H(T), \qquad \int_0^1 w(x)\psi(x)dx = H'(0) + \delta_2 H'(T). \tag{2.15}$$

Then the following statements are true:

(i) each classical solution $\{u(x,t), a(t)\}$ *of problem* (2.1)–(2.4) *is a solution of problem* (2.1)–(2.3), (2.14), *as well;*

(ii) each solution $\{u(x,t), a(t)\}$ *of problem* (2.1)–(2.3), (2.14) *under the circumstance*

$$\frac{(1+2\delta_1+3\delta_2+\delta_1\delta_2)T^2}{2(1+\delta_1)(1+\delta_2)} \|a(t)\|_{C[0,T]} < 1 \tag{2.16}$$

is a classical solution of problem (2.1)–(2.4).

Proof. Let $\{u(x,t), a(t)\}$ be a classical solution of problem (2.1)–(2.4). Multiplying the both sides of Eq.(2.1) by a special function $w(x)$ and integrating from 0 to 1 with respect to x gives

$$\begin{aligned}\frac{d^2}{dt^2}\int_0^1 w(x)u(x,t)dx - \int_0^1 w(x)u_{xx}(x,t)dx \\ = a(t)\int_0^1 w(x)u(x,t)dx + \int_0^1 w(x)f(x,t)dx, \qquad 0<t<T.\end{aligned} \tag{2.17}$$

Taking into account the condition $H(t) \in C^1[0,T] \cap C^2(0,T)$, and differentiating (2.4) twice, we have

$$\int_0^1 w(x)u_{tt}(x,t)dx = H''(t), \qquad 0<t<T. \tag{2.18}$$

From (2.17), taking into account (2.4) and (2.18) we arrive at (2.14).

Now, suppose that $\{u(x,t), a(t)\}$ is a solution to problem (2.1)–(2.3), (2.14). Then from (2.17), by allowing for (2.14), we find:

$$\frac{d^2}{dt^2}\left(\int_0^1 w(x)u(x,t)dx - H(t)\right) = a(t)\left(\int_0^1 w(x)u(x,t)dx - H(t)\right), \tag{2.19}$$

for $0<t<T$.

By using the initial conditions (2.2) and the compatibility conditions (2.15), we may write

$$\begin{aligned}&\int_0^1 w(x)u(x,0)dx - H(0) + \delta_1\left(\int_0^1 w(x)u(x,T)dx - H(T)\right)\\ &= \int_0^1 w(x)(u(x,0)+\delta_1 u(x,T))dx - (H(0)+\delta_1 H(T))\\ &= \int_0^1 w(x)\varphi(x)dx - (H(0)+\delta_1 H(T)) = 0,\end{aligned}$$

$$\begin{aligned}&\int_0^1 w(x)u_t(x,0)dx - H'(0) + \delta_2\left(\int_0^1 w(x)u_t(x,T)dx - H'(T)\right)\\ &= \int_0^1 w(x)(u_t(x,0)+\delta_2 u_t(x,T))dx - (H'(0)+\delta_2 H'(T))\\ &= \int_0^1 w(x)\psi(x)dx - (H'(0)+\delta_2 H'(T)) = 0.\end{aligned} \tag{2.20}$$

Lemma 2.1 enables us to conclude that the problem (2.19), (2.20) has only a trivial solution. Then, $\int_0^1 w(x)u(x,t)dx - H(t) = 0,\ 0 \le t \le T$, i.e., the condition (2.4) is satisfied. □

3 Existence and uniqueness of the solution of the inverse problem

We impose the following conditions on the numbers δ_1, δ_2, and the functions φ, ψ, f, w, and H:

$H_1)$ $\delta_1 \geq 0,\ \delta_2 \geq 0,\ 1+\delta_1\delta_2 > \delta_1+\delta_2;$

$H_2)$ $\varphi(x) \in C^2[0,1],\ \varphi'''(x) \in L_2(0,1),\ \varphi'(0) = \varphi(1) = \varphi''(1) = 0;$

$H_3)$ $\psi(x) \in C^1[0,1],\ \psi''(x) \in L_2(0,1),\ \psi'(0) = \psi(1) = 0;$

$H_4)$ $f(x,t), f_x(x,t) \in C(D_T),\ f_{xx}(x,t) \in L_2(D_T),\ f_x(0,t) = f(1,t) = 0,\ \ 0 \leq t \leq T;$

$H_5)$ $w(x) \in L_2(0,1),\ H(t) \in C^2[0,T],\ H(t) \neq 0,\ 0 \leq t \leq T.$

We seek the first component of solution $\{u(x,t), a(t)\}$ of the problem (2.1)–(2.3), (2.14) in the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\cos\lambda_k x, \qquad \lambda_k = \frac{\pi}{2}(2k-1), \tag{3.1}$$

where

$$u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx, \qquad k = 1,2,\ldots,$$

are twice-differentiable functions on an interval $[0,T]$.

Applying formal scheme of the Fourier method and using (2.1) and (2.2), we get

$$u_k''(t) + \lambda_k^2 u_k(t) = F_k(t;a,u), \qquad k = 1,2,\ldots; \qquad 0 < t < T, \tag{3.2}$$

$$u_k(0) + \delta_1 u_k(T) = \varphi_k, \qquad u_k'(0) + \delta_2 u_k'(T) = \psi_k, \qquad k = 1,2,\ldots, \tag{3.3}$$

where

$$F_k(t;u,a) = f_k(t) + a(t)u_k(t), \qquad f_k(t) = 2\int_0^1 f(x,t)\cos\lambda_k x dx,$$

$$\varphi_k = 2\int_0^1 \varphi(x)\cos\lambda_k x dx, \qquad \psi_k = 2\int_0^1 \psi(x)\cos\lambda_k x dx, \qquad k = 1,2,\ldots$$

Solving the problem (3.2),(3.3) gives

$$\begin{aligned} u_k(t) = {} & \frac{1}{\rho_k(T)}\left[\varphi_k(\cos\lambda_k t + \delta_2\cos\lambda_k(T-t)) + \frac{\psi_k}{\lambda_k}(\sin\lambda_k t - \delta_1\sin\lambda_k(T-t))\right] \\ & + \int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau, \end{aligned} \tag{3.4}$$

where

$$\rho_k(T) = 1 + (\delta_1+\delta_2)\cos\lambda_k T + \delta_1\delta_2, \tag{3.5}$$

$$G_k(t,\tau) = \begin{cases} -\frac{1}{\lambda_k\rho_k(T)}[\delta_1\sin\lambda_k(T-\tau)\cos\lambda_k t \\ \qquad + \delta_2\cos\lambda_k(T-\tau)\sin\lambda_k t + \delta_1\delta_2\sin\lambda_k(t-\tau)], & t \in [0,\tau], \\ -\frac{1}{\lambda_k\rho_k(T)}[\delta_1\sin\lambda_k(T-\tau)\cos\lambda_k t + \delta_2\cos\lambda_k(T-\tau)\sin\lambda_k t \\ \qquad + \delta_1\delta_2\sin\lambda_k(t-\tau)] + \frac{1}{\lambda_k}\sin\lambda_k(t-\tau), & t \in [\tau,T]. \end{cases} \tag{3.6}$$

Substituting the expression of (3.4) into (3.1), we find the component $u(x,t)$ of the classical solution to problem (2.1)–(2.3), (2.14) to be

$$u(x,t) = \sum_{k=1}^{\infty} \left\{ \frac{1}{\rho_k(T)} \left[\varphi_k(\cos\lambda_k t + \delta_2 \cos\lambda_k(T-t)) \right.\right.$$
$$\left.\left. + \frac{\psi_k}{\lambda_k}(\sin\lambda_k t - \delta_1 \sin\lambda_k(T-t)) \right] + \int_0^T G_k(t,\tau) F_k(\tau; u, a) d\tau \right\} \cos\lambda_k x. \quad (3.7)$$

Thus the problem (2.7), taking into account (2.14), yields

$$a(t) = [H(t)]^{-1} \left\{ H''(t) - \int_0^1 w(x) f(x,t) dx + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^2 u_k(t) w_k \right\}, \quad (3.8)$$

where

$$w_k = 2\int_0^1 w(x)\cos\lambda_k x dx, \qquad k = 1,2,\ldots$$

After substituting (3.4) into (3.8), we find the second component $a(t)$ of the solution to problem (2.1)–(2.3), (2.14) in the form

$$a(t) = [H(t)]^{-1} \left\{ H''(t) - \int_0^1 w(x) f(x,t) dx + \frac{1}{2} \sum_{k=1}^{\infty} w_k \lambda_k^2 \left(\frac{1}{\rho_k(T)} \left[\varphi_k(\cos\lambda_k t + \delta_2 \cos\lambda_k(T-t)) \right.\right.\right.$$
$$\left.\left.\left. + \frac{\psi_k}{\lambda_k}(\sin\lambda_k t - \delta_1 \sin\lambda_k(T-t)) \right] + \int_0^T G_k(t,\tau) F_k(\tau; u, a) d\tau \right) \right\}. \quad (3.9)$$

Thus the solution of problem (2.1)–(2.3), (2.14) was reduced to the solution of systems (3.7), (3.9) with respect to unknown functions $u(x,t)$ and $a(t)$.

The following lemma plays an important role in studying the uniqueness of the solution to problem (2.1)–(2.3), (2.14):

Lemma 3.1. *If $\{u(x,t), a(t)\}$ is any solution to problem (2.1)–(2.3), (2.14), then the functions*

$$u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx, \qquad k = 1,2,\ldots$$

satisfy the system (3.4) on an interval $[0,T]$.

Proof. Let $\{u(x,t), a(t)\}$ be any solution of the problem (2.1)–(2.3), (2.14). Multiplying both sides of the Eq. (2.1) by the special functions $2\cos\lambda_k x$ $(k = 1,2,\ldots)$, integrating from 0 to 1 with respect to x, and using the relations

$$2\int_0^1 u_{tt}(x,t)\cos\lambda_k x dx = \frac{d^2}{dt^2}\left(2\int_0^1 u(x,t)\cos\lambda_k x dx\right) = u_k''(t), \qquad k = 1,2,\ldots,$$

$$2\int_0^1 u_{xx}(x,t)\cos\lambda_k x dx = -\lambda_k^2\left(2\int_0^1 u(x,t)\cos\lambda_k x dx\right) = -\lambda_k^2 u_k(t), \qquad k = 1,2,\ldots,$$

we obtain that Eq. (3.2) is satisfied.

In like manner, it follows from (2.2) that condition (3.3) is also satisfied.

Thus, the system of functions $u_k(t)$ $(k = 1,2,\ldots)$ is a solution of problem (3.2), (3.3). From this fact it follows directly that the functions $u_k(t)$ $(k = 1,2,\ldots)$ also satisfy the system (3.4) on $[0,T]$. □

Obviously, if $u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx$, $k = 1,2,\ldots$, is a solution to system (3.4), then the pair $\{u(x,t), a(t)\}$ of functions $u(x,t) = \sum_{k=0}^{\infty} u_k(t)\cos\lambda_k x$ and $a(t)$ is also a solution to system (3.7), (3.9).

The next statement follows from Lemma 3.1.

Corollary 3.2. *Assume that the system (3.7), (3.9) has a unique solution. Then the problem (2.1)–(2.3), (2.14) has at most one solution, i.e., if the problem (2.1)–(2.3), (2.14) has a solution, then it is unique.*

Let us consider the functional space that is introduced in [1]. Denote by $B^3_{2,T}$ a set of all functions of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\cos\lambda_k x, \qquad \lambda_k = \frac{\pi}{2}(2k-1), \qquad k = 1,2,\ldots,$$

considered in D_T with the norm $\|u(x,t)\|_{B^3_{2,T}} = J_T(u)$, where $u_k(t) \in C[0,T]$ and

$$J_T(u) \equiv \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_k(t)\|_{C[0,T]}\right)^2 \right\}^{\frac{1}{2}} < +\infty.$$

Henceforth we shall denote by E^3_T the topological product of $B^3_{2,T} \times C[0,T]$, where the norm of an element $z = \{u, a\}$ is determined by the formula

$$\|z\|_{E^3_T} = \|u(x,t)\|_{B^3_{2,T}} + \|a(t)\|_{C[0,T]}.$$

It is known that the spaces $B^3_{2,T}$ and E^3_T are Banach spaces [27].

Let us now consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\},$$

in the space E^3_T, where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=1}^{\infty} \tilde{u}_k(t)\cos\lambda_k x, \qquad \Phi_2(u,a) = \tilde{a}(t),$$

and the functions $\tilde{u}_k(t)$ $(k = 1,2,\ldots)$ and $\tilde{a}(t)$ are equal to the right-hand sides of (3.4) and (3.9), respectively.

It is easy to see that under conditions $\delta_1 \geq 0$, $\delta_2 \geq 0$, $1 + \delta_1\delta_2 > \delta_1 + \delta_2$, we have

$$\frac{1}{\rho_k(T)} \leq \frac{1}{1 - (\delta_1 + \delta_2) + \delta_1\delta_2} \equiv \rho > 0.$$

Taking into account this relation, we obtain

$$\begin{aligned}
&\left\{ \sum_{k=1}^{\infty} \left(\lambda_k^3 \|\tilde{u}_k(t)\|_{C[0,T]}\right)^2 \right\}^{\frac{1}{2}} \\
&\quad \leq 2\rho(1+\delta_2)\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 |\varphi_k|\right)^2\right)^{\frac{1}{2}} + 2\rho(1+\delta_1)\left(\sum_{k=1}^{\infty} \left(\lambda_k^2 |\psi_k|\right)^2\right)^{\frac{1}{2}} \\
&\qquad + 2(1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2))\sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^2 |f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} \\
&\qquad + 2(1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2))T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},
\end{aligned} \tag{3.10}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le \left\|[H(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| H''(t) - \int_0^1 w(x) f(x,t)dx \right\|_{C[0,T]} \right.$$
$$+ \frac{1}{2}\left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \left[\rho(1+\delta_2)\left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_k|)^2\right)^{\frac{1}{2}} + \rho(1+\delta_1)\left(\sum_{k=1}^{\infty} (\lambda_k^2 |\psi_k|)^2\right)^{\frac{1}{2}} \right.$$
$$+ (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}}$$
$$\left.\left. +(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \right] \right\}. \quad (3.11)$$

Then from (3.10) and (3.11), respectively, we find that

$$\left\{\sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_k(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \le 4\sqrt{2}\rho(1+\delta_2)\|\varphi'''(x)\|_{L_2(0,1)} + 4\sqrt{2}\rho(1+\delta_1)\|\psi''(x)\|_{L_2(0,1)}$$
$$+ 4(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{2T}\|f_{xx}(x,t)\|_{L_2(D_T)}$$
$$+ 2(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3},$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le \left\|[H(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| H''(t) - \int_0^1 w(x) f(x,t)dx \right\|_{C[0,T]} \right.$$
$$+ \frac{1}{2}\left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \left[2\sqrt{2}\rho(1+\delta_2)\|\varphi'''(x)\|_{L_2(0,1)} \right.$$
$$+ 2\sqrt{2}\rho(1+\delta_1)\|\psi''(x)\|_{L_2(0,1)} + (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))2\sqrt{2T}\|f_{xx}(x,t)\|_{L_2(D_T)}$$
$$\left.\left. + (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3} \right] \right\},$$

or

$$\left\{\sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_k(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \le A_1(T) + B_1(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}, \quad (3.12)$$
$$\|\tilde{a}(t)\|_{C[0,T]} \le A_2(T) + B_2(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}, \quad (3.13)$$

where

$$A_1(T) = 4\sqrt{2}\rho(1+\delta_2)\|\varphi'''(x)\|_{L_2(0,1)} + 4\sqrt{2}\rho(1+\delta_1)\|\psi''(x)\|_{L_2(0,1)}$$
$$+ 4(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{2T}\|f_{xx}(x,t)\|_{L_2(D_T)},$$
$$B_1(T) = 2(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T,$$
$$A_2(T) = \left\|[H(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| H''(t) - \int_0^1 w(x) f(x,t)dx \right\|_{C[0,T]} \right.$$
$$+ \frac{1}{2}\left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \left[2\sqrt{2}\rho(1+\delta_2)\|\varphi'''(x)\|_{L_2(0,1)} + 2\sqrt{2}\rho(1+\delta_1)\|\psi''(x)\|_{L_2(0,1)} \right.$$
$$\left.\left. + (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))2\sqrt{2T}\|f_{xx}(x,t)\|_{L_2(D_T)} \right] \right\},$$

$$B_2(T) = \frac{1}{2}\left\|[H(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T.$$

Finally, from (3.12) and (3.13) we conclude:

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}, \tag{3.14}$$

where

$$A(T) = A_1(T) + A_2(T),\ B(T) = B_1(T) + B_2(T).$$

So, we can prove the following theorem.

Theorem 3.3. *Let the assumptions $H_1)$–$H_5)$ and the condition*

$$(A(T)+2)^2 B(T) < 1 \tag{3.15}$$

be satisfied. Then problem (2.1)–(2.3), (2.14) has a unique classical solution in the ball $K = K_R(\|z\|_{E^3_T} \leq R = A(T)+2)$ of the space E^3_T.

Remark 3.4. Inequality (3.15) is satisfied for sufficiently small values of T.

Proof. We consider the operator equation

$$z = \Phi z \tag{3.16}$$

in the space E^3_T, where $z = \{u, a\}$, and the components $\Phi_i(u,a),\ i = 1,2$ are defined by the right sides of equations (3.7) and (3.9), respectively.

Similar to (3.14) we obtain that for any $z, z_1, z_2 \in K_R$ the following inequalities hold

$$\|\Phi z\|_{E^3_T} \leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} \leq A(T) + B(T)(A(T)+2)^2, \tag{3.17}$$

$$\|\Phi z_1 - \Phi z_s\|_{E^3_T} \leq B(T)TR(\|a_1(t)-a_2(t)\|_{C[0,T]} + \|u_1(x,t)-u_2(x,t)\|_{B^3_{2,T}}). \tag{3.18}$$

Then by virtue of (3.15) from (3.17) and (3.18) it follows that the operator Φ acts in the ball $K = K_R$, and satisfy the conditions of the contraction mapping principle. Therefore, the operator Φ has a unique fixed point $\{u, a\}$ in the ball $K = K_R$, which is a solution of equation (3.16).

In this way we conclude that the function $u(x,t)$ as an element of space $B^3_{2,T}$ is continuous and has continuous derivatives $u(x,t)$ and $u_{xx}(x,t)$ in D_T.

From (3.2) it is easy to see that

$$\left(\sum_{k=1}^{\infty}(\lambda_k\,\|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$
$$\leq \sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \|\,\|f_x(x,t)+a(t)u_x(x,t)\|\,\|_{L_2(0,1)}\right].$$

Thus $u_{tt}(x,t)$ is continuous in the region D_T.

Further, it is possible to verify that Eq. (2.1) and conditions (2.2), (2.3), and (2.14) are satisfied in the usual sense. Consequently, $\{u(x,t), a(t)\}$ is a solution of (2.1)–(2.3), (2.14), and by Lemma 3.1 it is unique. □

On the basis of Theorem 2.2 it is easy to prove the following theorem.

Theorem 3.5. *Suppose that all assumptions of Theorem 3.3, and the conditions*

$$\frac{(1+2\delta_1+3\delta_2+\delta_1\delta_2)T^2(A(T)+2)}{2(1+\delta_1)(1+\delta_2)}<1,$$

$$\int_0^1 w(x)\varphi(x)dx = H(0)+\delta_1 H(T), \qquad \int_0^1 w(x)\psi(x)dx = H'(0)+\delta_2 H'(T)$$

hold. Then problem (2.1)–(2.4) has a unique classical solution in the ball $K=K_R(\|z\|_{E_T^3}\leq A(T)+2)$ of the space E_T^3.

4 Conclusion

The unique solvability of a time-nonlocal inverse boundary value problem for a second-order hyperbolic equation with an integral overdetermination condition is investigated. Considered problem was reduced to an auxiliary problem in a certain sense and using the contraction mappings principle a unique existence conditions for a solution of equivalent problem are established. Further, on the basis of the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse coefficient problem is proved for the smaller value of time.

Acknowledgements

The authors would like to express their profound thanks to the anonymous reviewers for careful reading of our manuscript and their insightful comments and helpful suggestions.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **29**, 1–12; https://doi.org/10.14232/ejqtde.2021.1.29 www.math.u-szeged.hu/ejqtde/

Singular Kneser solutions of higher-order quasilinear ordinary differential equations

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Received 19 November 2020, appeared 5 April 2021
Communicated by Zuzana Došlá

Abstract. In this paper we give a new sufficient condition in order that all nontrivial Kneser solutions of the quasilinear ordinary differential equation

$$D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x = (-1)^n p(t)|x|^{\beta} \operatorname{sgn} x, \quad t \geq a, \tag{1.1}$$

are singular. Here, $D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)$ is the nth-order iterated differential operator such that

$$D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x = D(\alpha_n)D(\alpha_{n-1}) \cdots D(\alpha_1)x$$

and, in general, $D(\alpha)$ is the first-order differential operator defined by $D(\alpha)x = (d/dt)\left(|x|^{\alpha} \operatorname{sgn} x\right)$ for $\alpha > 0$. In the equation (1.1), the condition $\alpha_1 \alpha_2 \cdots \alpha_n > \beta$ is assumed. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then one of the results of this paper yields a well-known theorem of Kiguradze and Chanturia.

Keywords: Kneser solutions, singular solutions, quasilinear equations.

2020 Mathematics Subject Classification: 34C11.

1 Introduction

For a positive constant α, let $D(\alpha)$ be the first-order differential operator defined by

$$D(\alpha)x = \frac{d}{dt}\left(|x|^{\alpha} \operatorname{sgn} x\right),$$

and for n positive constants $\alpha_1, \alpha_2, \dots, \alpha_n$ let $D(\alpha_i, \alpha_{i-1}, \dots, \alpha_1)$ be the ith-order iterated differential operator defined by

$$D(\alpha_i, \alpha_{i-1}, \dots, \alpha_1)x = D(\alpha_i)D(\alpha_{i-1}) \cdots D(\alpha_1)x, \quad i = 0, 1, 2, \dots, n.$$

Here, if $i = 0$, then $D(\alpha_i, \dots, \alpha_1)x$ is interpreted as x.

In this paper we consider nth-order quasilinear ordinary differential equations of the form

$$D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x = (-1)^n p(t)|x|^{\beta} \operatorname{sgn} x, \quad t \geq a, \tag{1.1}$$

where it is assumed that

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(a) $n \geq 2$ is an integer;

(b) $\alpha_1, \alpha_2, \dots, \alpha_n$ and β are positive constants;

(c) $p(t)$ is a continuous function on an interval $[a,\infty)$, and $p(t) \geq 0$ on $[a,\infty)$, and $p(t) \not\equiv 0$ on $[a_1,\infty)$ for any $a_1 \geq a$.

By a solution $x(t)$ of (1.1) on $[a,\infty)$ we mean that

$$\begin{aligned}&D(\alpha_1)x(t), \quad D(\alpha_2)D(\alpha_1)x(t) = D(\alpha_2,\alpha_1)x(t), \dots,\\ &\qquad D(\alpha_n)D(\alpha_{n-1})\cdots D(\alpha_1)x(t) = D(\alpha_n,\alpha_{n-1},\dots,\alpha_1)x(t)\end{aligned}$$

are well-defined and continuous on $[a,\infty)$ and $x(t)$ satisfies (1.1) at every point $t \in [a,\infty)$. A function $x(t)$ is said to be a *Kneser solution* of (1.1) on $[a,\infty)$ if $x(t)$ is a solution of (1.1) on $[a,\infty)$ and satisfies

$$(-1)^i D(\alpha_i,\dots,\alpha_1)x(t) \geq 0, \quad t \geq a, \quad i = 0,1,2,\dots,n-1. \tag{1.2}$$

To shorten notation, we set

$$D(\alpha_i,\dots,\alpha_1)x(t) = D_i x(t) \quad \text{for } i = 0,1,2,\dots,n.$$

Then, the equation (1.1) may be expressed as

$$D_n x = (-1)^n p(t)|x|^\beta \operatorname{sgn} x, \quad t \geq a, \tag{1.3}$$

and the condition (1.2) is rewritten in the form

$$(-1)^i D_i x(t) \geq 0, \quad t \geq a, \quad i = 0,1,2,\dots,n-1.$$

Suppose that $x(t)$ is a function on $[a,\infty)$ such that $D(\alpha)x(t)$, $\alpha > 0$, is well-defined and continuous on $[a,\infty)$. It is easily seen that if $D(\alpha)x(t) \geq 0$ [*resp.* $> 0, \leq 0, < 0$] on $[a,\infty)$, then $x(t)$ is increasing [*resp.* strictly increasing, decreasing, strictly decreasing] on $[a,\infty)$.

If $x(t)$ is a nonnegative solution of (1.3) on $[a,\infty)$, then $(-1)^n D_n x(t) = p(t)x(t)^\beta \geq 0$ on $[a,\infty)$. Therefore, if $x(t)$ is a Kneser solution of (1.3) on $[a,\infty)$, then $(-1)^i D_i x(t)$ is (nonnegative and) decreasing on $[a,\infty)$ $(i = 0,1,2,\dots,n-1)$.

Now, for the positive constants $\alpha_1, \alpha_2, \dots, \alpha_n$ appearing in (1.1), we put

$$\begin{aligned}\mu_n = \alpha_2 &+ \big(\alpha_2\alpha_3 + \alpha_3\big) + \big(\alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4 + \alpha_4\big)\\ &+ \cdots + \big(\alpha_2\alpha_3\cdots\alpha_n + \alpha_3\alpha_4\cdots\alpha_n + \cdots + \alpha_{n-1}\alpha_n + \alpha_n\big),\end{aligned} \tag{1.4}$$

$$\nu_n = \alpha_2\alpha_3\cdots\alpha_n + \alpha_3\alpha_4\cdots\alpha_n + \cdots + \alpha_{n-1}\alpha_n + \alpha_n, \tag{1.5}$$

$$\xi_n = \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 + \cdots + \alpha_1\alpha_2\cdots\alpha_{n-1} + \alpha_1\alpha_2\cdots\alpha_n. \tag{1.6}$$

Very recently, Naito and Usami ([6, Theorem 4.1]) have proved that, for each $A > 0$, the equation (1.1) has at least one Kneser solution $x(t)$ on $[a,\infty)$ such that $x(a) = A$. For the case $\alpha_1\alpha_2\cdots\alpha_n \leq \beta$, any nontrivial Kneser solution $x(t)$ of (1.1) on $[a,\infty)$ satisfies

$$(-1)^i D_i x(t) > 0 \qquad (t \geq a) \quad \text{for } i = 0,1,2,\dots,n-1$$

([6, the paragraph after the proof of Theorem 5.1]). However, for the case $\alpha_1\alpha_2\cdots\alpha_n > \beta$, a Kneser solution $x(t)$ of (1.1) on $[a,\infty)$ may be singular in the sense that

$$x(t) > 0 \quad (a \leq t < b) \quad \text{and} \quad x(t) = 0 \quad (t \geq b)$$

for some finite number $b > a$. Such a solution is often said to be a *first kind singular* solution of (1.1). It is known ([6, Theorem 6.1]) that if $\alpha_1\alpha_2\cdots\alpha_n > \beta$ and $p(b) > 0$ $(b > a)$, then (1.1) always has at least one singular Kneser solution $x(t)$ such that

$$\begin{cases} (-1)^i D_i x(t) > 0 \quad (a \le t < b) \quad \text{for } i = 0,1,2,\dots,n-1, \quad \text{and} \\ x(t) = 0 \quad (t \ge b). \end{cases} \tag{1.7}$$

In particular, if $p(t)$ is positive on $[a,\infty)$, then for any b $(> a)$ (1.1) has a singular Kneser solution $x(t)$ which satisfies (1.7). Note that, by putting $x_i = (D_{i-1}x)^{\alpha_i *}$ $(i = 1,2,\dots,n)$, the scalar equation (1.1) is equivalent to the n-dimensional system

$$\begin{cases} x_1' = x_2^{(1/\alpha_2)*}, \\ \quad \vdots \\ x_{n-1}' = x_n^{(1/\alpha_n)*}, \\ x_n' = (-1)^n p(t) x_1^{(\beta/\alpha_1)*}. \end{cases}$$

Then, applying Theorem 1 of Čanturia [2] to this n-dimensional system, we find that if $p(t)$ is positive on $[a,\infty)$, then for any b $(> a)$ there is a' $(a \le a' < b)$ such that (1.1) has a singular Kneser solution which is defined on $[a',\infty)$ and satisfies (1.7) with a replaced by a'. Theorem 6.1 of [6] shows that a' can be taken as $a' = a$.

If $p(t)$ is large enough in a neighborhood of ∞, then *all* nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular. In fact, making use of Theorem 2 of Čanturia [2], we have the following theorem.

Theorem A. *Let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. Let ν_n be the number defined by (1.5). If*

$$\liminf_{t\to\infty} t^{\nu_n+1} p(t) > 0, \tag{1.8}$$

then all nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular.

A different proof of Theorem A has been given by Naito and Usami [6, Theorem 6.8].

The main purpose of this paper is to show that Theorem A can be generalized as follows.

Theorem 1.1. *Let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. Let μ_n, ν_n and ξ_n be the numbers defined by (1.4), (1.5) and (1.6), respectively. Suppose that there exist $\sigma > 0$ and $\tau > 0$ such that*

$$(\nu_n + 1)\sigma - \mu_n\tau - 1 \ge 0, \tag{1.9}$$

$$\left(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\nu_n + 1\right)\sigma - \left(\mu_n - \frac{\nu_n\xi_n}{\alpha_1\alpha_2\cdots\alpha_n}\right)\tau - 1 \le 0, \tag{1.10}$$

and either

$$\int_{a^+}^{\infty} s^{-\mu_n\tau+(\nu_n+1)\sigma-1} p(s)^\sigma ds = \infty \quad (a^+ > \max\{a,0\}), \tag{1.11}$$

or

$$\limsup_{t\to\infty} t^{\mu_n\tau} \int_t^\infty s^{-\mu_n\tau+(\nu_n+1)\sigma-1} p(s)^\sigma ds > 0. \tag{1.12}$$

Then, all nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular.

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then

$$D_i x(t) = x^{(i)}(t) \quad (i = 0, 1, 2, \ldots, n),$$

and so (1.1) is reduced to

$$x^{(n)} = (-1)^n p(t)|x|^\beta \operatorname{sgn} x, \quad t \geq a. \tag{1.13}$$

If $n = 2$ and $\alpha_1 = 1$, $\alpha_2 = \alpha > 0$, then (1.1) is the second-order quasilinear differential equation

$$(|x'|^\alpha \operatorname{sgn} x')' = p(t)|x|^\beta \operatorname{sgn} x, \quad t \geq a. \tag{1.14}$$

Results on the problem of existence and asymptotic behavior of Kneser solutions of (1.13) are summarized and proved in the book of Kiguradze and Chanturia [3]. This problem has also been studied by Mizukami, Naito and Usami [4] for (1.14), and by Naito and Usami [6] for the general equation (1.1).

The proof of Theorem 1.1 is given in the next Section 2. In Section 3, Theorem 1.1 are restated in several ways, and some important corollaries are mentioned.

A function $x(t)$ is said to be a *strongly increasing solution* of the equation

$$D_n x = p(t)|x|^\beta \operatorname{sgn} x, \quad t \geq a, \tag{1.15}$$

on $[a, b)$ $(a < b \leq \infty)$ if $x(t)$ is a nontrivial solution of (1.15) on $[a, b)$ and satisfies

$$D_i x(t) \geq 0 \quad (a \leq t < b) \quad \text{for all} \quad i = 0, 1, 2, \ldots, n-1.$$

Suppose that $x(t)$ is a strongly increasing solution of (1.15) on $[a, b)$, and let $[a, b)$ be the maximal interval of existence of $x(t)$. If b is finite, then $x(t)$ is called *singular*. A singular strongly increasing solution is often said to be a *second kind singular* solution of (1.15). There is a remarkable duality between Kneser solutions of (1.3) and strongly increasing solutions of (1.15) (see [5, 6]). In the paper [7] we have established a new sufficient condition in order that all strongly increasing solutions of (1.15) are singular. The present paper corresponds to [7].

2 Proof of Theorem 1.1

Let us begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof is done by contradiction. Suppose that (1.1) has a Kneser solution $x(t)$ on $[a, \infty)$ such that $x(t) > 0$ for $t \geq a$. As mentioned in the preceding section, $(-1)^i D_i x(t)$ is decreasing on $[a, \infty)$ $(i = 0, 1, 2, \ldots, n-1)$. Furthermore, by (1.1), we easily see that

$$(-1)^i D_i x(t) > 0, \quad t \geq a \quad (i = 0, 1, 2, \ldots, n-1). \tag{2.1}$$

Define $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ and λ_n by

$$\begin{aligned}
\lambda_1 &= \frac{1}{\nu_n} \alpha_2 \cdots \alpha_n (1 - \sigma + \mu_n \tau) - \big(\alpha_2 + \alpha_2\alpha_3 + \cdots + \alpha_2 \cdots \alpha_{n-1}\alpha_n\big)\tau, \\
\lambda_2 &= \frac{1}{\nu_n} \alpha_3 \cdots \alpha_n (1 - \sigma + \mu_n \tau) - \big(\alpha_3 + \alpha_3\alpha_4 + \cdots + \alpha_3 \cdots \alpha_{n-1}\alpha_n\big)\tau, \\
&\;\;\vdots \\
\lambda_{n-1} &= \frac{1}{\nu_n} \alpha_n (1 - \sigma + \mu_n \tau) - \alpha_n \tau, \quad \text{and} \\
\lambda_n &= \sigma,
\end{aligned}$$

where σ and τ are positive constants satisfying (1.9) and (1.10). It is easy to see that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1, \quad \text{and} \tag{2.2}$$

$$\lambda_i - \alpha_{i+1}\lambda_{i+1} = -\alpha_{i+1}\tau \quad (i = 1, 2, \ldots, n-2). \tag{2.3}$$

We have

$$\lambda_i > 0 \quad (i = 1, 2, \ldots, n). \tag{2.4}$$

To see this, note that the condition (1.10) is rewritten as

$$\frac{\beta}{\alpha_1}\sigma + \tau - \lambda_1 \leq 0. \tag{2.5}$$

(The left-hand side of (1.10) multiplied by $(\alpha_2 \cdots \alpha_n)/\nu_n$ is equal to the left-hand side of (2.5).) It follows from (2.5) that

$$\lambda_1 \geq \frac{\beta}{\alpha_1}\sigma + \tau > 0.$$

By induction, (2.3) gives

$$\lambda_{i+1} = \frac{\lambda_i}{\alpha_{i+1}} + \tau > 0 \quad \text{for } i = 1, 2, \ldots, n-2.$$

Obviously, $\lambda_n = \sigma > 0$. Thus we have (2.4).

Next, define the function $y(t)$ by

$$y(t) = x(t)^{\alpha_1}[-D_1x(t)]^{\alpha_2}[D_2x(t)]^{\alpha_3} \cdots [(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n}$$

for $t \geq a$. By (2.1), we have $y(t) > 0$ $(t \geq a)$. It is easy to find that the derivative $y'(t)$ of $y(t)$ is calculated as

$$y'(t) = -\left[\frac{-D_1x(t)}{x(t)^{\alpha_1}} + \frac{D_2x(t)}{[-D_1x(t)]^{\alpha_2}} + \cdots + \frac{(-1)^nD_nx(t)}{[(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n}}\right]y(t), \quad t \geq a. \tag{2.6}$$

As a general inequality we have

$$u_1^{\lambda_1}u_2^{\lambda_2} \cdots u_n^{\lambda_n} \leq \lambda_1u_1 + \lambda_2u_2 + \cdots + \lambda_nu_n$$

for $u_i \geq 0$, $\lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$ (see, for example, [1, pp. 13–14]). This inequality may be written equivalently as

$$\Lambda v_1^{\lambda_1}v_2^{\lambda_2} \cdots v_n^{\lambda_n} \leq v_1 + v_2 + \cdots + v_n \quad \text{with} \quad \Lambda = \lambda_1^{-\lambda_1}\lambda_2^{-\lambda_2} \cdots \lambda_n^{-\lambda_n} \tag{2.7}$$

for $v_i \geq 0$, $\lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$. Therefore, by (2.6) and by (2.7) of the case

$$v_i = \frac{(-1)^iD_ix(t)}{[(-1)^{i-1}D_{i-1}x(t)]^{\alpha_i}} \quad (i = 1, 2, \ldots, n),$$

we get

$$\begin{aligned} y'(t) \leq & -\Lambda\left[\frac{-D_1x(t)}{x(t)^{\alpha_1}}\right]^{\lambda_1}\left[\frac{D_2x(t)}{[-D_1x(t)]^{\alpha_2}}\right]^{\lambda_2} \cdots \left[\frac{(-1)^nD_nx(t)}{[(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n}}\right]^{\lambda_n} y(t) \\ = & -\Lambda x(t)^{-\alpha_1\lambda_1}[-D_1x(t)]^{\lambda_1-\alpha_2\lambda_2} \cdots [(-1)^{n-2}D_{n-2}x(t)]^{\lambda_{n-2}-\alpha_{n-1}\lambda_{n-1}} \\ & \times [(-1)^{n-1}D_{n-1}x(t)]^{\lambda_{n-1}-\alpha_n\lambda_n}[(-1)^nD_nx(t)]^{\lambda_n}y(t) \end{aligned}$$

for $t \geq a$. Then, on account of (1.3) and (2.3), we see that

$$\begin{aligned} y'(t) \leq & -\Lambda x(t)^{-\alpha_1\lambda_1+\alpha_1\tau+\beta\lambda_n} x(t)^{-\alpha_1\tau}[-D_1x(t)]^{-\alpha_2\tau} \\ & \cdots [(-1)^{n-2}D_{n-2}x(t)]^{-\alpha_{n-1}\tau}[(-1)^{n-1}D_{n-1}x(t)]^{-\alpha_n\tau} \\ & \times [(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n\tau+\lambda_{n-1}-\alpha_n\lambda_n} p(t)^{\lambda_n} y(t), \end{aligned}$$

and, in consequence,

$$y'(t) \leq -\Lambda x(t)^{-\alpha_1\lambda_1+\alpha_1\tau+\beta\sigma}[(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n\tau+\lambda_{n-1}-\alpha_n\sigma} p(t)^{\sigma} y(t)^{1-\tau} \tag{2.8}$$

for $t \geq a$. Since $x(t)$ is decreasing on $[a, \infty)$ and $-\alpha_1\lambda_1 + \alpha_1\tau + \beta\sigma \leq 0$ (see (2.5)), we have

$$x(t)^{-\alpha_1\lambda_1+\alpha_1\tau+\beta\sigma} \geq x(a)^{-\alpha_1\lambda_1+\alpha_1\tau+\beta\sigma}, \quad t \geq a. \tag{2.9}$$

Next, we will claim that

$$\lim_{t\to\infty} t^{\nu_n/\alpha_n}[(-1)^{n-1}D_{n-1}x(t)] = 0, \tag{2.10}$$

or equivalently

$$\varepsilon_{n-1}(t) \equiv t^{\alpha_2\alpha_3\cdots\alpha_{n-1}+\alpha_3\cdots\alpha_{n-1}+\cdots+\alpha_{n-1}+1}[(-1)^{n-1}D_{n-1}x(t)] \to 0 \tag{2.11}$$

as $t \to \infty$. Let $i = 0, 1, 2, \ldots, n-1$. Since $(-1)^i D_i x(t)$ is positive and decreasing on $[a, \infty)$, the limit

$$\lim_{t\to\infty} (-1)^i D_i x(t) = \ell_i$$

exists and is nonnegative. Assume that $\ell_i > 0$ for some $i = 1, 2, \ldots, n-1$. Then it is easy to see that

$$\lim_{t\to\infty} \frac{[(-1)^{i-1}D_{i-1}x(t)]^{\alpha_i}}{t} = -\ell_i < 0.$$

This is a contradiction to the fact that $[(-1)^{i-1}D_{i-1}x(t)]^{\alpha_i}$ is positive on $[a, \infty)$. Hence we have

$$\lim_{t\to\infty} (-1)^i D_i x(t) = 0 \quad \text{for any } i = 1, 2, \ldots, n-1, \quad \text{and} \tag{2.12}$$

$$\lim_{t\to\infty} x(t) = \ell_0 \geq 0. \tag{2.13}$$

It follows from (2.13) that

$$x(t)^{\alpha_1} - \ell_0^{\alpha_1} = \int_t^{\infty} [-D_1x(s)]ds, \quad t \geq a,$$

and so

$$x(t)^{\alpha_1} - \ell_0^{\alpha_1} \geq \int_t^{2t} [-D_1x(s)]ds \geq t[-D_1x(2t)], \quad t \geq a^+. \tag{2.14}$$

Here, a^+ is a number such that $a^+ > \max\{a, 0\}$. In the same manner, it follows from (2.12) that

$$[(-1)^i D_i x(t)]^{\alpha_{i+1}} \geq t[(-1)^{i+1}D_{i+1}x(2t)], \quad t \geq a^+, \tag{2.15}$$

for $i = 1, 2, \ldots, n-2$. By (2.14) and (2.15), we can check with no difficulty that

$$\begin{aligned} & [x(t)^{\alpha_1} - \ell_0^{\alpha_1}]^{\alpha_2\alpha_3\cdots\alpha_{n-1}} \\ & \geq t^{\alpha_2\alpha_3\cdots\alpha_{n-1}}(2t)^{\alpha_3\cdots\alpha_{n-1}} \cdots (2^{n-3}t)^{\alpha_{n-1}}(2^{n-2}t)[(-1)^{n-1}D_{n-1}x(2^{n-1}t)] \\ & = 2^{\alpha_3\cdots\alpha_{n-1}} \cdots (2^{n-3})^{\alpha_{n-1}} 2^{n-2} t^{\alpha_2\alpha_3\cdots\alpha_{n-1}+\alpha_3\cdots\alpha_{n-1}+\cdots+\alpha_{n-1}+1}[(-1)^{n-1}D_{n-1}x(2^{n-1}t)] \end{aligned}$$

for $t \geq a^+$. Then, by (2.13), it is seen that (2.11) or equivalently (2.10) holds.

According to (2.10), there is $a_1 > a^+$ such that

$$(-1)^{n-1} D_{n-1}x(t) \leq t^{-\nu_n/\alpha_n}, \quad t \geq a_1. \tag{2.16}$$

Observe that (1.9) implies

$$\alpha_n \tau + \lambda_{n-1} - \alpha_n \sigma = -\frac{\alpha_n}{\nu_n}[(\nu_n + 1)\sigma - \mu_n \tau - 1] \leq 0,$$

and so (2.16) gives

$$[(-1)^{n-1} D_{n-1}x(t)]^{\alpha_n \tau + \lambda_{n-1} - \alpha_n \sigma} \geq t^{(\nu_n+1)\sigma - \mu_n \tau - 1}, \quad t \geq a_1. \tag{2.17}$$

Then it follows from (2.8), (2.9) and (2.17) that

$$y'(t) \leq -L t^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(t)^\sigma y(t)^{1-\tau}, \quad t \geq a_1,$$

where $L = \Lambda x(a)^{-\alpha_1 \lambda_1 + \alpha_1 \tau + \beta \sigma}$ is a positive constant. From this inequality it follows that

$$y(t')^\tau - y(t)^\tau \leq -\tau L \int_t^{t'} s^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(s)^\sigma ds$$

for any t and t' such that $a_1 \leq t \leq t'$. Then, letting $t' \to \infty$, we find that

$$\int_{a_1}^\infty s^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(s)^\sigma ds < \infty \tag{2.18}$$

and

$$y(t)^\tau \geq \tau L \int_t^\infty s^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(s)^\sigma ds, \quad t \geq a_1. \tag{2.19}$$

Of course, (2.18) contradicts (1.11). It will be showed that (2.19) is a contradiction to (1.12). By the definition of $y(t)$, the inequality (2.19) gives

$$\begin{aligned} &\Big[x(t)^{\alpha_1}[-D_1x(t)]^{\alpha_2}[D_2x(t)]^{\alpha_3} \cdots [(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n}\Big]^\tau \\ &\quad \geq \tau L \int_t^\infty s^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(s)^\sigma ds, \quad t \geq a_1. \end{aligned} \tag{2.20}$$

As in the proof of (2.11), we can find that

$$\begin{aligned} &\varepsilon_{n-2}(t) \equiv t^{\alpha_2\alpha_3\cdots\alpha_{n-2} + \alpha_3\cdots\alpha_{n-2} + \cdots + \alpha_{n-2} + 1}[(-1)^{n-2}D_{n-2}x(t)] \to 0, \\ &\quad\vdots \\ &\varepsilon_2(t) \equiv t^{\alpha_2+1}[(-1)^2 D_2x(t)] \to 0, \\ &\varepsilon_1(t) \equiv t[-D_1x(t)] \to 0, \end{aligned}$$

as $t \to \infty$. Set $\varepsilon_0(t) = x(t)$. From (2.20) and the definition of $\varepsilon_i(t)$ $(i = 0, 1, 2, \ldots, n-1)$ it follows that

$$\begin{aligned} &\Big[\varepsilon_0(t)^{\alpha_1}[t^{-1}\varepsilon_1(t)]^{\alpha_2}[t^{-\alpha_2-1}\varepsilon_2(t)]^{\alpha_3} \cdots [t^{-\alpha_2\alpha_3\cdots\alpha_{n-1} - \cdots - \alpha_{n-1} - 1}\varepsilon_{n-1}(t)]^{\alpha_n}\Big]^\tau \\ &\quad \geq \tau L \int_t^\infty s^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(s)^\sigma ds, \quad t \geq a_1, \end{aligned}$$

and, hence,

$$\begin{aligned}
&[\varepsilon_0(t)^{\alpha_1}\varepsilon_1(t)^{\alpha_2}\varepsilon_2(t)^{\alpha_3}\cdots\varepsilon_{n-1}(t)^{\alpha_n}]^\tau \\
&\quad\geq \tau L t^{[\alpha_2+(\alpha_2+1)\alpha_3+\cdots+(\alpha_2\alpha_3\cdots\alpha_{n-1}+\cdots+\alpha_{n-1}+1)\alpha_n]\tau}\int_t^\infty s^{(\nu_n+1)\sigma-\mu_n\tau-1}p(s)^\sigma ds \\
&\quad= \tau L t^{\mu_n\tau}\int_t^\infty s^{(\nu_n+1)\sigma-\mu_n\tau-1}p(s)^\sigma ds, \quad t\geq a_1.
\end{aligned}$$

Since $\varepsilon_0(t) = x(t)$ is bounded on $[a,\infty)$ and $\varepsilon_i(t) \to 0$ as $t\to\infty$ $(i = 1,2,\dots,n-1)$, we conclude that

$$\lim_{t\to\infty} t^{\mu_n\tau}\int_t^\infty s^{(\nu_n+1)\sigma-\mu_n\tau-1}p(s)^\sigma ds = 0,$$

which is a contradiction to (1.12). This finishes the proof of Theorem 1.1. □

For the case $n = 2$, $\alpha_1 = 1$ and $\alpha_2 = \alpha > 0$, the equation (1.1) becomes (1.14). In this case we have

$$\mu_2 = \alpha, \quad \nu_2 = \alpha \quad \text{and} \quad \xi_2 = 1+\alpha.$$

Therefore Theorem 1.1 gives an extension of Theorem 3.4 of [4]. The lim inf in the condition (3.3) of Theorem 3.4 of [4] can be replaced to lim sup.

Theorem A can easily be derived from Theorem 1.1. To see this, we first remark that

$$\nu_n\xi_n - \alpha_1\alpha_2\cdots\alpha_n\mu_n > 0, \tag{2.21}$$

where μ_n, ν_n and ξ_n are defined by (1.4), (1.5) and (1.6), respectively. Therefore the term $\mu_n - [(\nu_n\xi_n)/(\alpha_1\alpha_2\cdots\alpha_n)]$ appearing in (1.10) is a negative number. Then we find that the set of all pairs $(\sigma,\tau)\in(0,\infty)\times(0,\infty)$ satisfying (1.9) and (1.10) is nonempty. More precisely, the set is a triangle in the $\sigma\tau$ plane. Now, to prove Theorem A, suppose that (1.8) holds. There is a constant $c > 0$ such that $p(t)\geq ct^{-\nu_n-1}$ for all large t. Take a pair $(\sigma,\tau)\in(0,\infty)\times(0,\infty)$ satisfying (1.9) and (1.10). Then we get

$$t^{-\mu_n\tau+(\nu_n+1)\sigma-1}p(t)^\sigma \geq c^\sigma t^{-\mu_n\tau-1}$$

for all large t. If (1.11) does not hold, then the above inequality implies

$$\int_t^\infty s^{-\mu_n\tau+(\nu_n+1)\sigma-1}p(s)^\sigma ds \geq \frac{c^\sigma}{\mu_n\tau}t^{-\mu_n\tau}$$

for all large t, and, in consequence, the condition (1.12) is satisfied. Therefore we conclude from Theorem 1.1 that all nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular.

3 Other forms of Theorem 1.1

For simplicity, we put

$$\zeta_n = \frac{\nu_n\xi_n}{\alpha_1\alpha_2\cdots\alpha_n\mu_n} - 1.$$

By (2.21), ζ_n is a positive number.

Now, let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. It is easy to check that $\sigma > 0$ and $\tau > 0$ satisfy (1.9) and (1.10) if and only if

$$\frac{1}{\nu_n+1} < \sigma < \frac{1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\,\nu_n+1} \tag{3.1}$$

and

$$0 < \tau \le \frac{1}{\mu_n} \min\left\{(\nu_n + 1)\sigma - 1,\ \frac{1}{\zeta_n}\left[1 - \left(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\nu_n + 1\right)\sigma\right]\right\}. \tag{3.2}$$

Suppose that $\sigma > 0$ satisfies (3.1). Next, choose $\tau > 0$ so that the equality holds in the latter inequality of (3.2), and put $\tau = \tau(\sigma)$, that is to say, we define the number $\tau(\sigma)$ by

$$\tau(\sigma) = \frac{1}{\mu_n} \min\left\{(\nu_n + 1)\sigma - 1,\ \frac{1}{\zeta_n}\left[1 - \left(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\nu_n + 1\right)\sigma\right]\right\}. \tag{3.3}$$

For this choice, the conditions (1.11) and (1.12) become

$$\int_{a^+}^{\infty} s^{-\mu_n\tau(\sigma)+(\nu_n+1)\sigma-1} p(s)^{\sigma} ds = \infty \quad (a^+ > \max\{a, 0\}) \tag{3.4}$$

and

$$\limsup_{t\to\infty} t^{\mu_n\tau(\sigma)} \int_t^{\infty} s^{-\mu_n\tau(\sigma)+(\nu_n+1)\sigma-1} p(s)^{\sigma} ds > 0, \tag{3.5}$$

respectively. Therefore Theorem 1.1 produces the following result.

Theorem 3.1. *Let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. Suppose that σ satisfies (3.1). Define $\tau(\sigma)$ by (3.3). If either (3.4) or (3.5) holds, then all nontrivial Kneser solutions of (1.1) on $[a, \infty)$ are singular.*

As an example, consider the fourth-order equation

$$(|x''|^{\alpha}\operatorname{sgn} x'')'' = \kappa t^{-2(\alpha+1)}(1 + \sin t)|x|^{\beta}\operatorname{sgn} x, \quad t \ge 1, \tag{3.6}$$

where $\alpha > \beta > 0$, and κ is a positive constant. The equation (3.6) is a special case of (1.1) with $n = 4$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = \alpha$, $\alpha_4 = 1$, and $p(t) = \kappa t^{-2(\alpha+1)}(1 + \sin t)$. Then we have

$$\mu_4 = 2(2\alpha + 1), \quad \nu_4 = 2\alpha + 1, \quad \xi_4 = 2(\alpha + 1), \quad \zeta_4 = \frac{1}{\alpha}.$$

We can choose $\varepsilon_0 > 0$ sufficiently small so that

$$\frac{1}{2(\alpha+1)} < \frac{1+\varepsilon_0}{2(\alpha+1)} < \frac{1}{[\beta/\alpha](2\alpha+1)+1}$$

and

$$\varepsilon_0 < \alpha\left[1 - \left(\frac{\beta}{\alpha}(2\alpha+1)+1\right)\frac{1+\varepsilon_0}{2(\alpha+1)}\right].$$

For such $\varepsilon_0 > 0$, put

$$\sigma = \frac{1+\varepsilon_0}{2(\alpha+1)}.$$

Then, σ satisfies (3.1), and the number $\tau(\sigma)$ is given by

$$\begin{aligned}\tau(\sigma) &= \frac{1}{2(2\alpha+1)} \min\left\{2(\alpha+1)\sigma - 1,\ \alpha\left[1 - \left(\frac{\beta}{\alpha}(2\alpha+1)+1\right)\sigma\right]\right\} \\ &= \frac{\varepsilon_0}{2(2\alpha+1)}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&t^{\mu_4\tau(\sigma)} \int_t^{\infty} s^{-\mu_4\tau(\sigma)+(\nu_4+1)\sigma-1} p(s)^{\sigma} ds \\ &\quad = \kappa^{(1+\varepsilon_0)/[2(\alpha+1)]} t^{\varepsilon_0} \int_t^{\infty} s^{-1-\varepsilon_0}(1 + \sin s)^{(1+\varepsilon_0)/[2(\alpha+1)]} ds.\end{aligned}$$

If $m = 1, 2, \dots$, then

$$\begin{aligned}\int_{2m\pi}^{\infty} s^{-1-\varepsilon_0}(1+\sin s)^{(1+\varepsilon_0)/[2(\alpha+1)]}ds &\geq \sum_{i=0}^{\infty}\int_{2(m+i)\pi}^{(2(m+i)+1)\pi} s^{-1-\varepsilon_0}ds \\ &\geq \sum_{i=0}^{\infty}\left[(2(m+i)+1)\pi\right]^{-1-\varepsilon_0}\pi \geq \pi^{-\varepsilon_0}\int_m^{\infty}\frac{1}{(2s+1)^{1+\varepsilon_0}}ds \\ &= \frac{\pi^{-\varepsilon_0}}{2\varepsilon_0}(2m+1)^{-\varepsilon_0},\end{aligned}$$

and so

$$\liminf_{m\to\infty}(2m\pi)^{\varepsilon_0}\int_{2m\pi}^{\infty} s^{-1-\varepsilon_0}(1+\sin s)^{(1+\varepsilon_0)/[2(\alpha+1)]}ds \geq \frac{1}{2\varepsilon_0} > 0.$$

Consequently, we find that

$$\limsup_{t\to\infty} t^{\mu_4\tau(\sigma)}\int_t^{\infty} s^{-\mu_4\tau(\sigma)+(\nu_4+1)\sigma-1}p(s)^{\sigma}ds > 0.$$

By Theorem 3.1, it is concluded that all nontrivial Kneser solutions of (3.6) on $[1,\infty)$ are singular. Note that Theorem A cannot be applied to (3.6) since the lower limit as $t\to\infty$ of $t^{\nu_4+1}p(t)$ is equal to 0.

Now, let $\alpha_1\alpha_2\cdots\alpha_n > \beta$, and set

$$\sigma_n = \frac{\zeta_n+1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\nu_n+1+\zeta_n(\nu_n+1)}. \tag{3.7}$$

We have

$$\frac{1}{\nu_n+1} < \sigma_n < \frac{1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\,\nu_n+1}.$$

It is easily seen that if σ satisfies

$$\sigma_n \leq \sigma < \frac{1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\,\nu_n+1}, \tag{3.8}$$

then the number $\tau(\sigma)$ which is defined by (3.3) is

$$\tau(\sigma) = \frac{1}{\mu_n\zeta_n}\left[1-\left(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\nu_n+1\right)\sigma\right]. \tag{3.9}$$

Therefore Theorem 3.1 produces the following result.

Theorem 3.2. *Let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. Let σ be a number satisfying (3.8), where σ_n is given by (3.7), and define $\tau(\sigma)$ by (3.9). If either (3.4) or (3.5) holds, then all nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular.*

We have derived Theorem 3.1 from Theorem 1.1, and Theorem 3.2 from Theorem 3.1. We remark here that Theorem 1.1 can be derived from Theorem 3.2. In this sense, these three theorems are essentially identical. The following is a brief proof of the fact that Theorem 1.1 is derived from Theorem 3.2. Let $\sigma > 0$ and $\tau > 0$ be numbers which satisfy (1.9) and (1.10). As stated before, this is equivalent to the statement that σ and τ satisfy (3.1) and (3.2). Choose $\sigma^* > 0$ such that $\sigma = \sigma^*$ satisfies (3.8) and $\tau(\sigma^*)/\sigma^* < \tau/\sigma$ and $\sigma < \sigma^*$. Here, $\tau(\sigma^*)$ is given by (3.9) with $\sigma = \sigma^*$. If σ^* is taken sufficiently close to $1/\{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\,\nu_n + 1\}$, then it is possible to choose such a number σ^*. By the Höder inequality we find that

$$\int_{a^+}^{t} s^{-\mu_n\tau+(\nu_n+1)\sigma-1} p(s)^{\sigma} ds \le K_1 \left(\int_{a^+}^{t} s^{-\mu_n\tau(\sigma^*)+(\nu_n+1)\sigma^*-1} p(s)^{\sigma^*} ds \right)^{\sigma/\sigma^*}, \quad t \ge a^+,$$

and

$$t^{\mu_n\tau} \int_{t}^{\infty} s^{-\mu_n\tau+(\nu_n+1)\sigma-1} p(s)^{\sigma} ds \le K_2 \left(t^{\mu_n\tau(\sigma^*)} \int_{t}^{\infty} s^{-\mu_n\tau(\sigma^*)+(\nu_n+1)\sigma^*-1} p(s)^{\sigma^*} ds \right)^{\sigma/\sigma^*}, \quad t \ge a^+,$$

where K_1 and K_2 are certain positive constants. Therefore, (1.11) implies (3.4) with $\sigma = \sigma^*$, and (1.12) implies (3.5) with $\sigma = \sigma^*$. This means that Theorem 1.1 is derived from Theorem 3.2 of the case $\sigma = \sigma^*$.

It is also clear that if σ satisfies

$$\frac{1}{\nu_n + 1} < \sigma \le \sigma_n, \tag{3.10}$$

then the number $\tau(\sigma)$ defined by (3.3) is

$$\tau(\sigma) = \frac{1}{\mu_n}[(\nu_n + 1)\sigma - 1]. \tag{3.11}$$

Therefore, by Theorem 3.1, we have the following result.

Corollary 3.3. *Let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. Let σ be a number satisfying* (3.10)*, where σ_n is given by* (3.7)*, and define $\tau(\sigma)$ by* (3.11)*. If either* (3.4) *or* (3.5) *holds, then all nontrivial Kneser solutions of* (1.1) *on $[a,\infty)$ are singular.*

As mentioned before, if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then $D_i x(t) = x^{(i)}(t)$ $(i = 0, 1, 2, \ldots, n)$, and (1.1) is reduced to (1.13). Note that the singularity condition (1.7) is rewritten in the form

$$(-1)^i x^{(i)}(t) > 0 \quad \text{on } [a,b) \quad (i = 0, 1, 2, \ldots, n-1) \quad \text{and} \quad x(t) = 0 \quad (t \ge b).$$

Moreover, in the case $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, we have

$$\mu_n = \frac{n(n-1)}{2}, \quad \nu_n = n-1, \quad \xi_n = n, \quad \zeta_n = 1.$$

Therefore Theorem 3.2 yields the following result. For simplicity, we set $\rho_n(\sigma) = \mu_n\tau(\sigma)$.

Corollary 3.4. *Consider the equation* (1.13)*. Let $0 < \beta < 1$. Let σ be a number satisfying $2/[n + (n-1)\beta + 1] \le \sigma < 1/[(n-1)\beta + 1]$, and set $\rho_n(\sigma) = 1 - [(n-1)\beta + 1]\sigma$. If either*

$$\int_{a^+}^{\infty} s^{-\rho_n(\sigma)+n\sigma-1} p(s)^{\sigma} ds = \infty \quad (a^+ > \max\{a, 0\})$$

or

$$\limsup_{t\to\infty} t^{\rho_n(\sigma)} \int_{t}^{\infty} s^{-\rho_n(\sigma)+n\sigma-1} p(s)^{\sigma} ds > 0,$$

then all nontrivial Kneser solutions of (1.13) *on $[a,\infty)$ are singular.*

Corollary 3.4 has been formulated in the book of Kiguradze and Chanturia [3, Theorem 11.2 ($m = 0$, $k = 1$)].

By Corollary 3.3, we have the following result.

Corollary 3.5. *Consider the equation* (1.13)*. Let $0 < \beta < 1$. Let σ be a number satisfying $1/n < \sigma \le 2/[n + (n-1)\beta + 1]$. If either*

$$\int_{a}^{\infty} p(s)^{\sigma} ds = \infty \quad \textit{or} \quad \limsup_{t\to\infty} t^{n\sigma-1} \int_{t}^{\infty} p(s)^{\sigma} ds > 0,$$

then all nontrivial Kneser solutions of (1.13) *on $[a,\infty)$ are singular.*

Acknowledgements

The author would like to express his sincere gratitude to the referee for many valuable comments and suggestions.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **30**, 1–19; https://doi.org/10.14232/ejqtde.2021.1.30 www.math.u-szeged.hu/ejqtde/

Explicit solution and dynamical properties of atmospheric Ekman flows with boundary conditions

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Received 2 February 2020, appeared 5 April 2021
Communicated by Zuzana Došlá

Abstract. In this paper, we study the classical problem of the wind in the steady atmospheric Ekman layer with the constant eddy viscosity. Different from the previous work, we modify the boundary conditions and derive the explicit solution by using the notation of matrix cosine and matrix sine. For the arbitrary height-dependent eddy viscosity, we get the solution of the classical problem with zero velocity and acceleration at the bottom of the layer. In addition, uniqueness is shown and dynamical properties of solution are characterized.

Keywords: Ekman layer, variable eddy viscosity, explicit solutions, existence, dynamical properties.

2010 Mathematics Subject Classification: 34H05, 93B05.

1 Introduction

The Earth's atmosphere can be divided into several layers based on the behaviour of its temperature [11], these layers are, starting from ground level upwards, the troposphere, the stratosphere, the mesosphere and the thermosphere, A further region, beginning about 500 km above the ground level, is the exosphere, which fades away into the realm of interplanetary space. The troposphere contains more than 75% of all of the air in the atmosphere, and almost all of the water vapour (which forms clouds and rain). This is the region where the familiar weather phenomena occur. The lowest part-roughly the lower third-of the troposphere is called the atmospheric boundary layer, and it is here that friction plays an important role, while higher up, from the stratosphere upwards, the air flow is practically inviscid.

For a better understanding of the flow dynamics, it is useful to divide the atmospheric boundary layer into there parts [8, 11], i.e., the lamina sublayer, surface (Prandtl) layer and

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the Ekman layer(see Fig. 2.1), the lamina sublayer is only a few millimeters thick and is not relevant to the transfer of wind energy. Within the surface layer, confined to 20–100 meters of the atmosphere (above the lamina sublayer), the velocity profile is adjusted so that the horizontal frictional stress is nearly independent of height. In contrast to this, in the Ekman layer, located on top of the surface layer and extending to a height of about 1 km, on average, the flow is governed by a three-way balance among frictional effects, pressure gradient and the influence of the coriolis force [5,8,21]. Primarily the air flow is horizontal (the horizontal velocities are about 10^4 larger than the vertical velocity [20]).

The governing equations for mesoscale steady air flow at mid-latitudes in the Ekman layer are [8]

$$\begin{cases} f(v-v_g) = -\frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}), \\ f(u-u_g) = \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}), \end{cases} \tag{1.1}$$

where (u,v) represents the horizontal wind velocity, with zonal (West-to-East, in the sense of the Earth's rotation) component $u = u(t,x,y,z)$ and meridional (positive meaning towards the North Pole) component $v = v(t,x,y,z)$, u_g and v_g are the corresponding geostrophic wind component, k denotes the eddy viscosity, $f = 2\Omega\sin\theta$ is the Coriolis parameter at the fixed latitude θ in the Northern Hemisphere and $\Omega \approx 7.29\times10^{-5}s^{-1}$ is the angular speed of rotation of the Earth and $\theta \in (0,\pi/2]$ is the angle of latitude in right-handed rotating spherical coordinates ($\theta = 0$ corresponding to the Equator and $\theta = \pi/2$ to the North Pole).

The boundary conditions for the system (1.1) are

$$u = v = 0 \quad \text{at } z = 0, \tag{1.2}$$

and

$$u \to u_g, \qquad v \to v_g \quad \text{for } z \to \infty, \tag{1.3}$$

expressing the fact that, due to the frictional properties of the flow below the Ekman layer, a no-slip condition holds at the bottom $z = 0$ of the layer, while at the top of the Ekman layer the horizontal components of the wind must be in geostrophic balance: above the Ekman layer the flow is geostrophic (pressure-driven).

If k is a constant, then we can obtain the explicit formula of the solution to (1.1) with (1.2) and (1.3) by the classic Ekman theory, but this assumption is too restrictive. The dynamics of the atmospheric boundary-layer is very important in applications, for example, other than meteorology (weather prediction and climate studies), in the control and management of air pollution (since the dispersal of smog in urban environments depends strongly on meteorological conditions) and in agriculture (e.g. dewfall and frost formation). For this reason, it is important, both from the theoretical as well as from the practical point of view, to understand the flow dynamics of the atmospheric boundary-layer in the context of height-dependent eddy viscosities. The available explicit solutions for height-dependent eddy viscosities are very scare, being apparently restricted to special cases, for example, $k(z)$ denote linear and exponentially decaying functions [10,12] or $k(z)$ is a quadratic polynomial [15]. It is remarkable that Constantin and Johnson [2] studied the Atmospheric Ekman flows with variable eddy viscosity $k(z)$ which is a perturbation of the asymptotic and verify the existence of the solution by transforming the Ekman flows into a suitable integral equation and apply iterative technique to give an efficient approach to find the explicit solution, so that for other types of non-constant eddy viscosity we have to rely on case-by-case approximations and numerical simulations [4,6,9,13,14,16].

Remark 1.1. When $z = \pi\sqrt{\frac{2k}{f}}$, the wind (u, v) is parallel to and nearly equally to the geostrophic value (u_g, v_g), it is conventional to designate this level as the top of the Ekman layer [8], so we can change the condition (1.3) to

$$u = u_g, \qquad v = v_g, \quad \text{at } z = z_0, \tag{1.4}$$

where $z_0 > \pi\sqrt{\frac{2k}{f}}$.

For a constant eddy viscosity k, we can obtain the explicit formula of the solution to (1.1) with (1.2) and (1.3). Based on Remark 1.1, we consider (1.1) with (1.2) and (1.4). The first contribution of this paper is to apply the technique of second linear ODEs (using the notion of sin and cos matrix) to find the explicit solution of (1.1) with (1.2) and (1.4) and give a directly approach to compute the explicit solution.

If we assume the velocity and acceleration at the bottom of the layer are zero, then (1.2) is retained and (1.3) is changed into

$$u' = 0, \qquad v' = 0 \quad \text{at } z = 0, \tag{1.5}$$

so the second aim of this paper is to investigate the explicit solution of (1.1) with (1.2) and (1.5) for an arbitrary height-dependent eddy viscosity $k(z)$. We use the closed form of function series to give the representation of solutions. By using integral change and introducing Green function, a spectrum theorem of a corresponding anti-symmetric compact operator is used to deriving the uniqueness result. Finally, some dynamical properties of solution like asymptotic property, Lyapunov exponents, and stable manifold are characterized.

2 Model description

Motivated by [8], we give the details to derive (1.1) by dividing into four steps.

Step 1. We set up the momentum equation in rotating coordinates.

We derive the relationship between the total derivative of a vector in an inertial reference frame and the corresponding total derivative in a rotating system. Let $\overrightarrow{A}$ be an arbitrary vector whose Cartesian components in an inertial frame given by

$$\overrightarrow{A} = \overrightarrow{i'} A'_x + \overrightarrow{j'} A'_y + \overrightarrow{k'} A'_z$$

and whose components in a frame rotating with the angular velocity $\overrightarrow{\Omega}$ are

$$\overrightarrow{A} = \overrightarrow{i} A_x + \overrightarrow{j} A_y + \overrightarrow{k} A_z,$$

here $\overrightarrow{i}$, $\overrightarrow{j}$, $\overrightarrow{k}$ are unit vectors which are taken to be directed eastward, northward, and upward, respectively, $\overrightarrow{\Omega} = (0, \Omega \sin\phi, \Omega\cos\phi)$, ϕ is the latitude.

Letting $\frac{D_\alpha \overrightarrow{A}}{Dt}$ be the total derivative of $\overrightarrow{A}$ in the inertial frame, we can write

$$\begin{aligned}\frac{D_\alpha \overrightarrow{A}}{Dt} &= \overrightarrow{i'}\frac{DA'_x}{Dt} + \overrightarrow{j'}\frac{DA'_y}{Dt} + \overrightarrow{k'}\frac{DA'_z}{Dt} \\ &= \overrightarrow{i}\frac{Du}{Dt} + \overrightarrow{j}\frac{Dv}{Dt} + \overrightarrow{k}\frac{Dw}{Dt} + \frac{D_\alpha \overrightarrow{i}}{Dt}u + \frac{D_\alpha \overrightarrow{j}}{Dt}v + \frac{D_\alpha \overrightarrow{k}}{Dt}w,\end{aligned}$$

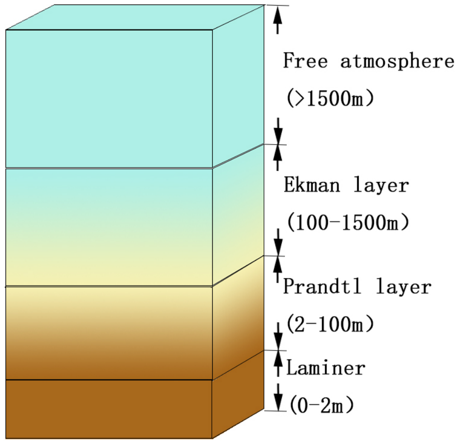

Figure 2.1: Ekman layer, surface layer and lamina sublayer are called the atmosphere boundary layer.

the first three terms on the left line above can be combined to give

$$\frac{D\overrightarrow{A}}{Dt}=\overrightarrow{i}\frac{DA_x}{Dt}+\overrightarrow{j}\frac{DA_y}{Dt}+\overrightarrow{k}\frac{DA_z}{Dt},$$

which is just the total derivative of $\overrightarrow{A}$ as viewed in the rotating coordinates. By direct calculation [8], we get

$$\frac{D_\alpha\overrightarrow{i}}{Dt}=\overrightarrow{\Omega}\times\overrightarrow{i},\qquad \frac{D_\alpha\overrightarrow{j}}{Dt}=\overrightarrow{\Omega}\times\overrightarrow{j},\qquad \frac{D_\alpha\overrightarrow{k}}{Dt}=\overrightarrow{\Omega}\times\overrightarrow{k},$$

there, the total derivative for $\overrightarrow{A}$ in an inertial frame is related to that in a rotating frame by

$$\frac{D_\alpha\overrightarrow{A}}{Dt}=\frac{D\overrightarrow{A}}{Dt}+\overrightarrow{\Omega}\times\overrightarrow{A}. \tag{2.1}$$

For a given air parcel the location (x,y,z) is a given function of t so that $x=x(t), y=y(t), z=z(t)$, let $\frac{Dx}{Dt}=u, \frac{Dy}{Dt}=v, \frac{Dz}{Dt}=w$, then u,v,w are the velocity components in the x,y,z directions, respectively, let $\overrightarrow{U}$ is the velocity vector , then $\overrightarrow{U}=\overrightarrow{i}u+\overrightarrow{j}v+\overrightarrow{k}w$.

In an inertial reference frame, Newton's second law of motion may be written as

$$\sum\overrightarrow{F}=\frac{D_\alpha\overrightarrow{U_\alpha}}{Dt},$$

here $\frac{D_\alpha\overrightarrow{U}_\alpha}{Dt}$ is the rate of change of the absolute velocity U_α. On the rotating Earth, if $\overrightarrow{r}$ is a position vector for an air parcel, from the (2.1), we get

$$\frac{D_\alpha\overrightarrow{r}}{Dt}=\frac{D\overrightarrow{r}}{Dt}+\overrightarrow{\Omega}\times\overrightarrow{r},$$

but $\frac{D_\alpha \vec{r}}{Dt} = \vec{U_\alpha}$, $\frac{D\vec{r}}{Dt} = \vec{U}$, so we obtain

$$\vec{U_\alpha} = \vec{U} + \vec{\Omega} \times \vec{r}. \tag{2.2}$$

We apply (2.1) to $\vec{U_\alpha}$ and obtain

$$\frac{D_\alpha \vec{U_\alpha}}{Dt} = \frac{D\vec{U_\alpha}}{Dt} + \vec{\Omega} \times \vec{U_\alpha}.$$

Using (2.2), we get

$$\begin{aligned}\frac{D_\alpha \vec{U}_\alpha}{Dt} &= \frac{D\vec{U_\alpha}}{Dt} + \vec{\Omega} \times \vec{U_\alpha} \\ &= \frac{D}{Dt}(\vec{U} + \vec{\Omega} \times \vec{r}) + \vec{\Omega} \times (\vec{U} + \vec{\Omega} \times \vec{r}) \\ &= \frac{D\vec{U}}{Dt} + 2\vec{\Omega} \times \vec{U} - \Omega^2 \vec{R},\end{aligned}$$

here $\vec{R}$ is a vector with direction perpendicular to the axis of rotation, and the magnitude equal to the distance to the axis of rotation.

If we assume that the only real forces acting on the atmosphere are the pressure gradient force $\vec{F_p}$, gravitation force $\vec{F_g}$ and friction force $\vec{F_r}$, then we have

$$\frac{D\vec{U}}{Dt} = \vec{F_g} + \vec{F_p} + \vec{F_r},$$

so we get

$$\frac{D\vec{U}}{Dt} = -2\vec{\Omega} \times \vec{U} + \Omega^2 \vec{R} + \vec{F_g} + \vec{F_p} + \vec{F_r}. \tag{2.3}$$

Step 2. We set up the component equations in spherical coordinates.

Let (λ, ϕ, z) be the spherical coordinates, λ is longitude, ϕ is latitude, and z is the vertical distance above the surface of the Earth, using the formula for the transformation of local rectangular coordinate system and spherical coordinate system, we can get the following relationships,

$$dx = a\cos\phi d\lambda, \qquad dy = ad\phi, \qquad dz = dr,$$

where a is the radius of the Earth, r is the distance to the center of the Earth, which is related to z by $r = a + z$.

The direction of the $\vec{i}$, $\vec{j}$, $\vec{k}$ unit vectors are not constant, they are the functions of position on the spherical Earth, thus we write

$$\frac{D\vec{U}}{Dt} = \vec{i}\frac{Du}{Dt} + \vec{j}\frac{Dv}{Dt} + \vec{k}\frac{Dw}{Dt} + u\frac{D\vec{i}}{Dt} + v\frac{D\vec{j}}{Dt} + w\frac{D\vec{k}}{Dt}, \tag{2.4}$$

from [8], we get

$$\frac{D\vec{i}}{Dt} = \frac{u}{a\cos\phi}\left(\vec{j}\sin\phi - \vec{k}\cos\phi\right), \qquad \frac{D\vec{j}}{Dt} = -\frac{u\tan\phi}{a}\vec{i} - \frac{v}{a}\vec{k}, \tag{2.5}$$

and

$$\frac{D\vec{k}}{Dt} = \frac{u}{a}\vec{i} + \frac{v}{a}\vec{j}, \tag{2.6}$$

substituting (2.5) and (2.6) into (2.4) and rearranging the terms, we obtain

$$\begin{aligned}\frac{D\vec{U}}{Dt} = & \left(\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a}\right)\vec{i} + \left(\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a}\right)\vec{j} \\ & + \left(\frac{Dw}{Dt} - \frac{u^2+v^2}{a}\right)\vec{k}.\end{aligned} \tag{2.7}$$

We know that

$$\Omega^2\vec{R} + \vec{F_g} = \vec{g}, \tag{2.8}$$

and

$$\begin{aligned}-2\vec{\Omega}\times\vec{U} &= -2\Omega\begin{bmatrix}\vec{i} & \vec{j} & \vec{k} \\ 0 & \cos\phi & \sin\phi \\ u & v & w\end{bmatrix} \\ &= -(2\Omega w\cos\phi - 2\Omega v\sin\phi)\vec{i} - 2\Omega u\sin\phi\vec{j} + 2\Omega u\cos\phi\vec{k}.\end{aligned} \tag{2.9}$$

We consider an infinitesimal volume element of air, $\delta V = \delta x\delta y\delta z$, center at the point (x_0, y_0, z_0) (see Fig. 2.2), so we can easily get the total pressure gradient force per unit mass is

$$\vec{F_p} = \frac{1}{\rho}\nabla\vec{p} = \vec{i}\frac{1}{\rho}\frac{\partial p}{\partial x} + \vec{j}\frac{1}{\rho}\frac{\partial P}{\partial y} + \vec{k}\frac{1}{\rho}\frac{\partial P}{\partial z}, \tag{2.10}$$

we know that

$$\vec{g} = -\vec{k}g, \tag{2.11}$$

and

$$\vec{F_r} = \vec{i}F_{rx} + \vec{j}F_{ry} + \vec{k}F_{rz}, \tag{2.12}$$

where

$$\begin{cases} F_{rx} = v[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}], \\ F_{ry} = v[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}], \\ F_{rz} = v[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}], \end{cases}$$

$v = \frac{\mu}{\rho}$ is the kinematic viscosity coefficient [8].

From (2.3) and using (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), we get the following equations

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{1}{\cos\phi}\frac{\partial P}{\partial\lambda} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + \frac{uv\tan\phi}{a} - \frac{uw}{a} + F_{rx}, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{1}{a}\frac{\partial P}{\partial\phi} - 2\Omega u\sin\phi - \frac{u^2\tan\phi}{a} - \frac{vw}{a} + F_{ry}, \\ \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g - 2\Omega u\cos\phi + \frac{u^2+v^2}{a} - \frac{uw}{a} + F_{rz}. \end{cases} \tag{2.13}$$

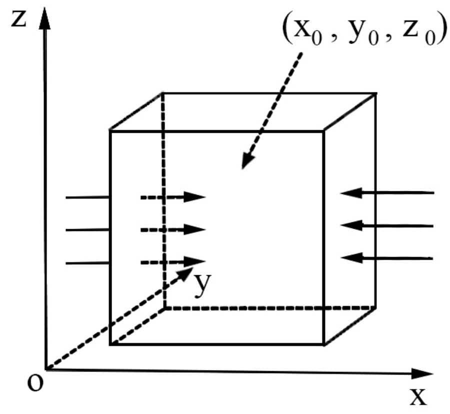

Figure 2.2: The x component of the pressure gradient forcer.

Step 3. We simplify (2.13) in local rectangular coordinates system.

The table 2.1 in [8] shows the terms proportional to $\frac{1}{a}$ on the above equations are unimportant for midlatitude synoptic scale motions, so we omit this terms and get

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega v \sin\phi - 2\Omega w \cos\phi + F_{rx}, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega u \sin\phi + F_{ry}, \\ \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g - 2\Omega u \cos\phi + F_{rz}. \end{cases}$$

As $u = u(t, x, y, z)$, and $\frac{Dx}{Dt} = u$, $\frac{Dy}{Dt} = v$, $\frac{Dz}{Dt} = w$, we get

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

$\frac{Dv}{Dt}$ and $\frac{Dw}{Dt}$ are similar.

For a wide range of air movements, $w \ll u, v$ [21], so we assume $w = 0$, for the atmosphere below 100km, kinematic viscosity coefficient is negligible except in a thin layer within a few centimeters of the Earth's surface where the vertical shear is very large [8], so $F_{rx} = 0$, $F_{ry} = 0$ in Ekman layer, as shown in chapter 3 in [8], the magnitude of w can be deduced from knowledge of the horizontal velocity u, v, so we omit the last equation of the system and get

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega v \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega u \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu. \end{cases}$$

Step 4. We set up the mean equations.

In a turbulent fluid, a field variable such as velocity measured at a point generally fluctuates rapidly in time as eddies of various scales pass the point, so we assume that the field variables can be separated into slowing varying turbulent components, for example, $u = \overline{u} + u'$, the corresponding means are indicated by overbars and the fluctuating component by primes. With the aid of the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

and the chain rule of the differentiation, we get

$$\begin{aligned}\frac{Du}{Dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ &= \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z}. \end{aligned} \tag{2.14}$$

Separating each dependant variable into mean and fluctuating parts, substituting into (2.14), and averaging then yields

$$\overline{\frac{Du}{Dt}} = \frac{\partial \overline{u}}{\partial t} + \frac{\partial}{\partial x}(\overline{u}\,\overline{u} + \overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u}\,\overline{v} + \overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u}\,\overline{w} + \overline{u'w'}).$$

Noting that the mean velocity fields satisfy the continuity equation, we get

$$\overline{\frac{Du}{Dt}} = \frac{\overline{D}\,\overline{u}}{Dt} + \frac{\partial}{\partial x}(\overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'}),$$

where

$$\frac{\overline{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z}$$

is the rate of change following the mean motion, the mean equations thus have the following form,

$$\begin{cases} \frac{\overline{D}\overline{u}}{Dt} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} + f\overline{v} - [\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}], \\ \frac{\overline{D}\overline{v}}{Dt} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial y} - f\overline{u} - [\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} + \frac{\partial \overline{v'w'}}{\partial z}]. \end{cases}$$

Away from region with horizontal inhomogeneities (e.g., shorelines terms, forest edges), we can assume turbulent fluxes are horizontally homogeneous because they are too small in comparison to the term involving vertical differentiation [8], so we assume $\frac{\partial \overline{u'u'}}{\partial x} = \frac{\partial \overline{u'v'}}{\partial y} = \frac{\partial \overline{u'v'}}{\partial x} = \frac{\partial \overline{v'v'}}{\partial y} = 0$.

Outside the boundary layer, the resulting approximation was geostrophic balance, i.e.,

$$\begin{cases} \frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} = f\overline{v_g}, \\ \frac{1}{\rho}\frac{\partial \overline{P}}{\partial y} = -f\overline{u_g}. \end{cases}$$

For midlatitude synoptic-scale motions, the inertial acceleration terms (the terms on the left of above equations) can be neglected compared to the Cariolis force and pressure gradient force terms [8], so we get

$$\begin{cases} f(\overline{v} - \overline{v_g}) - \frac{\partial \overline{u'w'}}{\partial z} = 0, \\ -f(\overline{u} - \overline{u_g}) - \frac{\partial \overline{v'w'}}{\partial z} = 0. \end{cases}$$

By the Flux-Gradient theory, we get

$$\begin{cases} \overline{u'w'} = -k(\frac{\partial \overline{u}}{\partial z}), \\ \overline{v'w'} = -k(\frac{\partial \overline{v}}{\partial z}), \end{cases}$$

where $k(m^2s^{-1})$ is the eddy viscosity coefficient, then we have

$$\begin{cases} f(\overline{v} - \overline{v_g}) = -\frac{\partial}{\partial z}(k\frac{\partial \overline{u}}{\partial z}), \\ f(\overline{u} - \overline{u_g}) = \frac{\partial}{\partial z}(k\frac{\partial \overline{v}}{\partial z}). \end{cases}$$

Finally, we omit the overbars for simplicity to obtain (1.1).

3 Main results

3.1 Existence of explicit solution

Note that if k reduces to a constant, then (1.1) reduces to

$$\begin{cases} \frac{d^2v}{dz^2} = \frac{f}{k}(u - u_g), \\ \frac{d^2u}{dz^2} = -\frac{f}{k}(v - v_g). \end{cases} \tag{3.1}$$

Based on Remark 1.1, we change the condition (1.3) to (1.4) in the following theorems, and we try to find explicit solution of (3.1) with (1.2) and (1.4) by using the notion of sin and cos matrices.

Definition 3.1 ((see [7]). It is well known that

$$\sin \Omega z = \Omega \frac{z}{1!} - \Omega^3 \frac{z^3}{3!} + \cdots + (-1)^k \Omega^{2k+1} \frac{z^{2k+1}}{(2k+1)!} + \cdots,$$
$$\cos \Omega z = I - \Omega^2 \frac{z^2}{2!} + \cdots + (-1)^k \Omega^{2k} \frac{z^{2k}}{(2k)!} + \cdots.$$

Theorem 3.2. *The solution of* (3.1) *with* (1.2) *and* (1.4) *can be expressed by the following formula*

$$\begin{bmatrix} v \\ u \end{bmatrix} = \cos \Omega z \begin{bmatrix} -v_g \\ -u_g \end{bmatrix} + \sin \Omega z \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} + \begin{bmatrix} v_g \\ u_g \end{bmatrix}, \tag{3.2}$$

where

$$\Omega = \begin{bmatrix} \sqrt{\frac{f}{2k}} & -\sqrt{\frac{f}{2k}} \\ \sqrt{\frac{f}{2k}} & \sqrt{\frac{f}{2k}} \end{bmatrix},$$

and

$$\begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} = (\sin \Omega z_0)^{-1} \cos \Omega z_0 \begin{bmatrix} v_g \\ u_g \end{bmatrix}.$$

Remark 3.3. Note that $(\sin\Omega z_0)^{-1}$ does exist because z_0 is a positive number, so using *Wolfram Mathematica*, $(\sin\Omega z_0)^{-1}\cos\Omega z_0$ can be solved by the following computations:

$$\sin\Omega z_0 = \begin{pmatrix} \sin\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0 & -\cos\sqrt{\frac{f}{2k}}z_0\sinh\sqrt{\frac{f}{2k}}z_0 \\ \cos\sqrt{\frac{f}{2k}}z_0\sinh\sqrt{\frac{f}{2k}}z_0 & \sin\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0 \end{pmatrix},$$

$$\det\sin\Omega z_0 = \frac{1}{2}\left(\cosh\left(2\sqrt{\frac{f}{2k}}z_0\right) - \cos\left(2\sqrt{\frac{f}{2k}}z_0\right)\right) > 0,$$

$$(\sin\Omega z_0)^{-1} = \begin{pmatrix} \frac{2\sin\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0}{\cosh(2\sqrt{\frac{f}{2k}}z_0)-\cos(2\sqrt{\frac{f}{2k}}z_0)} & \frac{2\cos(\sqrt{\frac{f}{2k}}z_0)\sinh\sqrt{\frac{f}{2k}}z_0}{\cosh(2\sqrt{\frac{f}{2k}}z_0)-\cos(2\sqrt{\frac{f}{2k}}z_0)} \\ \frac{2\cos\sqrt{\frac{f}{2k}}z_0\sinh\sqrt{\frac{f}{2k}}z_0}{\cos(2\sqrt{\frac{f}{2k}}z_0)-\cosh(2\sqrt{\frac{f}{2k}}z_0)} & \frac{2\sin\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0}{\cosh(2\sqrt{\frac{f}{2k}}z_0)-\cos(2\sqrt{\frac{f}{2k}}z_0)} \end{pmatrix},$$

and

$$\cos\Omega z_0 = \begin{pmatrix} \cos\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0 & \sin\sqrt{\frac{f}{2k}}z_0\sinh\sqrt{\frac{f}{2k}}z_0 \\ -\sin\sqrt{\frac{f}{2k}}z_0\sinh\sqrt{\frac{f}{2k}}z_0 & \cos\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0 \end{pmatrix},$$

$$\det\cos\Omega z_0 = \frac{1}{2}\left(\cosh\left(2\sqrt{\frac{f}{2k}}z_0\right) + \cos\left(2\sqrt{\frac{f}{2k}}z_0\right)\right) > 0,$$

$$(\sin\Omega z_0)^{-1}\cos\Omega z_0 = \begin{pmatrix} -\frac{\sin(2\sqrt{\frac{f}{2k}}z_0)}{\cos(2\sqrt{\frac{f}{2k}}z_0)-\cosh(2\sqrt{\frac{f}{2k}}z_0)} & -\frac{\sinh(2\sqrt{\frac{f}{2k}}z_0)}{\cos(2\sqrt{\frac{f}{2k}}z_0)-\cosh(2\sqrt{\frac{f}{2k}}z_0)} \\ \frac{\sinh(2\sqrt{\frac{f}{2k}}z_0)}{\cos(2k)-\cosh(2\sqrt{\frac{f}{2k}}z_0)} & -\frac{\sin(2\sqrt{\frac{f}{2k}}z_0)}{\cos(2\sqrt{\frac{f}{2k}}z_0)-\cosh(2\sqrt{\frac{f}{2k}}z_0)} \end{pmatrix},$$

and

$$\det\left((\sin\Omega z_0)^{-1}\cos\Omega z_0\right) = \frac{\cos\left(2\sqrt{\frac{f}{2k}}z_0\right) + \cosh\left(2\sqrt{\frac{f}{2k}}z_0\right)}{\cosh\left(2\sqrt{\frac{f}{2k}}z_0\right) - \cos\left(2\sqrt{\frac{f}{2k}}z_0\right)} > 0.$$

Proof. Let $U = u - u_g$, $V = v - v_g$ and $\bar{k} = \frac{f}{k}$. Then (3.1) becomes

$$\begin{cases} \frac{d^2V}{dz^2} = \bar{k}U, \\ \frac{d^2U}{dz^2} = -\bar{k}V, \end{cases} \tag{3.3}$$

and the conditions (1.2), (1.4) are transformed into the equivalent forms

$$U = -u_g, \quad V = -v_g \quad \text{at } z = 0, \tag{3.4}$$

$$U = 0, \quad V = 0 \quad \text{at } z = z_0. \tag{3.5}$$

From the (3.3), we get

$$\begin{bmatrix} V \\ U \end{bmatrix}'' + \begin{bmatrix} 0 & -\bar{k} \\ \bar{k} & 0 \end{bmatrix}\begin{bmatrix} V \\ U \end{bmatrix} = 0.$$

By using the matrix Ω, we obtain

$$\begin{bmatrix} V \\ U \end{bmatrix}'' + \Omega^2\begin{bmatrix} V \\ U \end{bmatrix} = 0. \tag{3.6}$$

So we get the solution of the (3.6) as following form,

$$\begin{bmatrix} V \\ U \end{bmatrix} = \cos\Omega z \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} + \sin\Omega z \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix}.$$

We determine the constants such that the initial conditions (3.4) and (3.5) are satisfied. Considering the condition (3.4), we get

$$C_{11} = -v_g, \qquad C_{12} = -u_g.$$

Considering the condition (3.5), we obtain

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \cos\Omega z_0 \begin{bmatrix} -v_g \\ -u_g \end{bmatrix} + \sin\Omega z_0 \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix}.$$

Because the matrix $\sin\Omega z_0$ is nonsingular, so we get

$$\begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} = (\sin\Omega z_0)^{-1} \cos\Omega z_0 \begin{bmatrix} v_g \\ u_g \end{bmatrix}.$$

As $U = u - u_g$, $V = v - v_g$, so we obtain (3.2). □

We recall the following result.

Lemma 3.4 (see [1,18]). *For the matrix equation*

$$\Phi'(t,t_0) = A(t)\Phi(t,t_0), \qquad t \in [t_0, t_\alpha]$$

with the initial boundary condition $\Phi(t_0,t_0) = I$, where the matrix $\Phi(t,t_0)$ and $A(t)$ are $n \times n$ matrices, $t_\alpha > t_0 \geq 0$, the solution $\Phi(t,t_0)$ is given by

$$\Phi(t,t_0) = I + \int_{t_0}^{t} A(\tau)d\tau + \int_{t_0}^{t} A(\tau_1)\left[\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2\right] d\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\int_{t_0}^{\tau_2} A(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \cdots$$

For (1.1), we assume the $k = k(z) \neq 0$, then we will get

$$\begin{cases} \frac{d^2 v}{dz^2} + \frac{k'(z)}{k(z)}\frac{dv}{dz} = \frac{f}{k(z)}(u - u_g), \\ \frac{d^2 u}{dz^2} + \frac{k'(z)}{k(z)}\frac{du}{dz} = -\frac{f}{k(z)}(v - v_g). \end{cases}$$

Let $u - u_g = U$, $v - v_g = V$, then we will get

$$\begin{cases} \frac{d^2 V}{dz^2} + \alpha(z)\frac{dV}{dz} = \beta(z)U, \\ \frac{d^2 U}{dz^2} + \alpha(z)\frac{dU}{dz} = -\beta(z)V, \end{cases}$$

where $\alpha(z) = \frac{k'(z)}{k(z)}$, $\beta(z) = \frac{f}{k(z)}$, and the conditions (1.2) and (1.5) will become

$$U(0) = -u_g, \qquad V(0) = -v_g, \tag{3.7}$$

and

$$U'(0) = 0, \qquad V'(0) = 0. \tag{3.8}$$

Let $V'(z) = w_1,\ U'(z) = w_2$, then we obtain

$$X'(z) = A(z)X(z), \qquad X(0) = X_0, \tag{3.9}$$

where

$$X = \begin{bmatrix} V \\ U \\ W_1 \\ W_2 \end{bmatrix}, \qquad X_0 = \begin{bmatrix} -v_g \\ -u_g \\ 0 \\ 0 \end{bmatrix},$$

and

$$A(z) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \beta(z) & -\alpha(z) & 0 \\ -\beta(z) & 0 & 0 & -\alpha(z) \end{bmatrix}.$$

We get the solution of (3.9) by using Lemma 3.4, that is

$$X(z) = \Phi(z, z_0)X_0,$$

where

$$\Phi(z, z_0) = I + \int_0^z A(\tau)d\tau + \int_0^z A(\tau_1)\left[\int_0^{\tau_1} A(\tau_2)d\tau_2\right] d\tau_1 + \cdots,$$

as $u - u_g = U, v - v_g = V$, so we get the solution of (1.1) with the conditions (1.2) and (1.5).

If $k(z)$ is constant, then we will solve (3.9) with the conditions (1.2) and (1.5).

Remark 3.5. If $k(z)$ is a constant k, then $\alpha(z) = 0$, $\beta(z) = \frac{f}{k}$, and (1.1) will become the following form,

$$X'(z) = AX(z), \tag{3.10}$$

the corresponding initial conditions are

$$X(0) = X_0,$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \end{bmatrix}. \tag{3.11}$$

The characteristic equation of (3.11) is

$$\lambda^4 + \beta^2 = 0,$$

so we get the four eigenvalues:

$$\lambda_1 = \sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}i, \quad \lambda_2 = -\sqrt{\frac{f}{2k}} - \sqrt{\frac{f}{2k}}i, \quad \lambda_3 = \sqrt{\frac{f}{2k}} - \sqrt{\frac{f}{2k}}i, \quad \lambda_4 = -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}i.$$

Let $\lambda = \lambda_1$, then we have

$$(A - \lambda_1 I) = \begin{bmatrix} -\lambda_1 & 0 & 1 & 0 \\ 0 & -\lambda_1 & 0 & 1 \\ 0 & \beta & -\lambda_1 & 0 \\ -\beta & 0 & 0 & -\lambda_1 \end{bmatrix},$$

so the corresponding eigenvector is

$$\xi_1 = \begin{bmatrix} 1 \\ i \\ \sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}} i \\ -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}} i \end{bmatrix},$$

thus we obtain

$$e^{\lambda_1 z}\xi_1 = e^{\sqrt{\frac{f}{2k}}z} \begin{bmatrix} \cos\sqrt{\frac{f}{2k}}z + i\sin\sqrt{\frac{f}{2k}}z \\ -\sin\sqrt{\frac{f}{2k}}z + i\cos\sqrt{\frac{f}{2k}}z \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) + \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) i \\ -\sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) + \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) i \end{bmatrix}.$$

The two linear independent solutions are obtained:

$$X_1(z) = e^{\sqrt{\frac{f}{2k}}z} \begin{bmatrix} \cos\sqrt{\frac{f}{2k}}z \\ -\sin\sqrt{\frac{f}{2k}}z \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) \\ -\sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) \end{bmatrix},$$

and

$$X_2(z) = e^{\sqrt{\frac{f}{2k}}z} \begin{bmatrix} \sin\sqrt{\frac{f}{2k}}z \\ \cos\sqrt{\frac{f}{2k}}z \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) \end{bmatrix}.$$

Similarly, let $\lambda_3 = -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}} i$, we will get the eigenvector

$$\xi_2 = \begin{bmatrix} 1 \\ -i \\ -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}} i \\ \sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}} i \end{bmatrix},$$

therefore we have

$$e^{\lambda_2 z}\xi_2 = e^{-\sqrt{\frac{f}{2k}}z}\begin{bmatrix} \cos\sqrt{\frac{f}{2k}}z + i\sin\sqrt{\frac{f}{2k}}z \\ \sin\sqrt{\frac{f}{2k}}z - i\cos\sqrt{\frac{f}{2k}}z \\ -\sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) + \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right)i \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) + \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right)i \end{bmatrix}.$$

The two linear independent solutions can be stated as follows,

$$X_3(z) = e^{-\sqrt{\frac{f}{2k}}z}\begin{bmatrix} \cos\sqrt{\frac{f}{2k}}z \\ \sin\sqrt{\frac{f}{2k}}z \\ -\sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) \end{bmatrix},$$

and

$$X_4(z) = e^{-\sqrt{\frac{f}{2k}}z}\begin{bmatrix} \sin\sqrt{\frac{f}{2k}}z \\ -\cos\sqrt{\frac{f}{2k}}z \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) \end{bmatrix}.$$

So the general solution of (3.9) is

$$X(z) = c_1X_1(z) + c_2X_2(z) + c_3X_3(z) + c_4X_4(z),$$

then

$$V = c_1e^{\sqrt{\frac{f}{2k}}z}\cos\sqrt{\frac{f}{2k}}z + c_2e^{\sqrt{\frac{f}{2k}}z}\sin\sqrt{\frac{f}{2k}}z + c_3e^{-\sqrt{\frac{f}{2k}}z}\cos\sqrt{\frac{f}{2k}}z + c_4e^{-\sqrt{\frac{f}{2k}}z}\sin\sqrt{\frac{f}{2k}}z, \tag{3.12}$$

and

$$\begin{aligned} U = & -c_1e^{\sqrt{\frac{f}{2k}}z}\sin\sqrt{\frac{f}{2k}}z + c_2e^{\sqrt{\frac{f}{2k}}z}\cos\sqrt{\frac{f}{2k}}z \\ & + c_3e^{-\sqrt{\frac{f}{2k}}z}\sin\sqrt{\frac{f}{2k}}z - c_4e^{-\sqrt{\frac{f}{2k}}z}\cos\sqrt{\frac{f}{2k}}z. \end{aligned} \tag{3.13}$$

By using the conditions (3.7), (3.8), we get $c_1 = c_3 = -\frac{1}{2}v_g$, $c_2 = -\frac{1}{2}u_g$, $c_4 = \frac{1}{2}u_g$, so the solution of (3.10) with the conditions (3.7), (3.8) is obtained.

Remark 3.6. From the above example, we know that the general solution of (3.10) is (3.12), (3.13), so if we use the traditional boundary conditions (1.2), (1.3), then we will get

$$c_1 = c_2 = 0, \qquad c_3 = -v_g, \qquad c_4 = u_g,$$

then we have

$$\begin{cases} V = e^{-\sqrt{\frac{f}{2k}}z}\left(u_g \sin\sqrt{\frac{f}{2k}}z\right) - v_g \cos\sqrt{\frac{f}{2k}}z, \\ U = e^{-\sqrt{\frac{f}{2k}}z}\left(-v_g \sin\sqrt{\frac{f}{2k}}z\right) - u_g \cos\sqrt{\frac{f}{2k}}z, \end{cases}$$

so the solution is

$$\begin{cases} v = v_g + e^{-\sqrt{\frac{f}{2k}}z}\left(u_g \sin\sqrt{\frac{f}{2k}}z\right) - v_g \cos\sqrt{\frac{f}{2k}}z, \\ u = u_g + e^{-\sqrt{\frac{f}{2k}}z}\left(-v_g \sin\sqrt{\frac{f}{2k}}z\right) - u_g \cos\sqrt{\frac{f}{2k}}z, \end{cases}$$

this coincides with the result in [8].

3.2 Uniqueness

For the constant k, the explicit solution of (1.1) with (1.2) and (1.4) is obtained by Theorem 3.2, in the following theorem, we try to find the uniqueness for $k(z)$.

Theorem 3.7. *Assume $f \neq 0$, then there is a unique solution of* (1.1) *with conditions* (1.2) *and* (1.4)*.*

Proof. Let $\hat{u}$ and $\hat{v}$ be the solutions of (1.1) for $f = 0$ with (1.2) and (1.4). Then

$$\hat{u}(z) = \frac{l(z)}{l(z_0)}u_g, \qquad \hat{v}(z) = \frac{l(z)}{l(z_0)}v_g$$

for $l(z) = \int_0^z \frac{ds}{k(s)}$. Thus using in (1.1) the exchange

$$u \leftrightarrow u + \hat{u}, \qquad v \leftrightarrow v + \hat{v},$$

we get

$$\begin{cases} f(v + \hat{v} - v_g) = -\frac{\partial}{\partial z}(k(z)\frac{\partial u}{\partial z}), \\ f(u + \hat{u} - u_g) = \frac{\partial}{\partial z}(k(z)\frac{\partial v}{\partial z}), \\ u = v = 0 \quad \text{at} \quad z = 0, z_0. \end{cases} \tag{3.14}$$

Introducing the corresponding Green function

$$G(z,s) = \begin{cases} l(s)\left(\frac{l(z)}{l(z_0)} - 1\right) & \text{for } 0 \le s \le z \le z_0, \\ l(z)\left(\frac{l(s)}{l(z_0)} - 1\right) & \text{for } 0 \le z \le s \le z_0, \end{cases}$$

(3.14) is rewritten as

$$\begin{cases} \hat{f}u(z) = -\int_0^{z_0} G(z,s)(v(s) + \hat{v}(s) - v_g)ds, \\ \hat{f}v(z) = \int_0^{z_0} G(z,s)(u(s) + \hat{u}(s) - u_g)ds \end{cases} \tag{3.15}$$

for $\hat{f} = f^{-1}$. Now we consider a Hilbert space $H = L^2(0, z_0)^2$ with an inner product

$$((u_1, v_1), (u_2, v_2)) = \int_0^{z_0} (u_1(z)v_1(z) + u_2(z)v_2(z))dz.$$

Next introducing a linear operator $A : H \to H$ by

$$A(u,v)(z) = \left(\int_0^{z_0} G(z,s)v(s)ds, -\int_0^{z_0} G(z,s)u(s)ds\right)$$

and functions

$$\tilde{u}(z) = -\int_0^{z_0} G(z,s)(\hat{v}(s) - v_g)ds,$$
$$\tilde{v}(z) = \int_0^{z_0} G(z,s)(\hat{u}(s) - u_g)ds,$$

(3.15) is equivalent to

$$\hat{f}(u,v) + A(u,v) = (\tilde{u}, \tilde{v}).$$

Since $G(z,s) = G(s,z)$, it is easy to see that A is anti-symmetric $A^* = -A$. It is also well-known that A is compact [1,19]. Thus a spectrum of A consists from isolated pure imaginary eigenvectors with a limit at the zero and the corresponding eigenvalues form an orthogonal bases of H. Consequently, for any $0 \neq f \in \mathbb{R}$, there is a unique solution of (3.14), and thus also for (1.1). Some approximations methods can be used for general $k(z)$ in order to construct these solutions. If $k(z)$ is constant then a method presented above is applied. □

3.3 Dynamical properties

Conditions (1.2) and (1.5) are Cauchy initial value conditions for (1.1), so they determine a unique solution on $\mathbb{R}_+ = [0, \infty)$. We will try to study the uniqueness of (1.1) with conditions (1.2) and (1.3).

Theorem 3.8. *For any constant $\bar{k} > 0$ there is an $\bar{\epsilon} > 0$ such that for any continuous function $k : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\sup_{z \in \mathbb{R}_+} |\bar{k} - k(z)| < \bar{\epsilon},$$

there is a unique solution of (1.1) with conditions (1.2) and (1.3).

Proof. To study conditions (1.2) and (1.3), we introduce

$$x = k\frac{\partial u}{\partial z},$$
$$y = k\frac{\partial v}{\partial z},$$

and (1.1) is replaced by

$$\begin{cases} \frac{\partial u}{\partial z} = \hat{k}x, \\ \frac{\partial v}{\partial z} = \hat{k}y, \\ \frac{\partial x}{\partial z} = -f(v - v_g), \\ \frac{\partial y}{\partial z} = f(u - u_g) \end{cases} \tag{3.16}$$

for $\hat{k} = \frac{1}{k}$. The affine system (3.16) has a unique equilibrium

$$(u_g, v_g, 0, 0)$$

with the linearization

$$\begin{cases} \frac{\partial u}{\partial z} = \hat{k}x, \\ \frac{\partial v}{\partial z} = \hat{k}y, \\ \frac{\partial x}{\partial z} = -fv, \\ \frac{\partial y}{\partial z} = fu. \end{cases} \tag{3.17}$$

If $\sup_{z\in\mathbb{R}_+} \hat{k}(z) < \infty$, then the asymptotic property of (3.17) is determined by its Lyapunov exponents. When k is a constant function, then the matrix

$$\begin{pmatrix} 0 & 0 & \hat{k} & 0 \\ 0 & 0 & 0 & \hat{k} \\ 0 & -f & 0 & 0 \\ f & 0 & 0 & 0 \end{pmatrix}$$

has eigenvalues

$$\lambda_1 = -\sqrt{\frac{f\hat{k}}{2}}(1+i), \qquad \lambda_2 = -\sqrt{\frac{f\hat{k}}{2}}(1-i), \qquad \lambda_3 = \sqrt{\frac{f\hat{k}}{2}}(1-i), \qquad \lambda_4 = \sqrt{\frac{f\hat{k}}{2}}(1+i)$$

with the corresponding eigenvectors

$$\left(-\frac{(1+i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{(1-i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, i, 1\right), \qquad \left(-\frac{(1-i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{(1+i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -i, 1\right),$$
$$\left(\frac{(1-i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{(1+i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -i, 1\right), \qquad \left(\frac{(1+i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{(1-i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, i, 1\right).$$

So the linear system (3.17) has a stable space

$$S = \begin{bmatrix} \left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, 0, 1\right) \\ \left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, 1, 0\right) \end{bmatrix}$$

and (3.16) has a stable manifold

$$W_s = (u_g, v_g, 0, 0) + S.$$

Thus condition (1.2) holds if [17,18]

$$(0,0) \in (u_g, v_g) + \begin{bmatrix} \left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}\right) \\ \left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}\right) \end{bmatrix},$$

which is uniquely satisfied

$$(0,0) = (u_g, v_g) + c_1\left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}\right) + c_2\left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}\right)$$
$$c_1 = \frac{\sqrt{\hat{k}}(u_g + v_g)}{\sqrt{2}\sqrt{f}}, \quad c_2 = \frac{\sqrt{\hat{k}}(u_g - v_g)}{\sqrt{2}\sqrt{f}}.$$

Consequently, there is a unique solution of (1.1) with conditions (1.2) and (1.3). This is already shown above in Remark 3.6. By using a roughness result [3, Proposition 1, p. 34], we see that for any constant $\bar{k} > 0$ there is an $\bar{\epsilon} > 0$ such that for any continuous function $k : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sup_{z\in\mathbb{R}_+} |\bar{k} - k(z)| < \bar{\epsilon},$$

there is a unique solution of (1.1) with conditions (1.2) and (1.3). $\bar{\epsilon}$ can be estimated in the term of $\bar{k}$ and f, but we do not go into details. Since $k(z)$ is just continuous, here we have a solution $u(z)$, $v(z)$ of (1.1) such that $u(z)$, $v(z)$, $\frac{\partial u(z)}{\partial z}$, $\frac{\partial v(z)}{\partial z}$, $\frac{\partial}{\partial z}(k(z)\frac{\partial u(z)}{\partial z})$ and $\frac{\partial}{\partial z}(k(z)\frac{\partial u(z)}{\partial z})$ exist and continuous on $\mathbb{R}_+$. The proof is complete. □

Acknowledgements

The authors are grateful to the referees for their careful reading of the manuscript and their valuable comments. We also thank the editor.

This work is partially supported by the National Natural Science Foundation of China (11661016), Training Object of High Level and Innovative Talents of Guizhou Province ((2016)4006), Department of Science and Technology of Guizhou Province (Fundamental Research Program [2018]1118), Guizhou Data Driven Modeling Learning and Optimization Innovation Team ([2020]5016), the Slovak Research and Development Agency under the contract No. APVV-18-0308, and the Slovak Grant Agency VEGA No. 1/0358/20 and No. 2/0127/20.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **31**, 1–21; https://doi.org/10.14232/ejqtde.2021.1.31 www.math.u-szeged.hu/ejqtde/

3D incompressible flows with small viscosity around distant obstacles

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Received 5 July 2020, appeared 10 April 2021
Communicated by Vilmos Komornik

Abstract. In this paper, we analyze the behavior of three-dimensional incompressible flows, with small viscosities $\nu > 0$, in the exterior of material obstacles $\Omega_R = \Omega_0 + (R, 0, 0)$, where Ω_0 belongs to a class of smooth bounded domains and $R > 0$ is sufficiently large. Applying techniques developed by Kato, we prove an explicit energy estimate which, in particular, indicates the limiting flow, when both $\nu \to 0$ and $R \to \infty$, as that one governed by the Euler equations in the whole space. According to this approach, it is natural to contrast our main result to that one already known in the literature for families of viscous flows in expanding domains.

Keywords: singular perturbation in context of PDEs, vanishing viscosity limit, Navier–Stokes equations, Euler equations.

2020 Mathematics Subject Classification: 35B25, 76B99, 35Q30, 35Q31.

1 Introduction

Let $\Omega_0 \subset \mathbb{R}^3$ be a smooth bounded domain, such that $\mathbb{R}^3 \setminus \overline{\Omega}_0$ is connected and simply connected. We also assume that $\mathbf{0} = (0, 0, 0)$ lies inside Ω_0. For each $R \geq 0$, let us set

$$\mathbf{R} = (R, 0, 0), \qquad \Omega_R = \Omega_0 + \mathbf{R}, \qquad \Pi_R = \mathbb{R}^3 \setminus \overline{\Omega}_R \quad \text{and} \quad \Gamma_R = \partial\Omega_R = \partial\Pi_R.$$

Under these notation, we recall the definition of some usual spaces related to incompressible fluids:

$$V(\Pi_R) = \{v \in (H^1(\Pi_R))^3 : \operatorname{div} v = 0 \text{ in } \Pi_R \text{ and } v = 0 \text{ on } \Gamma_R\} \tag{1.1}$$

and

$$H(\Pi_R) = \{v \in (L^2(\Pi_R))^3 : \operatorname{div} v = 0 \text{ in } \Pi_R \text{ and } v \cdot \mathbf{n} = 0 \; on \; \Gamma_R\}, \tag{1.2}$$

where $\mathbf{n}$ is the outward directed unit normal vector field to Γ_R.

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We fix an initial vorticity ω_0, which is a smooth, divergence-free and compactly supported vector field in $\mathbb{R}^3$. Since Π_R is simply connected, there exists a unique $v_{0,R} \in H(\Pi_R)$ such that $\operatorname{curl} v_{0,R} = \omega_0|_{\Pi_R}$ (see Proposition 3.4). In addition, let us denote by u_0 the velocity defined on $\mathbb{R}^3$ which is associated to the vorticity ω_0, as follows:

$$u_0(x) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times \omega_0(y) dy, \tag{1.3}$$

for each $x \in \mathbb{R}^3$, where $\times$ represents the cross product of vectors in $\mathbb{R}^3$. In this context, there exists $T^* > 0$ with the following property: for all $T \in (0, T^*)$, we can find a smooth solution $u = u(x,t)$ to the three-dimensional Euler equations in the whole space

$$\begin{cases} u_t + (u \cdot \nabla)u = -\nabla p, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0(x), \\ |u| \to 0 \text{ as } |x| \to +\infty, \end{cases} \tag{1.4}$$

defined on $\mathbb{R}^3 \times [0,T]$. In (1.4), u is understood as the velocity of an ideal incompressible fluid, while p denotes its pressure.

Taking $T \in (0, T^*)$ and a small viscosity $\nu > 0$, let us also consider the incompressible Navier–Stokes equations in Π_R, with initial data $v_{0,R}$, given by

$$\begin{cases} v_t^{\nu,R} + (v^{\nu,R} \cdot \nabla) v^{\nu,R} - \nu \Delta v^{\nu,R} + \nabla P^{\nu,R} = 0, & (x,t) \in \Pi_R \times (0,T), \\ \operatorname{div} v^{\nu,R} = 0, & (x,t) \in \Pi_R \times [0,T), \\ v^{\nu,R}(x,t) = 0, & (x,t) \in \partial\Pi_R \times (0,T), \\ v^{\nu,R}(x,0) = v_{0,R}(x), & x \in \Pi_R. \end{cases} \tag{1.5}$$

Above, $v^{\nu,R}$ represents the velocity of the particles of a viscous fluid and $P^{\nu,R}$ is its pressure. It is well-known that there exists a Leray–Hopf weak solution $v^{\nu,R} = v^{\nu,R}(x,t)$ to (1.5) (see Definition 4.1 and Theorem 4.2). We emphasize that, since we consider weak solutions to (1.5), there is no dependence of solution's existence time on the viscosity. Under all these notations we have just described, we are ready to state the main result of this paper.

Theorem 1.1. *As mentioned previously, let $\omega_0 \in (C_c^\infty(\mathbb{R}^3))^3$ be a divergence-free vector field in $\mathbb{R}^3$, and consider the smooth solution $u = u(x,t)$ of (1.4), defined on $\mathbb{R}^3 \times [0,T]$, with initial data given in (1.3). For $\nu > 0$ and $R > 0$, let $v^{\nu,R}$ be a weak solution of (1.5) in $\Pi_R \times [0,T)$, with initial data $v_{0,R}$, where $v_{0,R}$ is the L^2-orthogonal projection of $u_0|_{\Pi_R}$ on $H(\Pi_R)$. Then, there exist $C = C(T, \Omega_0, \omega_0) > 0$ and $R_0 > 0$ such that, for all $R > R_0$, we have*

$$\|v^{\nu,R} - u\|_{L^\infty([0,T];[L^2(\Pi_R)]^3)} \leq C \left(\frac{1}{R} + \sqrt{\nu} \right). \tag{1.6}$$

At this moment, we would like to list some papers where asymptotic behavior of incompressible flows under singular domain perturbation has been considered. Initially, we recall the study of incompressible flows in the presence of small obstacles, presented in [7] and [6]. In [7], it was investigated the asymptotic behavior of 2D incompressible ideal flows in the exterior of a single smooth obstacle that shrinks homothetically to a point. The work developed in [7] allowed to identify the equation satisfied by the limit flow. In fact, if γ is the circulation around the obstacle and $\gamma = 0$, then the limit velocity verifies the Euler equations in the full-plane, with the same initial vorticity. On the other hand, when $\gamma \neq 0$, the limit equation involves a new forcing term, with an initial vorticity that acquires a pointwise Dirac mass. In a similar analysis for the 2D Navier–Stokes equations, considered in [6], it was proved that, if the circulation is sufficiently small, then the limit equation is the Navier–Stokes equations in the whole space, but an additional pointwise Dirac mass still appears in the vorticity of the limit equation. In [4], the corresponding problem was considered in the three-dimensional case, where it was established that the limit velocity is a solution of the Navier–Stokes equations in the full-space. Later, in [1], the research proceeded with the asymptotic behavior of solutions of the incompressible 2D Euler equations on a bounded domain with a finite number of holes, assuming that the size of one of them vanishes. In that situation, the limit flow was identified as a modified Euler system in the domain without its small hole.

In [5], incompressible flows around a small obstacle, with small viscosity, are considered. Under specific assumptions, it can be seen that solutions of the Navier–Stokes system in exterior domains converge to solutions of the Euler system in the full space when both viscosity and the size of the obstacle vanish. In the proof of this result, it is presented a rate of convergence in terms of the viscosity and the size of the obstacle. In addition, the complementary situation was treated in [9], where 2D Euler and Navier–Stokes systems were analyzed in expanding domains. To be more precise, such asymptotic analysis also pointed out that solutions in large domains converge to the corresponding solution in the full plane.

As we can see, in the context of fluid dynamics, limits of singularly perturbed domain have been extensively studied over the last years. Last but not least, we would like to highlight [10], where Kelliher, Lopes Filho and Nussenzveig Lopes examined, in dimensions 2 and 3, the limiting behavior of incompressible flows with small viscosity inside expanding domains. Based on energy estimates developed by Kato in [8], these three authors identified conditions under which the limit velocity satisfies the Euler system in the whole space when both viscosity vanishes and the domain becomes large. We are supposed to remark that their analysis also exhibits a rate of convergence which takes into account the small viscosity of the fluid and the enlarged boundary domain. The current work intends to be part of the list of papers we have just mentioned. However, our purpose here is closer to [10]. In fact, we study, in dimension 3, the limiting behavior of incompressible flows, with small viscosity, around far obstacles. In this sense, here the boundary domain becomes distant through the translation of $\Gamma_0 = \partial\Pi_0$ by $\mathbf{R} = (R, 0, 0)$, while, in [10], the boundary goes far by dilatation. In both cases, there is some effect of distant boundaries in the vanishing viscosity limit. Thus, Theorem 1.1 should be contrasted with the corresponding three-dimensional main result of [10].

The remainder of this paper is organized as follows: in Section 2, we deal with the behavior of smooth solutions of (1.4) at infinity. In Section 3, we set some suitable approximate solutions to the Euler equations and, applying the decay results obtained in Section 2, some indispensable estimates are achieved in Propositions 3.3 and 3.4. Section 4 is devoted to a brief discussion about the Navier–Stokes in exterior domains. At this point, we must emphasize Proposition 4.5, where a well-known relation involving weak solutions to (4.1) is extended to a larger class of test vector fields. In Section 5 we prove Theorem 1.1, our main result. In Section 6, we make further comments about some aspects related to this work. At the end, for the sake of clarity, there is an appendix, where we list domains, differential operators, function spaces and notations related to the PDEs mentioned throughout the development of this work.

2 Incompressible inviscid flow in the whole space

At the beginning of this section, we would like to have a few words on the local well-posedness for the 3D Euler system in the whole space. As said before, we fix an initial vorticity $\omega_0 \in (C_c^\infty(\mathbb{R}^3))^3$, which is divergence-free, and take the associated velocity u_0, expressed in (1.3). Under these assumptions, it was proved in [12] that, for sufficiently small times, there exists a smooth solution of (1.4), with u_0 as the initial data. It means that there exists $T^* > 0$, depending on u_0, such that, for all $T \in (0, T^*)$, there exists a unique smooth solution (u, p) of (1.4), defined on $\mathbb{R}^3 \times [0, T]$.

Additionally, for each $t \in [0, T]$, the vector field $\omega = \operatorname{curl}(u(\cdot, t))$ is compactly supported and there exists $r > 0$ such that

$$\operatorname{supp}(\omega(u)) \subset B_r(\mathbf{0}) \times [0, T] \tag{2.1}$$

what can be seen in [10], for example. It is important to notice that ω and u solve the system

$$\begin{cases} \omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, & (x,t) \in \mathbb{R}^3 \times (0,T), \\ \operatorname{div} \omega = 0, & (t,x) \in \mathbb{R}^3 \times [0,T], \\ \omega = \operatorname{curl} u, & (x,t) \in \mathbb{R}^3 \times [0,T], \end{cases} \tag{2.2}$$

and, due to the second PDE in (2.2), it is true that $u = \operatorname{curl} \Psi$, where Ψ is the vector-valued stream function given by

$$\Psi(x,t) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(y,t)}{|x-y|} dy. \tag{2.3}$$

As a consequence, for all $(x, t) \in \mathbb{R}^3 \times [0, T]$, u can be recovered from ω throughout the *Biot–Savart* law

$$u(x,t) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times \omega(y,t) dy, \tag{2.4}$$

which we had already stated in (1.3), for $t = 0$.

In the rest of this section, we will focus our attention on the behavior of the smooth solution (u, p) of (1.4) at infinity.

Lemma 2.1. *Let $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in (C^\infty(\mathbb{R}^3))^3$ be a compactly supported vector field and consider $M > 0$ such that $\operatorname{supp} \Phi \subset \bar{B}_M(\mathbf{0})$. Then, there exists $C > 0$ such that, for any $x \in \mathbb{R}^3 \setminus B_{2M}(\mathbf{0})$, we have*

$$\left| \int_{\mathbb{R}^3} \frac{\Phi(y)}{|x-y|} dy - \frac{1}{|x|} \int_{\mathbb{R}^3} \Phi(y) dy \right| \leq \frac{C}{|x|^2}. \tag{2.5}$$

Additionally, if $\int_{\mathbb{R}^3} \Phi(y) dy = \mathbf{0}$, then the inequality

$$\left| \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times \Phi(y) dy \right| \leq \frac{C}{|x|^3} \tag{2.6}$$

also holds.

Proof. We start proving (2.6). Consider the vector field $g = (g_1, g_2, g_3) \in (C^\infty(\mathbb{R}^3 \setminus \{\mathbf{0}\}))^3$, given by $g(x) = \frac{x}{|x|^3}$ for each $x \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Let us take $x \in \mathbb{R}^3 \setminus B_{2M}(\mathbf{0})$ and $y \in \bar{B}_M(\mathbf{0})$. Since

$$\{(1-t)x + t(x-y) : t \in [0,1]\} \subset \mathbb{R}^3 \setminus \{\mathbf{0}\},$$

applying the mean value theorem, we obtain $\theta_i \in (0,1)$ such that

$$g_i(x-y) = g_i(x) + Dg_i(x-\theta_i y)(-y),$$

where $i \in \{1,2,3\}$. Using that $\int_{\mathbb{R}^3} \Phi(y)dy = \mathbf{0}$ and $|x-\theta_i y| \geq |x| - \theta_i|y| \geq |x| - \frac{|x|}{2} = \frac{|x|}{2}$, we easily check that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} \left[\frac{(x_i-y_i)}{|x-y|^3}\Phi_j(y) - \frac{(x_j-y_j)}{|x-y|^3}\Phi_i(y) \right] dy \right| \\
&\quad \leq \left| \int_{\bar{B}_M(\mathbf{0})} Dg_i(x-\theta_i y)(-y)\Phi_j(y)dy \right| + \left| \int_{\bar{B}_M(\mathbf{0})} Dg_j(x-\theta_j y)(-y)\Phi_i(y)dy \right| \\
&\quad \leq C\left(\int_{\bar{B}_M(\mathbf{0})} \frac{|\Phi_j(y)|}{|x-\theta_i y|^3}dy + \int_{\bar{B}_M(\mathbf{0})} \frac{|\Phi_i(y)|}{|x-\theta_j y|^3}dy \right) \leq \frac{C}{|x|^3},
\end{aligned}$$

for all $i,j \in \{1,2,3\}$. From this, (2.6) follows.

The proof of (2.5) is analogous, but the condition $\int_{\mathbb{R}^3} \Phi(y)dy = \mathbf{0}$ is not required. In fact, we can find $\lambda \in (0,1)$ such that

$$\left| \int_{\mathbb{R}^3} \frac{\Phi(y)}{|x-y|}dy - \frac{1}{|x|}\int_{\mathbb{R}^3} \Phi(y)dy \right| = \left| \int_{\bar{B}_M(\mathbf{0})} \frac{(x-\lambda y)\cdot y}{|x-\lambda y|^3}\Phi(y)dy \right| \leq \int_{\bar{B}_M(\mathbf{0})} \frac{M|\Phi(y)|}{|x-\lambda y|^2}dy \leq \frac{C}{|x|^2},$$

following the desired conclusion. □

Next, we apply Lemma 2.1 in order to state the decay of u and its derivatives, as $|x| \to \infty$.

Proposition 2.2. *Consider u and ω as mentioned above and take $M > 0$ such that $\operatorname{supp}(\omega(\cdot,t)) \subset \bar{B}_M(\mathbf{0})$ for all $t \in [0,T]$. Then, there exists $C > 0$ such that*

$$|u(x,t)| \leq \frac{C}{|x|^2}, \qquad |u_t(x,t)| \leq \frac{C}{|x|^3} \quad \text{and} \quad |\nabla u(x,t)| \leq \frac{C}{|x|^3}, \tag{2.7}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \bar{B}_{2M}(\mathbf{0})) \times [0,T]$.

Proof. During this proof, suppose that $(x,t) \in (\mathbb{R}^3 \setminus \bar{B}_{2M}(\mathbf{0})) \times [0,T]$ is fixed. Thus, we easily get

$$|u(x,t)| = \left| \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times \omega(y,t)dy \right| \leq \left(\frac{1}{\pi}\int_{\bar{B}_M(\mathbf{0})} |\omega(y,t)|dy \right) \frac{1}{|x|^2} \leq \frac{C}{|x|^2}.$$

For the second desired estimate, we use (2.2) and (2.4) in order to obtain

$$u_t(x,t) = \frac{-1}{4\pi}\int_{\bar{B}_M(\mathbf{0})} \frac{(x-y)}{|x-y|^3} \times [(\omega\cdot\nabla)u - (u\cdot\nabla)\omega](y,t)dy.$$

Recalling that u and ω are two divergence-free vector fields, we take

$$\int_{\mathbb{R}^3} [(\omega\cdot\nabla)u - (u\cdot\nabla)\omega](y,t) = \mathbf{0}.$$

Hence, applying the inequality (2.6) from Lemma 2.1, we have

$$|u_t(x,t)| \leq \frac{C}{|x|^3}.$$

In the last part of this proof, we will estimate $\nabla u = [\partial_i u_j]_{i,j=1}^3$. Notice that, if $0 < \varepsilon < M$ and $y \in \overline{B}_M(\mathbf{0})$, we clearly get $|x-y| \geq \frac{|x|}{2} > M > \varepsilon$ for all $y \in \bar{B}_M(0)$. Consequently, for any 3×1 matrix B, we take

$$\begin{aligned}
|[\nabla u(x,t)]B| &= \lim_{\varepsilon \to 0} \left| \int_{|y-x| \geq \varepsilon} \left(\frac{\omega(y,t) \times B}{4\pi |x-y|^3} + \frac{3\{[(x-y) \times \omega(y,t)] \otimes (x-y)\}B}{4\pi |x-y|^5} \right) dy \right| \\
&\leq \int_{\bar{B}_M(\mathbf{0})} \left| \frac{\omega(y,t) \times B}{4\pi |x-y|^3} + \frac{3\{[(x-y) \times \omega(y,t)] \otimes (x-y)\}B}{4\pi |x-y|^5} \right| dy \\
&\leq \left(\frac{1}{4\pi} \int_{\bar{B}_M(\mathbf{0})} \frac{|\omega(y,t)|}{|x-y|^3} dy \right) |B| \\
&\leq \frac{C}{|x|^3} |B|,
\end{aligned}$$

where $h \otimes k$ denotes the 3×3 matrix $[h_i k_j]_{i,j=1}^3$ for each $h, k \in \mathbb{R}^3$. It completes the proof. □

In the next two results, we will specify the decay of the scalar pressure p, given in (1.4), as $|x| \to \infty$.

Lemma 2.3. *Let (u,p) be the solution of (1.4) and consider $\bar{y} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. The following properties hold:*

(a) There exists $C > 0$ such that

$$|\nabla p(x,t)| \leq \frac{C}{|x|^3}$$

for any $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$.

(b) For each $t \in [0,T]$, there exists

$$L(t) = \lim_{\theta \to \infty} p(\theta \bar{y}, t).$$

Proof. The pointwise estimate for ∇p comes immediately from Proposition 2.2.

Let us prove the second part of the result. Let $(\theta_n)_{n=1}^\infty$ be a sequence of positive real numbers which tends to infinity. Since

$$|p(\theta_m \bar{y}, t) - p(\theta_n \bar{y}, t)| = \left| \int_{\theta_n}^{\theta_m} \nabla p(s\bar{y}, t) \cdot \bar{y} ds \right| \leq \frac{C}{2|\bar{y}|^2} \left| \frac{1}{\theta_n^2} - \frac{1}{\theta_m^2} \right| \tag{2.8}$$

for all positive integers m and n, we conclude that, for each $t \in [0,T]$, the sequence $(p(\theta_n \bar{y}, t))_{n=1}^\infty$ converges as $n \to \infty$. Analogously, if $(\lambda_n)_{n=1}^\infty$ is another sequence of positive real numbers which tends to infinity, we take

$$|p(\theta_n \bar{y}, t) - p(\lambda_n \bar{y}, t)| \leq \frac{C}{2|\bar{y}|^2} \left| \frac{1}{\theta_n^2} - \frac{1}{\lambda_n^2} \right|$$

for all positive integers m and n, and $t \in [0,T]$. It means that there exists $\lim_{\theta \to \infty} p(\theta \bar{y}, t)$, as desired. □

Next, we will see that Lemma 2.3 allows us to collect some properties of the pressure p.

Proposition 2.4. *Let (u,p) be the solution (1.4) and consider $\bar{y}, \bar{z} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Then*

(a) $\lim_{\theta \to \infty} p(\theta \bar{y}, t) = \lim_{\theta \to \infty} p(\theta \bar{z}, t)$ for each $t \in [0,T]$;

(b) *there exists a continuous function* $p_\infty : [0,T] \longrightarrow \mathbb{R}$ *and* $C > 0$ *such that*

$$|p(x,t) - p_\infty(t)| \leq \frac{C}{|x|^2} \tag{2.9}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$.

Proof. Firstly, let us focus on the proof of (a). Without loss of generality, we can assume that $\bar{z} \notin \mathbb{R}\bar{y}$ and $|\bar{y}| \geq |\bar{z}|$. Take $\theta > 0$ and consider the sphere

$$\mathcal{S} = \{x \in \mathbb{R}^3 : |x| = \theta|\bar{z}|\}.$$

Let $\sigma : [s_1, s_2] \subset [0, 2\pi] \longrightarrow \mathcal{S}$ be the geodesic on $\mathcal{S}$ from $\theta\bar{z}$ to $q = \frac{\theta|\bar{z}|}{|\bar{y}|}\bar{y}$, given by

$$\sigma(s) = (\sin s)q + (\cos s)\theta|\bar{z}|v,$$

where v belongs to the tangent plane to $\mathcal{S}$ at q. Thus, from Lemma 2.3, we obtain

$$\begin{aligned}|p(\theta\bar{y},t) - p(\theta\bar{z},t)| &\leq |p(\theta\bar{y},t) - p(q)| + |p(q) - p(\theta\bar{z},t)| \\ &= \left|\int_{s_1}^{s_2} \nabla p(\sigma(s)) \cdot \sigma'(s)ds\right| + \left|\int_{\frac{\theta|\bar{z}|}{|\bar{y}|}}^{\theta} \nabla p(s\bar{y}) \cdot \bar{y}ds\right| \leq \left[\frac{2\pi C}{|\bar{z}|^2} + \frac{C}{2|\bar{y}|}\left(\frac{|\bar{y}|^2}{|\bar{z}|^2} - 1\right)\right]\frac{1}{\theta^2}.\end{aligned}$$

Therefore, $\lim_{\theta\to\infty} p(\theta\bar{y},t) = \lim_{\theta\to\infty} p(\theta\bar{z},t)$ for each $t \in [0,T]$.

Secondly, we must prove the part (b). Let us set the scalar function $p_\infty : [0,T] \longrightarrow \mathbb{R}$ by $p_\infty(t) = \lim_{\theta\to\infty} p(\theta\bar{y},t)$. Arguing as in (2.8), we easily check that,

$$|p(x,t) - p_\infty(t)| = \lim_{\theta\to\infty} |p(x,t) - p(\theta x,t)| \leq \lim_{\theta\to\infty} \frac{C}{|x|^2}\left(1 - \frac{1}{\theta^2}\right) = \frac{C}{|x|^2}.$$

This completes the proof of Proposition 2.4. □

In the last result of this section, we will make use of Propositions 2.2 and 2.4 in order to describe how the stream vector field Ψ behaves at infinity.

Proposition 2.5. *Let* (u,p) *be the solution of* (1.4) *and* Ψ *be the associated stream vector field given in* (2.3). *Consider* $M > 0$ *such that* $\operatorname{supp}(\omega(\cdot,t)) \subset \bar{B}_M(\mathbf{0})$ *for all* $t \in [0,T]$. *Then, there exists* $C > 0$ *such that*

$$|\Psi(x,t)| \leq \frac{C}{|x|}, \qquad |\Psi_t(x,t)| \leq \frac{C}{|x|^2} \quad \text{and} \quad |\nabla\Psi(x,t)| \leq \frac{C}{|x|^2}, \tag{2.10}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \bar{B}_{2M}(\mathbf{0})) \times [0,T]$.

Proof. The first inequality in (2.10) is straightforward. Now, take $(x,t) \in (\mathbb{R}^3 \setminus \bar{B}_{2M}(\mathbf{0})) \times [0,T]$. Since

$$\Psi_t(x,t) = \frac{1}{4\pi}\int_{\bar{B}_M(\mathbf{0})} \frac{(\omega\cdot\nabla)u - (u\cdot\nabla)\omega}{|x-y|}(y,t)dy$$

and $\int_{\mathbb{R}^3}[(\omega\cdot\nabla)u - (u\cdot\nabla)\omega](y,t) = \mathbf{0}$, the inequality (2.5) gives us the second estimate in (2.10). Finally, for each $i \in \{1,2,3\}$, we notice that

$$|\partial_i\Psi(x,t)| = \left|\frac{1}{4\pi}\int_{\bar{B}_M(\mathbf{0})} \frac{x_i - y_i}{|x-y|^3}\omega(y,t)dy\right| \leq \int_{\bar{B}_M(\mathbf{0})} \frac{|\omega(y,t)|}{|x-y|^2}dy,$$

and thus the third estimate in (2.10) also holds. □

3 Approximate inviscid solutions

Firstly, let us recall some notations given in Section 2. Consider $\omega_0 \in (C_c^\infty(\mathbb{R}^3))^3$ and let (u,p) be the smooth solution of (1.4) in $\mathbb{R}^3 \times [0,T]$, with initial data

$$u_0 = u_0(x) = \frac{-1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega_0(y)dy. \tag{3.1}$$

Also, let Ψ be the stream vector field associated to u, given in (2.3).

In this section, we intend to approximate the solution u by an appropriate net $(u^R)_{R>0}$ of divergence-free vector fields. Suppose that $\bar{\Omega}_0 \cup \operatorname{supp}\omega(\cdot,t) \subset B_{M_0}(\mathbf{0})$ for all $t \in [0,T]$, where $M_0 > 0$, and let us take $\chi \in C^\infty(\mathbb{R})$ satisfying $0 \le \chi \le 1$, $\chi \equiv 1$ in $\mathbb{R} \setminus (-2M_0, 2M_0)$, and $\chi \equiv 0$ in $[-M_0, M_0]$. For each $R > 0$, let us set $\chi^R(x) = \chi(|x - \mathbf{R}|)$ and

$$u^R(x,t) := \operatorname{curl}(\chi^R(\Psi + C_R)) = \nabla\chi^R \times (\Psi + C_R) + \chi^R u, \tag{3.2}$$

where $(x,t) \in \mathbb{R}^3 \times [0,T]$ and $C_R = \frac{-1}{4\pi R}\int_{\mathbb{R}^3}\omega_0(y)dy$. Clearly, each u^R is a smooth and divergence-free vector field in $\mathbb{R}^3$, which vanishes in the neighborhood of $\Gamma_R = \partial(\Omega_R)$, where $\Omega_R = \Omega_0 + \mathbf{R}$. Besides, taking

$$\overline{p} = p - p_\infty, \tag{3.3}$$

where the function $p_\infty : [0,T] \longrightarrow \mathbb{R}$ was obtained in Proposition 2.4, we also have

$$u_t^R = \nabla\chi^R \times \Psi_t - \chi^R (u\cdot\nabla)u - \chi^R \nabla\bar{p} \tag{3.4}$$

in $\Pi_R \times (0,T)$, recalling that $\Pi_R = \mathbb{R}^3 \setminus \overline{\Omega}_R$.

Next, we will prove some important estimates involving $(u^R)_{R>0}$, which will allow us to obtain Theorem 1.1.

Lemma 3.1. *Let us consider Ψ, C_R and $M_0 > 0$ as mentioned at the beginning of this section. Then there exist $C > 0$ and $R_0 > 0$ such that*

$$\sup_{|y| \in [M_0, 2M_0]} |\Psi(y + \mathbf{R}, t) + C_R| \le \frac{C}{R^2} \tag{3.5}$$

for all $R > R_0$ and $t \in [0,T]$.

Proof. Firstly, we observe that, for all $x_1, x_2 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ satisfying $|x_1| \ge 2|x_2|$, we can apply the mean value theorem in order to obtain

$$\left|\frac{1}{|x_1 - x_2|} - \frac{1}{|x_1|}\right| \le \frac{4|x_2|}{|x_1|^2}. \tag{3.6}$$

Let us fix $y \in \mathbb{R}^3$ such that $|y| \in [M_0, 2M_0]$. Since $\int_{\mathbb{R}^3}\omega(z,t)dz = \int_{\mathbb{R}^3}\omega_0(z)dz$, for any $t \in [0,T]$, we have

$$\begin{aligned}
|\Psi(y+\mathbf{R},t) + C_R| &= \left|\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\omega(z,t)}{|(y+\mathbf{R})-z|}dz - \frac{1}{4\pi R}\int_{\mathbb{R}^3}\omega_0(z)dz\right| \\
&\le \left|\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\omega(z,t)}{|(y+\mathbf{R})-z|}dz - \frac{1}{4\pi|y+\mathbf{R}|}\int_{\mathbb{R}^3}\omega(z,t)dz\right| \\
&\quad + \frac{1}{4\pi}\left|\frac{1}{|y+\mathbf{R}|} - \frac{1}{|\mathbf{R}|}\right|\int_{\mathbb{R}^3}|\omega_0(z)|dz \\
&=: A + B.
\end{aligned}$$

Taking $R_0 = 4M_0$, it is clear that, if $z \in B_{M_0}(\mathbf{0})$ and $R > R_0$, then $|(y+\mathbf{R}) - z| \geq R - 3M_0 > 0$ and $|y + \mathbf{R}| \geq R - 2M_0 > 0$. As a consequence,

$$\begin{aligned}
A &\leq \frac{1}{4\pi}\int_{B_{M_0}(\mathbf{0})} \left| \frac{1}{|(y+\mathbf{R}) - z|} - \frac{1}{|y+\mathbf{R}|} \right| |\omega(z,t)| dz \\
&\leq \frac{1}{4\pi}\int_{B_{M_0}(\mathbf{0})} \frac{|z|}{|(y+\mathbf{R}) - z||y+\mathbf{R}|} |\omega(z,t)| dz \\
&\leq \frac{C}{(R-3M_0)^2} \\
&\leq \frac{C}{R^2}.
\end{aligned}$$

for all $R > R_0$. Finally, applying (3.6) with $x_1 = -\mathbf{R}$ and $x_2 = y$, we also obtain

$$B \leq \frac{|y|}{\pi R^2}\int_{B_{M_0}(\mathbf{0})} |\omega_0(z)| dz \leq \frac{C}{R^2}$$

for $R > R_0$. Hence, (3.5) follows. □

Remark 3.2. The estimates proved in Propositions 2.2 and 2.5 are valid for any $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$.

Proposition 3.3. *Under the previous notation, there exist two constants $C = C(\Omega_0, T) > 0$ and $R_0 > 0$ such that, for all $R > R_0$, we have:*

(a) $\|u^R - u\|_{L^\infty([0,T];L^2(\Pi_R))} + \|u^R - \chi^R u\|_{L^\infty([0,T];L^2(\Pi_R))} \leq C/R^2$;

(b) $\|\nabla u^R - \nabla u\|_{L^\infty([0,T];L^2(\Pi_R))} \leq C/R^2$;

(c) $\|\bar{p}\nabla\chi^R\|_{L^\infty([0,T];L^2(\Pi_R))} + \|\nabla\chi^R \times \Psi_t\|_{L^\infty([0,T];L^2(\Pi_R))} \leq C/R^2$;

(d) $\|u^R\|_{L^\infty([0,T];L^\infty(\Pi_R))} + \|\nabla u^R\|_{L^\infty([0,T];L^\infty(\Pi_R))} + \|\nabla u^R\|_{L^\infty([0,T];L^2(\Pi_R))} \leq C$.

Proof. **ESTIMATE (a)**: In the first place, using (3.2) and (3.5), we get

$$\begin{aligned}
\|u^R(\cdot,t) - \chi^R u(\cdot,t)\|^2_{L^2(\Pi_R)} &= \|\nabla\chi^R \times (\Psi + C_R)\|^2_{L^2(\Pi_R)} \\
&= \int_{|x-\mathbf{R}|\in[M_0,2M_0]} |\chi'(|x-\mathbf{R}|)|^2 |\Psi(x,t) + C_R|^2 dx \\
&= \int_{|y|\in[M_0,2M_0]} |\chi'(|y|)|^2 |\Psi(y+\mathbf{R},t) + C_R|^2 dy \\
&\leq C \sup_{|y|\in[M_0,2M_0]} |\Psi(y+\mathbf{R},t) + C_R|^2 \\
&\leq \frac{C}{R^4}
\end{aligned} \tag{3.7}$$

for all $t \in [0,T]$ and $R > 0$ sufficiently large. On the other hand, recalling Proposition 2.2, we also obtain

$$
\begin{aligned}
\|(\chi^R-1)u(\cdot,t)\|^2_{L^2(\Pi_R)} &= \int_{B_{2M_0}(\mathbf{R})\cap\Pi_R} |\chi(|x-\mathbf{R}|)-1|^2|u(x,t)|^2dx \\
&= \int_{B_{2M_0}(\mathbf{0})\cap\Pi_0} |\chi(|y|)-1|^2|u(y+\mathbf{R},t)|^2dy \\
&\le C\int_{B_{2M_0}(\mathbf{0})\cap\Pi_0} \frac{|\chi(|y|)-1|^2}{|y+\mathbf{R}|^4}dy \\
&\le \frac{C}{(R-2M_0)^4} \\
&\le \frac{C}{R^4}
\end{aligned}
\tag{3.8}
$$

for all $t\in[0,T]$ and $R>0$ sufficiently large. As a result, desired estimate holds.

ESTIMATE (b): From (3.2), we know that

$$\partial_i u^R-\partial_i u=\partial_i(\nabla\chi^R)\times(\Psi+C_R)+\nabla\chi^R\times\partial_i\Psi+(\partial_i\chi^R)u+(\chi^R-1)\partial_i u.$$

Hence, using Propositions 2.2 and 2.5, and Lemma 3.1, we can argue as in (3.7) and (3.8) in order to prove that

$$\|\nabla u^R-\nabla u\|_{L^\infty([0,T];L^2(\Pi_R))}\le CR^{-2}$$

for all $R>0$ sufficiently large.

ESTIMATE (c): The proof of the third desired estimate is very similar to the last ones. In fact, it is a consequence of (2.9) and (2.10), given in Propositions 2.4 and 2.5, respectively.

ESTIMATE (d): Let us prove the last estimate. Since the inviscid velocity u is a smooth vector field, Proposition 2.2 assures that $\|\nabla u^R\|_{L^\infty([0,T];L^2(\Pi_R))}\le C$ for $R>0$ is sufficiently large. Finally, for all $i\in\{1,2,3\}$, it is clear that

$$|\chi^R(x)|+|\partial_i\chi^R(x)|+|\partial_i\partial_j\chi^R(x)|\le C,$$

for all $x\in\mathbb{R}^3$ and $R>0$. It implies that $\|u^R\|_{L^\infty([0,T];L^\infty(\Pi_R))}$ and $\|\nabla u^R\|_{L^\infty([0,T];L^\infty(\Pi_R))}$ are uniformly bounded with respect to $R>0$. This ends the proof. □

Next, we prove the last result of this section, which yields a suitable convergence related to the initial data.

Proposition 3.4. *As before, let $\omega_0\in(C_c^\infty(\mathbb{R}^3))^3$ be a divergence-free vector field and consider the initial velocity u_0 as in (3.1). Then, for each $R>0$, there exists a unique $v_{0,R}\in H(\Pi_R)$ such that* $\operatorname{curl} v_{0,R}=\omega_0|_{\Pi_R}$*. In addition, there exist $C>0$ and $R_0>0$ such that*

$$\|\tilde v_{0,R}-u_0\|_{L^2(\mathbb{R}^3)}\le\frac{C}{R^2}\tag{3.9}$$

for all $R>R_0$, where $\tilde v_{0,R}$ vanishes on $\overline{\Omega}_R$ and equals $v_{0,R}$ on Π_R.

Proof. For each $R>0$, the existence of exactly one vector field $v_{0,R}\in H(\Pi_R)$ satisfying $\operatorname{curl} v_{0,R}=\omega_0|_{\Pi_R}$ was given in [4], where the authors have used the Leray–Helmholtz–Weyl orthogonal decomposition as well as the simple connectedness of Π_R.

In order to prove (3.9), we observe that

$$\|u_0|_{\Pi_R}-v_{0,R}\|_{L^2(\Pi_R)}\le\|u_0|_{\Pi_R}-w\|_{L^2(\Pi_R)}$$

for all $w \in H(\Pi_R)$. In particular, taking $w(x) = u^R(x,0)$, where $x \in \mathbb{R}^3$, and applying Proposition 3.3, we obtain

$$\begin{aligned}\|\tilde{v}_{0,R} - u_0\|^2_{L^2(\mathbb{R}^3)} &= \|v_{0,R} - u_0|_{\Pi_R}\|^2_{L^2(\Pi_R)} + \|u_0\|^2_{L^2(\bar{\Omega}_R)} \\ &\leq \|u_0|_{\Pi_R} - u^R(\cdot,0)\|^2_{L^2(\Pi_R)} + \|u_0\|^2_{L^2(\bar{\Omega}_R)} \\ &\leq \frac{C}{R^4} + \|u_0\|^2_{L^2(\bar{\Omega}_R)},\end{aligned}$$

for all $R > 0$ sufficiently large. Besides, taking $R > 2M_0$, we observe that $\bar{\Omega}_R \cap \operatorname{supp} \omega_0 = \emptyset$. As a consequence, for any $x \in \overline{\Omega}_R$ and $y \in \operatorname{supp} \omega_0$, we have

$$|x - y| \geq R - |y| - |x - \mathbf{R}| \geq R - 2M_0 > 0.$$

Therefore,

$$\|u_0\|^2_{L^2(\bar{\Omega}_R)} \leq C \int_{B_{M_0}(\mathbf{0})} \frac{|\omega_0(y)|^2}{|x-y|^4} dy dx \leq \frac{C}{(R-2M_0)^4} \leq \frac{C}{R^4},$$

and (3.9) holds. □

4 Leray–Hopf solutions in exterior domains

Throughout this section, let $\Pi = \mathbb{R}^3 \setminus \overline{\Omega} \subset \mathbb{R}^3$ be a smooth exterior domain, which means that Ω is a smooth compact set in $\mathbb{R}^3$. Given $T > 0$ and $v_0 \in H(\Pi)$, we consider the Navier–Stokes system

$$\begin{cases} v_t + (v \cdot \nabla)v - \nu\Delta v + \nabla P = 0, & (x,t) \in \Pi \times (0,T), \\ \operatorname{div} v = 0, & (x,t) \in \Pi \times [0,T), \\ v(x,t) = \mathbf{0}, & (x,t) \in \partial\Pi \times (0,T), \\ v(x,0) = v_0(x), & x \in \Pi, \end{cases} \tag{4.1}$$

where $v = v(x,t)$ is the velocity field evaluated at the point $x \in \Pi$ and at the time $t \in [0,T]$, $P = P(x,t)$ is the related scalar pressure field, and $\nu > 0$ is the kinematic viscosity.

Definition 4.1. Under the notation above, a measurable vector field

$$v \in L^2(0,T; V(\Pi)) \cap L^\infty(0,T; H(\Pi))$$

is said to be a weak solution of (4.1) in $\Pi \times [0,T)$ if, for any $\Phi \in \mathcal{D}_T(\Pi)$, we have

$$\int_0^T \int_\Pi [v \cdot \Phi_t - \nu(\nabla v \cdot \nabla\Phi) - (v \cdot \nabla)v \cdot \Phi](x,t) dx dt = -\int_\Pi v_0 \cdot \Phi(x,0) dx. \tag{4.2}$$

The next result assures the existence of a weak solution to (4.1) satisfying a very important additional estimate, which is called a *Leray–Hopf solution* of (4.1).

Theorem 4.2. *Given $T > 0$ and $v_0 \in H(\Pi)$, the system (4.1) there exists a weak solution $v : \Pi \times [0,T) \longrightarrow \mathbb{R}^3$ which satisfies the energy estimate*

$$\|v(\cdot,t)\|^2_{L^2(\Pi)} + 2\nu \int_0^t \|\nabla v(\cdot,\tau)\|^2_{L^2(\Pi)} d\tau \leq \|v_0\|^2_{L^2(\Pi)} \tag{4.3}$$

for all $t \in [0,T]$

Proof. The proof of this result can be found in [2]. □

Remark 4.3. Taking $T > 0$ and $v_0 \in H(\Pi)$, let us consider a weak solution $v : \Pi \times [0,T) \longrightarrow \mathbb{R}^3$ of (4.1). It is known that, for any $\Phi \in \mathcal{D}(\Pi \times [0,T))$, v satisfies

$$\begin{aligned}\int_\Pi (v\cdot\Phi)(x,t)dx - \int_\Pi (v\cdot\Phi)(x,0)dx \\ = \int_0^t \int_\Pi [v\cdot\Phi_t - \nu(\nabla v\cdot\nabla\Phi) - (v\cdot\nabla)v\cdot\Phi](x,\tau)d\tau \end{aligned} \tag{4.4}$$

for all $t \in [0,T)$ (see [3], for instance).

Later, we will apply the relation (4.4) replacing Φ by each approximate inviscid solutions u^R, where $R > 0$. For this reason, we are supposed to prove that (4.4) remains valid when Φ decays sufficiently fast at infinity, but is not compactly supported.

Let $\eta : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function which satisfies $0 \le \eta \le 1$ in $\mathbb{R}^3$, $\eta \equiv 1$ in $B_1(\mathbf{0})$, and $\eta \equiv 0$ in $\mathbb{R}^3 \setminus B_2(\mathbf{0})$. For each $s > 0$, we set $\eta_s(x) = \eta(s^{-1}x)$, where $x \in \mathbb{R}$. Under these notations, we are ready to present the next two results.

Lemma 4.4. *Let $F : \Pi \longrightarrow \mathbb{R}^3$ and $G : \Pi \times [0,T] \longrightarrow \mathbb{R}^3$ be two smooth vector fields, with $F \in H^1(\Pi)$. Also, suppose that there exist $C > 0$ and $\alpha > 0$ such that*

$$|G(x,t)| \le \frac{C}{|x|^\alpha} \tag{4.5}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$. The following properties hold:

(a) $\|\eta_s F - F\|_{H^1(\Pi)} \to 0$ *as* $s \to \infty$;

(b) If $a \in (3,\infty]$, *then* $\|\nabla\eta_s\|_{L^a(\Pi)} \to 0$ *as* $s \to \infty$;

(c) $\|\partial_i\eta_s G(\cdot,t)\|_{L^2(\Pi)} \le \frac{C}{s^{\alpha-1/2}}$ *for all* $t \in [0,T]$ *and* $i \in \{1,2,3\}$;

(d) $\|\partial^2_{ij}\eta_s G(\cdot,t)\|_{L^2(\Pi)} \le \frac{C}{s^{\alpha+1/2}}$ *for all* $t \in [0,T]$ *and* $i,j \in \{1,2,3\}$;

(e) If $\alpha > \frac{3}{2}$, *then* $\|\eta_s G(\cdot,t) - G(\cdot,t)\|_{L^2(\Pi)} \le \frac{C}{s^{\alpha-3/2}}$ *for all* $t \in [0,T]$.

Proof. Take $d > 0$ such that $\Omega \subset B_d(\mathbf{0})$, where $\Pi = \mathbb{R}^3 \setminus \overline{\Omega}$. In this proof, we will only consider the functions $\eta_s : \mathbb{R}^3 \longrightarrow \mathbb{R}$, with $s \ge d$.

PART (a): It follows immediately from Lebesgue's dominated convergence theorem.

PART (b): Let $i \in \{1,2,3\}$. If $a \in (3,+\infty)$, the desired convergence is a consequence of

$$\int_\Pi |\partial_i\eta_s(x)|^a dx = \int_{\mathbb{R}^3} \left|\partial_i\eta\left(\frac{x}{s}\right)\right|^a \frac{1}{s^a}dx = \frac{1}{s^{a-3}}\|\partial_i\eta\|^a_{L^a}.$$

On the other hand, the estimate $\|\partial_i\eta_s\|_{L^\infty(\Pi)} \le \frac{C}{s}$ gives us the complete conclusion.

PARTS (c) and (d): Let us take $t \in [0,T]$ and $i,j \in \{1,2,3\}$. Thus, using (4.5), we take

$$\begin{aligned}\int_\Pi |\partial_i\eta_s(x)G(x.t)|^2dx &= \int_{|x|\in[s,2s]} \frac{1}{s^2}\left|\partial_i\eta\left(\frac{x}{s}\right)\right|^2 |G(x,t)|^2dx \\ &= \int_{|y|\in[1,2]} s\,|\partial_i\eta(y)|^2\,|G(sy,t)|^2dy \\ &\le \frac{C}{s^{2\alpha-1}}\left(\int_{|y|\in[1,2]} |\partial_i\eta(y)|^2dy\right)\end{aligned}$$

and

$$\begin{aligned}
\int_{\Pi} |\partial_{ij}^2 \eta_s(x) G(x,t)|^2 dx &= \int_{|x|\in[s,2s]} \frac{1}{s^4} \left|\partial_{ij}^2 \eta\left(\frac{x}{s}\right)\right|^2 |G(x,t)|^2 dx \\
&= \int_{|y|\in[1,2]} \frac{1}{s} \left|\partial_{ij}^2 \eta(y)\right|^2 |G(sy,t)|^2 dy \\
&\leq \frac{C}{s^{2\alpha+1}} \left(\int_{|y|\in[1,2]} |\partial_{ij}^2 \eta(y)|^2 dy \right).
\end{aligned}$$

PART (e): For the last estimate, we assume that $\alpha > 3/2$. Once again, applying (4.5), we have

$$\begin{aligned}
\int_{\Pi} |\eta_s(x) G(x,t) - G(x,t)|^2 dx &= \int_{|x|\geq s} \left| \left[\eta\left(\frac{x}{s}\right) - 1 \right] G(x,t) \right|^2 dx \\
&= \int_{|y|\geq 1} |\eta(y) - 1|^2 |G(sy,t)|^2 s^3 dy \\
&\leq \frac{C}{(2\alpha - 3) s^{2\alpha-3}}
\end{aligned}$$

for each $t \in [0,T]$. It concludes the proof. $\square$

The following result is the last one of this section. We emphasize that its content brings the information that (4.4) holds for a larger class of test functions.

Proposition 4.5. *Let $\tilde{\Psi} : \mathbb{R}^3 \times [0,T] \longrightarrow \mathbb{R}^3$ and $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be two smooth vector fields satisfying:*

(a) $\operatorname{supp}(\tilde{\Psi}) \subset (\mathbb{R}^3 \setminus \bar{B}) \times [0,T]$, *where B is an open ball containing Ω;*

(b) F is divergence-free and $\operatorname{supp}(F)$ *is a compact subset of Π;*

(c) There exists $C_1 > 0$ such that

$$|\tilde{\Psi}(x,t)| \leq \frac{C_1}{|x|}, \qquad |\tilde{\Psi}_t(x,t)| \leq \frac{C_1}{|x|^2} \quad \text{and} \quad |\nabla \tilde{\Psi}(x,t)| \leq \frac{C_1}{|x|^2}, \tag{4.6}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$.

Additionally, consider $\tilde{\Phi} := \operatorname{curl}(\tilde{\Psi}) + F$ and suppose that there exists $C_2 > 0$ such that

$$|\tilde{\Phi}(x,t)| \leq \frac{C_2}{|x|^2}, \qquad |\tilde{\Phi}_t(x,t)| \leq \frac{C_2}{|x|^3} \quad \text{and} \quad |\nabla \tilde{\Phi}(x,t)| \leq \frac{C_2}{|x|^3}, \tag{4.7}$$

for all $(x,t) \in (\mathbb{R}^3 \setminus \{\mathbf{0}\}) \times [0,T]$. Then, a weak solution v of (4.1), with initial data $v_0 \in H(\Pi)$, satisfies

$$\begin{aligned}
&\int_{\Pi} (v \cdot \tilde{\Phi})(x,t) dx - \int_{\Pi} v_0(x) \cdot \tilde{\Phi}(x,0) dx \\
&\qquad = \int_0^t \int_{\Pi} [v \cdot \tilde{\Phi}_t - \nu(\nabla v \cdot \nabla \tilde{\Phi}) + (v \cdot \nabla)\tilde{\Phi} \cdot v](x,\tau) d\tau.
\end{aligned} \tag{4.8}$$

Proof. As in the proof of Lemma 2.10, consider $\Pi = \mathbb{R}^3 \setminus \overline{\Omega}$ and take $d > 0$ such that $\Omega \subset B_d(\mathbf{0})$. For each $s \geq d$, define $\Phi^s \in \mathcal{D}(\Pi \times [0, T))$ given by

$$\Phi^s := \operatorname{curl}(\eta_s \tilde{\Psi}) + F = \nabla \eta_s \times \tilde{\Psi} + \eta_s \operatorname{curl} \tilde{\Psi} + F.$$

From (4.4), we obtain

$$\begin{aligned}\int_\Pi (v \cdot \Phi^s)(x,t)dx - \int_\Pi (v \cdot \Phi^s)(x,0)dx \\ = \int_0^t \int_\Pi [v \cdot (\Phi^s)_t - \nu(\nabla v \cdot \nabla \Phi^s) - (v \cdot \nabla)v \cdot \Phi^s](x,\tau)d\tau\end{aligned} \tag{4.9}$$

for all $t \in [0, T)$.

Next, we fix $t \in [0, T)$, in order to pass to the limit in (4.9) as $s \to \infty$. Firstly, using (4.3) and Lemma 4.4, we take

$$\begin{aligned}\left| \int_\Pi (v \cdot \Phi^s)(x,t)dx - \int_\Pi (v \cdot \tilde{\Phi})(x,t)dx \right| &\leq \|v(\cdot,t)\|_{L^2(\Pi)} \|\nabla \eta_s \times \tilde{\Psi} + (\eta_s - 1) \operatorname{curl} \tilde{\Psi}\|_{L^2(\Pi)} \\ &\leq \frac{C\|v_0\|_{L^2(\Pi)}}{s^{1/2}}\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\left| \int_0^t \int_\Pi (v \cdot (\Phi^s)_t) dx d\tau - \int_0^t \int_\Pi (v \cdot \tilde{\Phi}_t) dx d\tau \right| &\leq \int_0^t \int_\Pi |v||\nabla \eta_s \times \tilde{\Psi}_t + (\eta_s - 1)(\operatorname{curl} \tilde{\Psi})_t|(x,\tau)dxd\tau \\ &\leq \|v_0\|_{L^2(\Pi)} \int_0^t (\|\nabla \eta_s \times \tilde{\Psi}_t\|_{L^2(\Pi)} + \|(\eta_s - 1)(\operatorname{curl} \tilde{\Psi})_t\|_{L^2(\Pi)})(\tau)d\tau \\ &\leq \frac{CT\|v_0\|_{L^2(\Pi)}}{s^{3/2}}.\end{aligned} \tag{4.11}$$

Likewise, using Lemma 4.4, (4.3) and

$$\partial_i \Phi^s - \partial_i \tilde{\Phi} = \partial_i(\nabla \eta_s) \times \tilde{\Psi} + \nabla \eta_s \times \partial_i \tilde{\Psi} + (\partial_i \eta_s) \operatorname{curl} \tilde{\Psi} + (\eta_s - 1)\partial_i(\operatorname{curl} \tilde{\Psi})$$

where $i \in \{1, 2, 3\}$, we obtain the estimate

$$\begin{aligned}&\left| \int_0^t \int_\Pi (\nabla v \cdot \nabla \Phi^s)(x,\tau)dxd\tau - \int_0^t \int_\Pi (\nabla v \cdot \nabla \tilde{\Phi})(x,\tau)dxd\tau \right| \\ &\leq \sum_{i=1}^3 \int_0^t \|\partial_i v(\cdot,\tau)\|_{L^2(\Pi)} \|(\partial_i \Phi^s - \partial_i \tilde{\Phi})(\cdot,\tau)\|_{L^2(\Pi)} d\tau \\ &\leq \frac{\|v_0\|_{L^2(\Pi)}}{\sqrt{2\nu}} \sum_{i=1}^3 \left(\int_0^t \|(\partial_i \Phi^s - \partial_i \tilde{\Phi})(\cdot,\tau)\|^2_{L^2(\Pi)} d\tau \right)^{1/2} \\ &\leq \frac{C}{s^{1/2}}.\end{aligned} \tag{4.12}$$

To conclude the proof, we use

$$\int_0^t \int_\Pi [(v \cdot \nabla)v] \cdot \Phi^s(x.\tau)dxd\tau = -\int_0^t \int_\Pi [(v \cdot \nabla)\Phi^s] \cdot v(x.\tau)dxd\tau,$$

as well as Lemma 4.4 and (4.3) in order to get

$$\begin{aligned}
&\left|\int_0^t \int_\Pi [(v\cdot\nabla)v]\cdot\Phi^s(x.\tau)dxd\tau + \int_0^t\int_\Pi [(v\cdot\nabla)\tilde{\Phi}]\cdot v(x.\tau)dxd\tau\right| \\
&\quad\leq \int_0^t\int_\Pi |v|^2|\nabla\Phi^s - \nabla\tilde{\Phi}|^2(x,\tau)dxd\tau \\
&\quad\leq \int_0^t \|v(\cdot,\tau)\|^2_{L^4(\Pi)}\|(\nabla\Phi^s-\nabla\tilde{\Phi})(\cdot,\tau)\|_{L^2(\Pi)}d\tau \\
&\quad\leq \frac{C}{s^{1/2}}\int_0^t \|v(\cdot,\tau)\|^2_{H^1(\Pi)}d\tau \\
&\quad\leq \frac{C}{s^{1/2}}. \qquad (4.13)
\end{aligned}$$

Therefore, from (4.10), (4.11), (4.12) and (4.13), the relation (4.8) holds. □

5 Proof of Theorem 1.1

This section is devoted to the main result of this paper. Let us recall its hypotheses:

- ω_0 is a smooth, compactly supported and divergence-free vector field in $\mathbb{R}^3$;
- (u,p) is the smooth solution of (1.4), defined in $\mathbb{R}^3\times(0,T)$, with initial data u_0, as in (1.3).

Taken $T\in(0,T^*)$, $R>0$ and $\nu>0$, we also consider:

- u^R as defined in (3.2);
- $v_{0,R}$ is the L^2-orthogonal projection of $u_0|_{\Pi_R}$ on $H(\Pi_R)$, mentioned in Proposition 3.4;
- $v^{\nu,R}$ is a weak solution of (4.1) in $\Pi_R\times[0,T)$, with initial data $v_{0,R}$, given by Theorem 4.2.

Proof of Theorem 1.1. **CLAIM**: There exist $C=C(T,\Omega_0,\omega_0)>0$ and $R_0>0$ such that, if $R>R_0$, then

$$\|v^{\nu,R}(\cdot,t)-u^R(\cdot,t)\|^2_{L^2(\Pi_R)}+\nu\int_0^t\|\nabla v^{\nu,R}(\cdot,\tau)-\nabla u^R(\cdot,\tau)\|^2_{L^2(\Pi_R)}d\tau\leq C\left(\frac{1}{R^2}+\nu\right) \qquad (5.1)$$

for almost every $t\in[0,T]$. As a consequence, Theorem 1.1 holds.

From now on, we will focus on the verification of (5.1). To do so, the main arguments used below take into account those presented in [8]. Let us fix $t\in[0,T]$. From (4.3),

$$\frac{1}{2}\|v^{\nu,R}(\cdot,t)\|^2_{L^2(\Pi_R)}+\nu\int_0^t\|\nabla v^{\nu,R}(\cdot,\tau)\|^2_{L^2(\Pi_R)}d\tau\leq\frac{1}{2}\|v_{0,R}\|^2_{L^2(\Pi_R)}=\frac{1}{2}\|\tilde{v}_{0,R}\|^2_{L^2(\mathbb{R}^3)}, \qquad (5.2)$$

where $\tilde{v}_{0,R}$ equals $v_{0,R}$ on Π_R and vanishes otherwise.

Next, due to (2.7) and (2.10), we can apply Proposition 4.5, with $v = v^{\nu,R}$ and $\tilde{\Phi} = u^R$, in order to obtain

$$\begin{aligned}
&-\int_{\Pi_R}(v^{\nu,R}\cdot u^R)(x,t)dx+\int_{\Pi_R}(v_{0,R}\cdot u^R)(x,0)dx\\
&\quad=-\int_0^t\int_{\Pi_R}[v^{\nu,R}\cdot u^R_t-\nu\nabla v^{\nu,R}\cdot\nabla u^R](x,\tau)dxd\tau-\int_0^t\int_{\Pi_R}[(v^{\nu,R}\cdot\nabla)u^R]\cdot v^{\nu,R}dxd\tau\\
&\quad=-\int_0^t\int_{\Pi_R}v^{\nu,R}\cdot[\nabla\chi^R\times\Psi_t-\chi^R(u\cdot\nabla)u-\chi^R\nabla\overline{p}](x,\tau)dxd\tau\\
&\qquad+\nu\int_0^t\int_{\Pi_R}(\nabla v^{\nu,R}\cdot\nabla u^R)(x,\tau)dxd\tau-\int_0^t\int_{\Pi_R}[(v^{\nu,R}\cdot\nabla)u^R]\cdot v^{\nu,R}(x,\tau)dxd\tau\\
&\quad=-\int_0^t\int_{\Pi_R}v^{\nu,R}\cdot[\nabla\chi^R\times\Psi_t-\chi^R\nabla\overline{p}](x,\tau)dxd\tau+\nu\int_0^t\int_{\Pi_R}(\nabla v^{\nu,R}\cdot\nabla u^R)(x,\tau)dxd\tau\\
&\qquad+\int_0^t\int_{\Pi_R}[(\chi^R u-u^R)\nabla u]\cdot v^{\nu,R}(x,\tau)dxd\tau+\int_0^t\int_{\Pi_R}[(u^R\cdot\nabla)(u-u^R)]\cdot v^{\nu,R}(x,\tau)dxd\tau\\
&\qquad+\int_0^t\int_{\Pi_R}[(v^{\nu,R}-u^R)\nabla u^R]\cdot(v^{\nu,R}-u^R)(x,\tau)dxd\tau. \qquad (5.3)
\end{aligned}$$

Besides, recalling that the kinetic energy $E(t)=\frac{1}{2}\int_{\mathbb{R}^3}|u(x,t)|^2dx$ is a conserved quantity in time, we also take

$$\begin{aligned}
\frac{1}{2}\|u^R(\cdot,t)\|^2_{L^2(\Pi_R)}&\le\frac{1}{2}\|(\nabla\chi^R\times(\Psi+C_R))(\cdot,t)\|^2_{L^2(\Pi_R)}\\
&\quad+\|(\nabla\chi^R\times(\Psi+C_R))(\cdot,t)\|_{L^2(\Pi_R)}\|\chi^R u\|_{L^2(\Pi_R)}+\frac{1}{2}\|\chi^R u(\cdot,t)\|^2_{L^2(\Pi_R)}\\
&\le\frac{1}{2}\|(u^R-\chi^R u)(\cdot,t)\|^2_{L^2(\Pi_R)}\\
&\quad+\|(u^R-\chi^R u)(\cdot,t)\|_{L^2(\Pi_R)}\|\chi^R u\|_{L^2(\Pi_R)}+\frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^3)}. \qquad (5.4)
\end{aligned}$$

As a result, from (5.2), (5.3) and (5.4) , we conclude that

$$\begin{aligned}
&\frac{1}{2}\|v^{\nu,R}-u^R\|^2_{L^2(\Pi_R)}+\nu\int_0^t\|(\nabla v^{\nu,R}-\nabla u^R)(\cdot,\tau)\|^2_{L^2(\Pi_R)}d\tau\\
&\quad\le\left[\frac{1}{2}\|\tilde{v}_{0,R}-u_0\|^2_{L^2(\mathbb{R}^3)}+\int_{\Pi_R}v_{0,R}\cdot(u_0-u^R)(x,0)dx\right]\\
&\quad-\left[\int_0^t\int_{\Pi_R}v^{\nu,R}\cdot(\nabla\chi^R\times\Psi_t-\chi^R\nabla\overline{p})(x,\tau)dxd\tau\right]\\
&\quad+\nu\left[\int_0^t\|\nabla u^R\|^2_{L^2(\Pi_R)}-\int_0^t\int_{\Pi_R}(\nabla v^{\nu,R}\cdot\nabla u^R)(x,\tau)dxd\tau\right]\\
&\quad+\left[\int_0^t\int_{\Pi_R}[(\chi^R u-u^R)\nabla u]\cdot v^{\nu,R}(x,\tau)dxd\tau+\int_0^t\int_{\Pi_R}[(u^R\cdot\nabla)(u-u^R)]\cdot v^{\nu,R}(x,\tau)dxd\tau\right]\\
&\quad+\left[\int_0^t\int_{\Pi_R}[(v^{\nu,R}-u^R)\nabla u^R]\cdot(v^{\nu,R}-u^R)(x,\tau)dxd\tau\right]\\
&\quad+\left[\frac{1}{2}\|(u^R-\chi^R u)(\cdot,t)\|^2_{L^2(\Pi_R)}+\|(u^R-\chi^R u)(\cdot,t)\|_{L^2(\Pi_R)}\|\chi^R u\|_{L^2(\Pi_R)}\right]\\
&\quad=:A_1+A_2+A_3+A_4+A_5+A_6. \qquad (5.5)
\end{aligned}$$

Firstly, using Proposition 3.3(d), we easily get

$$|A_5| \leq C\int_0^t \|v^{\nu,R} - u^R\|^2_{L^2(\Pi_R)}(x,\tau)d\tau.$$

Likewise, A_6 was completely analyzed in Proposition 3.3(a). Thus, in the rest of this proof, we will estimate each A_i, with $i \in \{1,2,3,4\}$, in terms of $R > 0$ and $\nu > 0$.

To see that

$$|A_1| \leq \frac{C}{R^2},$$

we just apply Propositions 3.3(a) and 3.4. In fact, observe that

$$\begin{aligned}\int_{\Pi_R} v_{0,R}\cdot(u_0 - u^R)(x,0)dx &\leq \|v_{0,R}\|_{L^2(\Pi_R)}\|(u_0 - u^R)(\cdot,0)\|_{L^2(\Pi_R)}\\ &\leq \|u_0\|_{L^2(\mathbb{R}^3)}\|(u_0 - u^R)(\cdot,0)\|_{L^2(\Pi_R)}\\ &\leq \frac{C}{R^2},\end{aligned}$$

and recall (3.9).

Next, using the relation

$$-\int_0^t\int_{\Pi_R} v^{\nu,R}\cdot(\chi^R\nabla\overline{p})dxd\tau = \int_0^t\int_{\Pi_R} v^{\nu,R}\cdot(\overline{p}\nabla\chi^R)dxd\tau,$$

and Proposition 3.3(c), we obtain

$$\begin{aligned}|A_2| &= \left|\int_0^t\int_{\Pi_R} v^{\nu,R}\cdot(\nabla\chi^R\times\Psi_t + \overline{p}\nabla\chi^R)(x,\tau)dxd\tau\right|\\ &\leq \int_0^t \|v^{\nu,R}\|_{L^2(\Pi_R)}(\|\nabla\chi^R\times\Psi_t\|_{L^2(\Pi_R)} + \|\overline{p}\nabla\chi^R\|_{L^2(\Pi_R)})(\cdot,\tau)dxd\tau\\ &\leq T\|v_{0,R}\|_{L^2(\Pi_R)}(\|\nabla\chi^R\times\Psi_t\|_{L^\infty([0,T];L^2(\Pi_R))} + \|\overline{p}\nabla\chi^R\|_{L^\infty([0,T];L^2(\Pi_R))})\\ &\leq \frac{CT\|u_0\|_{L^2(\mathbb{R}^3)}}{R^2}.\end{aligned}$$

Besides, using Young's inequality and Proposition 3.3(d), we can easily check that

$$\begin{aligned}|A_3| &= \left|\nu\int_0^t\int_{\Pi_R}\nabla u^R\cdot(\nabla u^R - v^{\nu,R})(x,\tau)dxd\tau\right|\\ &\leq \frac{\nu}{2}\int_0^t\|\nabla u^R\|^2_{L^2(\Pi_R)}(\tau)d\tau + \frac{\nu}{2}\int_0^t\|\nabla v^{\nu,R} - \nabla u^R\|^2_{L^2(\Pi_R)}(\tau)d\tau\\ &\leq C\nu + \frac{\nu}{2}\int_0^t\|\nabla v^{\nu,R} - \nabla u^R\|^2_{L^2(\Pi_R)}(\tau)d\tau.\end{aligned}$$

At last, the estimate

$$\begin{aligned}|A_4| \leq \int_0^t &\|\chi^R u - u^R\|_{L^2(\Pi_R)}\|\nabla u\|_{L^\infty(\Pi_R)}\|v^{\nu,R}\|_{L^2(\Pi_R)}d\tau\\ &+\int_0^t\|u^R\|_{L^\infty(\Pi_R)}\|\nabla u^R - \nabla u\|_{L^2(\Pi_R)}\|v^{\nu,R}\|_{L^2(\Pi_R)}d\tau\\ &\leq \frac{CT}{R^2}\end{aligned}$$

comes from Theorem 4.2 and Proposition 3.3(a),(b),(d).

Therefore, there exist $K > 0$ and $L > 0$, independent of time t, such that

$$\frac{1}{2}\|v^{\nu,R}(\cdot,t) - u^R(\cdot,t)\|^2_{L^2(\Pi_R)} + \frac{\nu}{2}\int_0^t \|\nabla v^{\nu,R} - \nabla u^R\|^2_{L^2(\Pi_R)}(\tau)d\tau$$
$$\leq K\left(\frac{1}{R^2} + \nu\right) + L\int_0^t \|v^{\nu,R} - u^R\|^2_{L^2(\Pi_R)}(\tau)d\tau$$

for any $t \in [0,T]$. Thus, the integral form of Gronwall's inequality allows us to achieve the estimate (5.1) . □

6 Some additional comments

1. In order to obtain Proposition 3.4, the circulation of the velocity on the boundary Γ_R is not required, since each Π_R is a 3D simply connected domain.

2. In this paper, we deal with three-dimensional incompressible flows, with small viscosity, around distant obstacles. Perhaps, the analogous two-dimensional case can also be studied. To be more precise, let $\mathcal{U}_0 \subset \mathbb{R}^2$ be a smooth bounded domain, which is also connected and simply connected. For each $R \geq 0$, let us set

$$\mathcal{U}_R = \mathcal{U}_0 + (R,0), \qquad \mathcal{V}_R = \mathbb{R}^2 \setminus \overline{\mathcal{U}}_R \quad \text{and} \quad \mathcal{C}_R = \partial\mathcal{U}_R = \partial\Pi_R.$$

Let us consider $\omega^0 \in C_c^\infty(\mathbb{R}^2)$ and $\gamma \in \mathbb{R}$, which are both independent of $R > 0$. Set

$$y_{0,R} = K_R[\omega^0] + (\gamma + m)H_R,$$

where $m = \int_{\mathbb{R}^2}\omega^0 dx$, $K_R[\omega^0] = K_R[\omega^0](x)$ is the Biot–Savart operator in $\mathcal{V}_R$ and $H_R = H_R(x)$ is the generator of the harmonic vector fields in $\mathcal{V}_R$. Thanks to Lemma 2.2 and Proposition 2.1 of [7], for each $R > 0$, we have

$$\begin{cases} \operatorname{div} y_{0,R} = 0, & \text{in } \mathcal{V}_R, \\ \operatorname{curl} y_{0,R} = \omega^0, & \text{in } \mathcal{V}_R, \\ y_{0,R} \cdot \hat{n} = 0, & \text{on } \mathcal{C}_R, \\ |y_{0,R}(x)| \to 0, & \text{as } |x| \to +\infty, \\ \int_{\mathcal{C}_R} u_{0,R} \cdot ds = \gamma. \end{cases}$$

Thus, it would be very nice to understand the asymptotic behavior of the family of 2D incompressible flows, with small viscosity, around distant obstacles, governed by

$$\begin{cases} y_t^{\nu,R} + (y^{\nu,R} \cdot \nabla)y^{\nu,R} - \nu\Delta y^{\nu,R} + \nabla\pi^{\nu,R} = 0, & (x,t) \in \mathcal{V}_R \times (0,T), \\ \operatorname{div} y^{\nu,R} = 0, & (x,t) \in \mathcal{V}_R \times [0,T), \\ y^{\nu,R}(x,t) = 0, & (x,t) \in \mathcal{C}_R \times (0,T), \\ y^{\nu,R}(x,0) = y_{0,R}(x), & x \in \mathcal{V}_R. \end{cases} \tag{6.1}$$

It means that we have fixed ω^0 as an initial vorticity and γ as the circulation of the initial flow around each obstacle $\mathcal{U}_R$. We should notice that $y_{0,R} \in L^{2,\infty}(\Pi_R)$, but $y_{0,R} \notin L^2(\Pi_R)$. In this case, it seems to us that Kato's argument can not be applied following the same structure of the proof of Theorem 1.1. However, another approach may work if we take mild solutions of (6.1), obtained [11].

3. In order to simplify calculations and estimates, we have considered material obstacles of the form $\Omega_R = \Omega_0 + \mathbf{R} = \Omega_0 + R\mathbf{1}$, where $\mathbf{1} = (1,0,0)$. However, our main result remains valid if we replace Ω_R by $\Omega_0 + R\vec{z}$, for each $R > 0$, where $\vec{z}$ is another unit vector in $\mathbb{R}^3$.

4. We believe that Theorem 1.1 is related to that one obtained in [10], in the three-dimensional case. Let $\Omega^0 \subset \mathbb{R}^3$ be a smooth and simply connected bounded domain $\Omega^0 \subset \mathbb{R}^3$. In that work, the authors started with a smooth initial vorticity $\omega_0 \in C_c^\infty(\mathbb{R}^3)$, which is divergence-free and, for each $L > 0$, they took u_0^L as the unique divergence-free vector field in $\Omega^L = L\Omega^0$ that is tangent to $\partial\Omega^L$ and has a curl equal to ω_0 in Ω^L. At this point, they studied the limiting behavior of a family
$$\{u^{\nu,L} : \nu > 0 \text{ and } L > 0\},$$
consisting of weak solutions to the Navier–Stokes system
$$\begin{cases} u_t^{\nu,L} + (u^{\nu,L} \cdot \nabla)u^{\nu,L} - \nu\Delta u^{\nu,L} + \nabla\pi^{\nu,L} = 0, & (x,t) \in \Omega^L \times (0,T), \\ \operatorname{div} u^{\nu,L} = 0, & (x,t) \in \Omega^L \times [0,T), \\ u^{\nu,L}(x,t) = 0, & (x,t) \in \partial\Omega^L \times (0,T), \\ u^{\nu,L}(x,0) = u_0^L(x), & x \in \Omega^L. \end{cases} \tag{6.2}$$
Summarizing, it is proved in Theorem 1.2 of [10] that
$$\|u^{\nu,L} - u\|_{L^\infty([0,T];L^2(\Omega^L))} \leq \left[C\left(\nu + \frac{1}{\sqrt{L}}\right) + \|u_0^L - u_0\|_{L^2(\Omega^L)} \right] e^{CT}, \tag{6.3}$$
where $u = u(x,t)$ is a smooth solution to (1.4), with initial data u_0, given in (1.3). Above, we observe that
$$\|u_0^L - u_0\|_{L^2(\Omega^L)} \leq CL^{-1/2},$$
which allows us to contrast the current main result of this paper with (6.3). In both cases, there are results for 3D incompressible flows, with small viscosity, in domains with distant boundaries. Thus, it seems to us that the rates of convergence
$$R^{-1} \quad \text{and} \quad L^{-1/2}$$
on the right side of (1.6) and (6.3), respectively, suggest that the boundary $\Gamma_R = \partial\Pi_R$, in (1.5), produces a bigger effect in the vanishing viscosity limit, when compared to the same effect produced by $\partial\Omega^L$, in (6.2).

Acknowledgements

This research was supported by CNPq and CAPES. I would like to thank H. J. Nussenzveig Lopes and M. C. Lopes Filho for some fruitful discussions. Also, I would like to thank the referee for revising my manuscript and suggesting improvements.

Appendix: List of notations

This appendix reunites some notations and definitions that we have used in the previous sections. Our purpose here is to become the reading of this article easier.

Domains

- $\Omega_0 \subset \mathbb{R}^3$ be a smooth bounded domain, such that $\mathbb{R}^3 \setminus \overline{\Omega_0}$ is connected and simply connected;
- For each $R \geq 0$,

$$\mathbf{R} := (R, 0, 0), \qquad \Omega_R := \Omega_0 + \mathbf{R}, \qquad \Pi_R := \mathbb{R}^3 \setminus \overline{\Omega}_R \quad \text{and} \quad \Gamma_R := \partial\Omega_R = \partial\Pi_R.$$

Operators

For a smooth vector field $F = (F_1, F_2, F_3)$ in $\mathbb{R}^3$, we denote:

- $\operatorname{div} F := \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$;
- $\operatorname{curl} F := (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1)$;
- $(F \cdot \nabla) F := F_1 \partial_1 F + F_2 \partial_2 F + F_3 \partial_3 F$.

Function spaces

Given an open set $\mathcal{O} \subset \mathbb{R}^3$ and a function $f : \mathcal{O} \longrightarrow \mathbb{R}$, the *support* of f is the closed set

$$\operatorname{supp} f := \overline{\{x \in \mathcal{O} : f(x) \neq 0\}}.$$

Under these notations,

- $L^{2,\infty}(\mathcal{O})$ denotes the Lorentz space of f satisfying $\sup_{s>0} A_s < +\infty$, where

$$A_s := s^2 \mu(\{x \in \mathcal{O} : |f(x)| > s\})$$

  and μ is the Lebesgue measure in $\mathbb{R}^3$.
- $C_c^\infty(\mathcal{O})$ denotes the space of all infinitely differentiable real functions with compact support in $\mathcal{O}$;
- $\mathcal{D}(\mathcal{O}) := \{\Psi \in (C_c^\infty(\mathcal{O}))^3 : \operatorname{div} \Psi = 0 \text{ in } \mathcal{O}\}$;
- $V(\mathcal{O}) := \{\Psi \in (H^1(\mathcal{O}))^3 : \operatorname{div} \Psi = 0 \text{ in } \mathcal{O} \text{ and } u = 0 \text{ on } \partial\mathcal{O}\}$;
- $H(\mathcal{O}) := \{\Psi \in (L^2(\mathcal{O}))^3 : \operatorname{div} \Psi = 0 \text{ in } \mathcal{O} \text{ and } \Psi \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}$, where $\mathbf{n}$ is the outward directed unit normal vector field to $\partial\mathcal{O}$;
- Given $T > 0$, $\mathcal{D}_T(\mathcal{O})$ denotes the set of all $\varphi \in (C_c^\infty(\mathcal{O} \times [0, T)))^3$ such that

$$\operatorname{div}_x \varphi(x, t) := \partial_1 \varphi_1 + \partial_2 \varphi_2 + \partial_3 \varphi_3 = 0 \quad \text{in } \Pi \times [0, T).$$

Solutions and initial data

- $\omega_0 \in \mathcal{D}(\mathbb{R}^3)$ denotes a fixed initial vorticity;
- u_0 represents the velocity defined on $\mathbb{R}^3$, associated to the vorticity ω_0, as in (1.3);
- (u, p) denotes a smooth solution of (1.4), with initial data u_0;
- (v, P) denotes a Leray–Hopf solution of (4.1);
- Given $R > 0$ and $\nu > 0$, $(v^{\nu,R}, P^{\nu,R})$ denotes a Leray–Hopf solution of (1.5).

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **32**, 1–20; https://doi.org/10.14232/ejqtde.2021.1.32 www.math.u-szeged.hu/ejqtde/

Infinitely many nodal solutions for a class of quasilinear elliptic equations

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Received 19 December 2020, appeared 11 April 2021
Communicated by Patrizia Pucci

Abstract. In this paper, we study the existence of infinitely many nodal solutions for the following quasilinear elliptic equation

$$\begin{cases} -\nabla\cdot[\phi'(|\nabla u|^2)\nabla u]+|u|^{\alpha-2}u=f(u), & x\in\mathbb{R}^N,\\ u(x)\to 0, \quad \text{as } |x|\to\infty, \end{cases}$$

where $N\geq 2$, $\phi(t)$ behaves like $t^{q/2}$ for small t and $t^{p/2}$ for large t, $1<p<q<N$, $f\in\mathcal{C}^1(\mathbb{R}^+,\mathbb{R})$ is of subcritical, $q\leq\alpha\leq p^*q'/p'$, let $p^*=\frac{Np}{N-p}$, p' and q' be the conjugate exponents respectively of p and q. For any given integer $k\geq 0$, we prove that the equation has a pair of radial nodal solution with exactly k nodes.

Keywords: quasilinear elliptic equation, nodal solutions, multiple solutions.

2020 Mathematics Subject Classification: 35A15, 35J62, 46E30.

1 Introduction

In this paper, we consider the following quasilinear elliptic equation

$$-\nabla\cdot[\phi'(|\nabla u|^2)\nabla u]+|u|^{\alpha-2}u=f(u), \qquad x\in\mathbb{R}^N, \tag{1.1}$$

where $N\geq 2$, $\phi\in\mathcal{C}^2(\mathbb{R}_+,\mathbb{R}_+)$ has a different growth near zero and infinity. Quasilinear equation of form (1.1) can be transformed into different differential equations corresponding to various types of ϕ. For example, when $\phi(t)=2[(1+t)^{\frac{1}{2}}-1]$ and $\alpha=2$, equation (1.1) corresponds to the prescribed mean curvature equation or the capillary surface equation

$$-\nabla\cdot\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)+u=f(u), \qquad x\in\mathbb{R}^N.$$

Such problem has been deeply studied since last century, under different assumptions on the nonlinearity f, the existence and nonexistence of solutions have been investigated by many authors, see [3,5,8,27] for example.

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Equation (1.1) also related to (p,q)-Laplacian equations. In fact, if $\phi(t) = \frac{2}{p}t^{\frac{p}{2}} + \frac{2}{q}t^{\frac{q}{2}}$, then equation (1.1) becomes

$$-\Delta_p u - \Delta_q u + |u|^{\alpha-2}u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < q < N$ and $\alpha > 2$ satisfies some conditions. Equation (1.2) originates from the following reaction diffusion system

$$\frac{\partial u}{\partial t} = \operatorname{div}\big[D(u)\nabla u\big] + c(x,u), \tag{1.3}$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. This system has a wide range of application in physics and related sciences such as biophysics, plasma physics and chemical reaction design. In such applications, the function u describes a concentration; the first term on the right hand side of (1.3) corresponds to diffusion with a diffusion coefficient $D(u)$, whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,u)$ has a polynomial form with respect to the concentration u. For more mathematical and physical background of equations (1.2)–(1.3), we refer the reader to the papers [9, 24, 25, 31] and the references therein. In particular, when $p = q = \alpha = 2$, equation (1.2) reduced to

$$-\Delta u + u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

There has been plenty of results on the existence, nonexistence and multiplicity of positive or sign-changing solutions for equation (1.4), see [2, 6, 7, 10, 17] and the references therein.

If $p = q = \alpha \neq 2$, then equation (1.2) becomes into the following general p-Laplacian equation

$$-\Delta_p u + |u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

which was studied by many authors. Many results for equation (1.4) has been extended to equation (1.5). Deng, Guo and Wang in [12] proved the existence of nodal solutions for p-Laplacian equations with critical growth. Recently in [13], Deng, Li and Shuai studied the existence of solutions for a class of p-Laplacian equations with critical growth and potential vanishing at infinity.

Recently, Azzollini et al. [1] studied the following quasilinear elliptic equation

$$\begin{cases} -\nabla \cdot [\phi'(|\nabla u|^2)\nabla u] + |u|^{\alpha-2}u = |u|^{s-2}u, & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{1.6}$$

where $N \geq 2$, $\phi(t)$ behaves like $t^{q/2}$ for small t and $t^{p/2}$ for large t, $1 < p < q < N$, $1 < \alpha \leq p^*q'/p'$ and $\max\{q,\alpha\} < s < p^* = \frac{Np}{N-p}$, with being p', q' are the conjugate exponents of p, q respectively. The authors in [1] found a sort of Orlicz–Sobolev space in which the energy functional is well defined. They also proved that the Orlicz–Sobolev space compactly embedded into certain Lebesgue spaces. Then, they obtained the existence of a sequence of nontrivial radial solutions for equation (1.6) besides a nontrivial non-negative radial solution. General quasilinear elliptic problems of (1.1) have been intensively studied, see for example, [1, 11, 15, 16, 18, 28] and the references therein.

Motivated by the above results, in this paper, we intend to find nodal solutions for the following quasilinear elliptic equation

$$\begin{cases} -\nabla \cdot \big[\phi'(|\nabla u|^2)\nabla u\big] + |u|^{\alpha-2}u = f(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{1.7}$$

where $N \geq 2$, $\phi(t)$ behaves like $t^{q/2}$ for small t and $t^{p/2}$ for large t, $1 < p < q < N$, $q \leq \alpha \leq p^* q'/p'$, and the function f satisfies some conditions given by (f_1)-(f_3) in this paper. Similar as [1], we impose some restrictions on ϕ, let $\phi \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R}_+)$ such that

$(\Phi 1)$ $\phi(0) = 0$;

$(\Phi 2)$ there exists a positive constant C such that

$$\begin{cases} Ct^{\frac{p}{2}-1} \leq \phi'(t), & \text{if } t \geq 1, \\ Ct^{\frac{q}{2}-1} \leq \phi'(t), & \text{if } 0 \leq t \leq 1; \end{cases}$$

$(\Phi 3)$ there exists a positive constant C such that

$$\begin{cases} \phi(t) \leq Ct^{\frac{p}{2}}, & \text{if } t \geq 1, \\ \phi(t) \leq Ct^{\frac{q}{2}}, & \text{if } 0 \leq t \leq 1; \end{cases}$$

$(\Phi 4)$ there exists $\alpha < \theta$ such that $\phi'(t)/t^{\frac{\theta-2}{2}}$ is strictly decreasing for all $t > 0$;

$(\Phi 5)$ the map $t \mapsto \phi(t^2)$ is convex.

We also assume the nonlinearity f satisfies:

(f_1) $f(t) = o(t^{\alpha-1})$, as $t \to 0^+$;

(f_2) $f(t) = o(t^{p^*-1})$, as $t \to +\infty$;

(f_3) there exists $\alpha < \theta$ such that

$$0 < (\theta - 1) f(t) \leq f'(t)t, \quad \text{for all } t > 0.$$

Before we present our main result, we give some notions and definitions. In the following, we use $\|u\|_q$ to denote the $L^q(\mathbb{R}^N)$ norm.

Definition 1.1 (See [1, Definition 2.1])**.** Let $1 < p < q$ and $\Omega \subset \mathbb{R}^N$. Denote $L^p(\Omega) + L^q(\Omega)$ the completion of $\mathcal{C}_c^\infty(\Omega, \mathbb{R})$ in the norm

$$\|u\|_{L^p(\Omega)+L^q(\Omega)} = \inf\left\{\|v\|_{L^p(\Omega)} + \|w\|_{L^q(\Omega)} \mid v \in L^p(\Omega),\ w \in L^q(\Omega), u = v + w\right\}.$$

Next, we denote $\|u\|_{p,q} = \|u\|_{L^p(\mathbb{R}^N)+L^q(\mathbb{R}^N)}$. Moreover, in [4], it has shown that $L^p(\Omega) + L^q(\Omega)$ can be characterized as an Orlicz spaces.

Definition 1.2 (See [1, Definition 2.3])**.** Let $\alpha > 1$, the Orlicz–Sobolev space $\mathcal{W}(\mathbb{R}^N)$ is the completion of $\mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R})$ in the norm

$$\|u\| = \|u\|_\alpha + \|\nabla u\|_{p,q}.$$

By Theorem 2.8 of [1], the space $\mathcal{W}(\mathbb{R}^N)$ can be precise description by

$$\mathcal{W}(\mathbb{R}^N) = \left\{u \in L^\alpha(\mathbb{R}^N) \cap L^{p^*}(\mathbb{R}^N) \mid \nabla u \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)\right\}.$$

In the following, we define

$$\left(\mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R})\right)_r = \left\{u \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}) \mid u \text{ is radially symmetric}\right\}.$$

Then $\mathcal{W}_r(\mathbb{R}^N)$ is the completion of $\left(\mathcal{C}_c^\infty(\mathbb{R}^N,\mathbb{R})\right)_r$ in the norm $\|\cdot\|$, namely

$$\mathcal{W}_r(\mathbb{R}^N) = \overline{\left(\mathcal{C}_c^\infty(\mathbb{R}^N,\mathbb{R})\right)_r}^{\|\cdot\|}.$$

Thus, $\mathcal{W}_r(\mathbb{R}^N)$ coincides with the set of radial functions of $\mathcal{W}(\mathbb{R}^N)$. Define the energy functional I corresponding to equation (1.7) by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)\,\mathrm{d}x + \frac{1}{\alpha}\int_{\mathbb{R}^N} |u|^\alpha\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x, \qquad u \in \mathcal{W}_r(\mathbb{R}^N),$$

where $F(u) = \int_0^u f(z)\,\mathrm{d}z$. The well-posedness and regularity of $I(u)$ are given by Proposition 3.1 in [1] and hypotheses (f_1)–(f_2).

A function $u \in \mathcal{W}_r(\mathbb{R}^N)$ is called a weak solution of equation (1.7) if for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N,\mathbb{R})$, it holds

$$\int_{\mathbb{R}^N} \phi'(|\nabla u|^2)\nabla u\nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{\alpha-2}u\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N} f(u)\varphi\,\mathrm{d}x = 0.$$

In particular, for $u \in \mathcal{W}_r(\mathbb{R}^N)$, we denote

$$\gamma(u) = \langle I'(u), u\rangle = \int_{\mathbb{R}^N} \phi'(|\nabla u|^2)|\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^\alpha\,\mathrm{d}x - \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x.$$

Now we state our main result. We denote $u^+ = \max\{u,0\}$ and $u^- = \min\{u,0\}$.

Theorem 1.3. *Suppose* $1 < p < q < \min\{N,p^*\}$, $q \le \alpha \le p^*q'/p'$, $(\Phi1)$–$(\Phi5)$ *and* (f_1)–(f_3) *hold, then there exists a pair of radial solutions* $u_k^\pm$ *of equation* (1.7) *with the following properties:*

(i) $u_k^-(0) < 0 < u_k^+(0)$,

(ii) $u_k^\pm$ *possess exactly* k *nodes* r_i *with* $0 < r_1 < r_2 < \cdots < r_k < +\infty$, *and* $u_k^\pm(x)|_{|x|=r_i} = 0$, $i = 1,2,\dots,k$.

Remark 1.4. The solutions u_k obtained in Theorem 1.3, as we will see, is the least energy radial solution of equation (1.7) and changes sign exactly k ($k \in \{0,1,2,\dots\}$) times. We should point out that $\alpha < p^*$. The existence of u_0 had been proved by the Mountain Pass Theorem in [1].

Remark 1.5. Like [1], a specific example of the function $\phi(t)$ is

$$\phi(t) = \frac{2}{p}\left[\left(1 + t^{\frac{q}{2}}\right)^{\frac{p}{q}} - 1\right].$$

In this paper, we prove by constrained minimization method in a special space in which each function changes sign $k (k \in \{0,1,2,\dots\})$ times. We first prove the existence of minimizer and then verify that the minimizer is indeed a solution to equation (1.7) by analyzing the least energy related to the minimizer. Here, we have to point out that it is hard to obtain radial solutions with a prescribed number of nodes by gluing method as in Bartsch–Willem [6] and Cao–Zhu [10]. Because, we obtain that all weak solutions of (1.7) by Lemma 2.7 are only of class $\mathcal{C}^{1,\gamma}_{loc}(\mathbb{R}^N)$, and it is not enough to glue the functions in each annuli by matching the normal derivative at each junction point. We will follow the approach explored by Z. Liu and Z.-Q. Wang [21,22], see Section 3 for more details. Moreover, we introduce some new analysis techniques and establish better inequalities.

This paper is organized as follows. In Section 2, we give some preliminary results, which are crucial to prove our main results. In Section 3, we will prove our main theorem.

Throughout this paper, we denote " $\to$ " and " $\rightharpoonup$ " as the strong convergence and the weak convergence, respectively. We use $\langle\cdot,\cdot\rangle$ to denote the duality pairing between $\mathcal{W}_r(\mathbb{R}^N)$ and $\mathcal{W}_r'(\mathbb{R}^N)$. We employ C or C_j, $j = 1,2,\dots$ to denote the generic constant which may vary from line to line.

2 Some preliminary lemmas

In this section, let us first recall some known facts about (1.7). From [1], we introduce the embedding result on $\mathcal{W}_r(\mathbb{R}^N)$ and a uniform decaying estimate on the functions of $\mathcal{W}_r(\mathbb{R}^N)$. The proof of lemmas can be found in the corresponding references.

Lemma 2.1 (see [1, Remark 2.7]). *If $1 < p < \min\{q, N\}$ and $1 < p^* \frac{q'}{p'}$, then for every $\alpha \in \left(1, p^* \frac{q'}{p'}\right]$, $\mathcal{W}_r(\mathbb{R}^N)$ is continuously embedded into $L^\tau(\mathbb{R}^N)$ with $\alpha \le \tau \le p^*$.*

Lemma 2.2 (see [1, Theorem 2.11]). *If $1 < p < q < N$ and $1 < p^* \frac{q'}{p'}$, then for every $\alpha \in \left(1, p^* \frac{q'}{p'}\right]$, $\mathcal{W}_r(\mathbb{R}^N)$ is compactly embedded into $L^\tau(\mathbb{R}^N)$ with $\alpha < \tau < p^*$.*

Lemma 2.3 (see [1, Lemma 2.13]). *If $1 < p < q < N$, there exists $C > 0$ such that for every $u \in \mathcal{W}_r(\mathbb{R}^N)$*

$$|u(x)| \le \frac{C}{|x|^{\frac{N-q}{q}}} \|\nabla u\|_{p,q}, \quad \textit{for } |x| \ge 1.$$

Let Ω be one of the following domains:

$$\{x \in \mathbb{R}^N : |x| < R_1\}, \qquad \{x \in \mathbb{R}^N : 0 < R_2 \le |x| < R_3 < \infty\}, \qquad \{x \in \mathbb{R}^N : |x| \ge R_4 > 0\}.$$

Thus, we first consider the existence of positive least energy solution for

$$\begin{cases} -\nabla \cdot [\phi'(|\nabla u|^2)\nabla u] + |u|^{\alpha-2}u = f(u), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.1}$$

Define

$$I_\Omega(u) = \frac{1}{2}\int_\Omega \phi(|\nabla u|^2)\,\mathrm{d}x + \frac{1}{\alpha}\int_\Omega |u|^\alpha\,\mathrm{d}x - \int_\Omega F(u)\,\mathrm{d}x,$$

$$\gamma_\Omega(u) = \langle I'_\Omega(u), u\rangle = \int_\Omega \phi'(|\nabla u|^2)|\nabla u|^2\,\mathrm{d}x + \int_\Omega |u|^\alpha\,\mathrm{d}x - \int_\Omega f(u)u\,\mathrm{d}x$$

and

$$\mathrm{M}(\Omega) = \left\{u \in \mathcal{W}_r(\Omega) : u \not\equiv 0, u|_{\partial\Omega} = 0,\ \gamma_\Omega(u) = 0\right\}.$$

Then we have the following lemmas.

Lemma 2.4. *Suppose $1 < p < q < \min\{N, p^*\}$, $q \le \alpha \le p^*q'/p'$, ($\Phi$1)–($\Phi$5) and (f_1)–(f_3) hold and $u \in \mathcal{W}_r(\Omega)$. Then there exists a unique $t > 0$ such that $tu \in \mathrm{M}(\Omega)$.*

Proof. For fixed $u \in \mathcal{W}_r(\Omega)$ with $u \not\equiv 0$, tu is contained in $\mathrm{M}(\Omega)$ if and only if

$$\gamma_\Omega(tu) = \int_\Omega \phi'(|t\nabla u|^2)|t\nabla u|^2\,\mathrm{d}x + \int_\Omega |tu|^\alpha\,\mathrm{d}x - \int_\Omega f(tu)tu\,\mathrm{d}x = 0. \tag{2.2}$$

Hence, the problem is reduced to verify that there is only one solution of equation (2.2) with $t > 0$. Since $1 < p < q \le \alpha$ and

$$\phi(t^2) \simeq \begin{cases} t^p, & \text{if } |t| \gg 1, \\ t^q, & \text{if } |t| \ll 1. \end{cases}$$

By (f_1)–(f_2), for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ and $\alpha < s < p^*$ such that

$$f(u)u \le \varepsilon|u|^\alpha + C_\varepsilon|u|^s. \tag{2.3}$$

It is easy to see that $I_\Omega(tu) \to 0$ as $t \to 0$ and $I_\Omega(tu) \to -\infty$ as $t \to +\infty$. We have that I_Ω possesses a global maximum point $t \in (0, +\infty)$, i.e., $tu \in \mathrm{M}(\Omega)$.

It remains to show the uniqueness of t. We shall divide our proof into two cases.

Case 1. $u \in \mathrm{M}(\Omega)$. First of all, we note that it follows from $\gamma_\Omega(u) = 0$ that

$$\int_\Omega \phi'(|\nabla u|^2)|\nabla u|^2\,\mathrm{d}x + \int_\Omega |u|^\alpha\,\mathrm{d}x - \int_\Omega f(u)u\,\mathrm{d}x = 0. \tag{2.4}$$

We now prove that $t = 1$ is the unique number such that $tu \in \mathrm{M}(\Omega)$. In fact, if $t > 0$ such that $\gamma_\Omega(tu) = 0$, then we have

$$\int_\Omega \phi'(|t\nabla u|^2)|t\nabla u|^2\,\mathrm{d}x + \int_\Omega |tu|^\alpha\,\mathrm{d}x - \int_\Omega f(tu)tu\,\mathrm{d}x = 0. \tag{2.5}$$

Furthermore, combining equation (2.4) and (2.5), we have

$$\begin{aligned}\int_\Omega \left[\phi'(t^2|\nabla u|^2)t^2|\nabla u|^2 - t^\theta\phi'(|\nabla u|^2)|\nabla u|^2\right]\mathrm{d}x \\ + \int_\Omega \left[(t^\alpha - t^\theta)|u|^\alpha + \left(t^\theta f(u) - f(tu)tu\right)\right]\mathrm{d}x = 0.\end{aligned} \tag{2.6}$$

On one hand, by (f_3), we can get that

$$\frac{f(t)}{t^{\theta-1}}$$

is increasing for all $t > 0$. On the other hand, by $(\Phi4)$, we can deduce that

$$\frac{\phi'(t^2)}{t^{\theta-2}}$$

is strictly decreasing for all $t > 0$. Assume $t > 1$ for a while, then we get

$$\frac{f(u)}{u^{\theta-1}} \le \frac{f(tu)}{|tu|^{\theta-1}}, \qquad \frac{\phi'(t^2|\nabla u|^2)}{t^{\theta-2}|\nabla u|^{\theta-2}} < \frac{\phi'(|\nabla u|^2)}{|\nabla u|^{\theta-2}},$$

that is

$$t^\theta f(u) - f(tu)tu \le 0 \tag{2.7}$$

and

$$\phi'(t^2|\nabla u|^2)t^2|\nabla u|^2 - t^\theta\phi'(|\nabla u|^2)|\nabla u|^2 < 0. \tag{2.8}$$

Since $\alpha < \theta$, the left side of equation (2.6) is negative, which gives a contradiction. With a similar argument, the case $t < 1$ is also contradictory. Thus we deduce that $t = 1$.

Case 2. $u \notin \mathrm{M}(\Omega)$. If there exist $t_1, t_2 > 0$ such that $t_1u, t_2u \in \mathrm{M}(\Omega)$, we have

$$\frac{t_2}{t_1}(t_1u) = t_2u \in \mathrm{M}(\Omega).$$

Noticing $t_1u \in \mathrm{M}(\Omega)$, by Case 1, we obtain $t_1 = t_2$. This completes the proof of Lemma 2.4. □

Lemma 2.5. *Suppose $1 < p < q < \min\{N, p^*\}$, $q \le \alpha \le p^*q'/p'$, $(\Phi1)$–$(\Phi5)$ and (f_1)–(f_3) hold and $u \in \mathrm{M}(\Omega)$, $t \in (0, \infty)$ and $t \neq 1$, then $I_\Omega(tu) < I_\Omega(u)$.*

Proof. Define a function in $(0,\infty)$ by $g(t) = I_\Omega(tu)$

$$g(t) = I_\Omega(tu) = \frac{1}{2}\int_\Omega \phi(t^2|\nabla u|^2)\,\mathrm{d}x + \frac{t^\alpha}{\alpha}\int_\Omega |u|^\alpha\,\mathrm{d}x - \int_\Omega F(tu)\,\mathrm{d}x.$$

Then

$$g'(t) = \int_\Omega t\phi'(t^2|\nabla u|^2)|\nabla u|^2\,\mathrm{d}x + t^{\alpha-1}\int_\Omega |u|^\alpha\,\mathrm{d}x - \int_\Omega f(tu)u\,\mathrm{d}x.$$

By the fact $u \in \mathrm{M}(\Omega)$, i.e.,

$$\int_\Omega \phi'(|\nabla u|^2)|\nabla u|^2\,\mathrm{d}x + \int_\Omega |u|^\alpha\,\mathrm{d}x - \int_\Omega f(u)u\,\mathrm{d}x = 0,$$

using a similar argument to Lemma 2.4, we obtain $g'(t) > 0$ for $0 < t < 1$ and $g'(t) < 0$ for $t > 1$. Hence $g(t) < g(1)$, that is $I_\Omega(tu) < I_\Omega(u)$ for $t \in (0,\infty)$ and $t \neq 1$. $\square$

Next we consider the following minimization problem

$$\tilde{c} = \inf_{\mathrm{M}(\Omega)} I_\Omega(u).$$

$\mathrm{M}(\Omega)$ is nonempty in $\mathcal{W}_r(\Omega)$ by Lemma 2.4. Here we denote

$$\|u\|_\Omega = \|u\|_{L^\alpha(\Omega)} + \|\nabla u\|_{L^p(\Omega)+L^q(\Omega)},$$

and

$$\Lambda_u = \{x \in \Omega : |u| > 1\}, \qquad \Lambda_u^c = \{x \in \Omega : |u| \le 1\}.$$

Lemma 2.6. *Suppose* $1 < p < q < \min\{N, p^*\}$, $q \le \alpha \le p^*q'/p'$, $(\Phi1)$–$(\Phi5)$ *and* (f_1)–(f_3) *hold, then* $\tilde{c}$ *can be achieved by some positive function* $\tilde{u}$ *which is a solution of equation* (2.1).

Proof. We use the minimization method. The proof can be divided into two steps.

Step 1. $\tilde{c}$ is attained. By the definition of $\tilde{c}$, there exists a sequence $\{\tilde{u}_n\} \subset \mathrm{M}(\Omega)$ such that

$$I_\Omega(\tilde{u}_n) = \tilde{c} + o(1), \qquad \gamma_\Omega(\tilde{u}_n) = 0,$$

i.e.,

$$I_\Omega(\tilde{u}_n) = \frac{1}{2}\int_\Omega \phi(|\nabla\tilde{u}_n|^2)\,\mathrm{d}x + \frac{1}{\alpha}\int_\Omega |\tilde{u}_n|^\alpha\,\mathrm{d}x - \int_\Omega F(\tilde{u}_n)\,\mathrm{d}x = \tilde{c} + o(1),$$

$$\int_\Omega \phi'(|\nabla\tilde{u}_n|^2)|\nabla\tilde{u}_n|^2\,\mathrm{d}x + \int_\Omega |\tilde{u}_n|^\alpha\,\mathrm{d}x - \int_\Omega f(\tilde{u}_n)\tilde{u}_n\,\mathrm{d}x = 0.$$

By the Proposition 2.2 of [1], we have

$$\|\tilde{u}_n\|_{L^p(\Omega)+L^q(\Omega)} \le \max\left\{\|\tilde{u}_n\|_{L^p(\Lambda_{\tilde{u}_n})}, \|\tilde{u}_n\|_{L^q(\Lambda^c_{\tilde{u}_n})}\right\}.$$

It follows from $(\Phi4)$ that $\phi''(t)t < \frac{\theta-2}{2}\phi'(t)$ for all $t > 0$. Moreover, $\phi(0) = 0$, we see that $\phi'(t)t < \frac{\theta}{2}\phi(t)$. There exists $0 < \mu < 1$ such that

$$\phi'(t)t \le \frac{\theta\mu}{2}\phi(t), \quad \text{for all } t \ge 0.$$

Thus, by $(\Phi 2)$ and the fact that $\tilde{u}_n \in L^p(\Lambda_{\tilde{u}_n}) \cap L^q(\Lambda^c_{\tilde{u}_n})$ (see Proposition 2.2 (iv) in [1]), we get

$$
\begin{aligned}
\tilde{c} + o(1) &= I_\Omega(\tilde{u}_n) - \frac{1}{\theta}\langle I'_\Omega(\tilde{u}_n), \tilde{u}_n\rangle \\
&\geq \int_\Omega \Big[\frac{1}{2}\phi(|\nabla \tilde{u}_n|^2) - \frac{1}{\theta}\phi'(|\nabla \tilde{u}_n|^2)|\nabla \tilde{u}_n|^2\Big]\,\mathrm{d}x + \Big(\frac{1}{\alpha} - \frac{1}{\theta}\Big)\int_\Omega |\tilde{u}_n|^\alpha\,\mathrm{d}x \\
&\geq \frac{1-\mu}{2}\int_\Omega \phi(|\nabla \tilde{u}_n|^2)\,\mathrm{d}x + \Big(\frac{1}{\alpha} - \frac{1}{\theta}\Big)\int_\Omega |\tilde{u}_n|^\alpha\,\mathrm{d}x \\
&\geq C_1\int_{\Lambda^c_{\nabla \tilde{u}_n}} |\nabla \tilde{u}_n|^q\,\mathrm{d}x + C_2\int_{\Lambda_{\nabla \tilde{u}_n}} |\nabla \tilde{u}_n|^p\,\mathrm{d}x + \Big(\frac{1}{\alpha} - \frac{1}{\theta}\Big)\int_\Omega |\tilde{u}_n|^\alpha\,\mathrm{d}x \\
&\geq C\Big[\min\big\{\|\nabla \tilde{u}_n\|^q_{L^p(\Omega)+L^q(\Omega)}, \|\nabla \tilde{u}_n\|^p_{L^p(\Omega)+L^q(\Omega)}\big\} + \|\tilde{u}_n\|^\alpha_{L^\alpha(\Omega)}\Big] \\
&\geq C\|\tilde{u}_n\|^\alpha_\Omega.
\end{aligned}
\tag{2.9}
$$

Since $C > 0$, it is easy to verify $\{\tilde{u}_n\}$ is bounded in $\mathrm{M}(\Omega)$. Then by Proposition 2.5 of [1] and Lemma 2.1, there exists $\tilde{u} \in \mathcal{W}_r(\Omega)$ such that

$$
\begin{aligned}
&\tilde{u}_n \rightharpoonup \tilde{u}, \quad \text{weakly in } \mathcal{W}_r(\Omega), \\
&\tilde{u}_n \to \tilde{u}, \quad \text{in } L^s(\Omega), \\
&\tilde{u}_n \to \tilde{u}, \quad \text{a.e. in } \Omega,
\end{aligned}
$$

where $\alpha < s < p^*$. By Theorem A.2 in [34], we can deduce that

$$
f(\tilde{u}_n)\tilde{u}_n \to f(\tilde{u})\tilde{u} \quad \text{in } L^1(\Omega).
$$

Since $\gamma_\Omega(\tilde{u}_n) = 0$, we first prove $\tilde{u} \not\equiv 0$. In fact, by equation (2.3) , Lemma 2.1 and inequality (2.9), we have

$$
C_\varepsilon\|\tilde{u}_n\|^s_\Omega + \varepsilon\|\tilde{u}_n\|^\alpha_\Omega \geq \int_\Omega f(\tilde{u}_n)\tilde{u}_n\,\mathrm{d}x = \int_\Omega \phi'(|\nabla \tilde{u}_n|^2)|\nabla \tilde{u}_n|^2\,\mathrm{d}x + \int_\Omega |\tilde{u}_n|^\alpha\,\mathrm{d}x \geq C\|\tilde{u}_n\|^\alpha_\Omega. \tag{2.10}
$$

Since $s > \alpha$, we have $\|\tilde{u}_n\|_\Omega \geq C_3 > 0$. Hence

$$
\begin{aligned}
C_\varepsilon\|\tilde{u}\|^s_\Omega + \varepsilon\|\tilde{u}\|^\alpha_\Omega + o(1) \geq o(1) + \int_\Omega f(\tilde{u})\tilde{u}\,\mathrm{d}x = &\int_\Omega \phi'(|\nabla \tilde{u}_n|^2)|\nabla \tilde{u}_n|^2\,\mathrm{d}x + \int_\Omega |\tilde{u}_n|^\alpha\,\mathrm{d}x \\
&\geq C\|\tilde{u}_n\|^\alpha_\Omega \geq C_3,
\end{aligned}
$$

we get $\tilde{u} \not\equiv 0$.

According to Lemma 2.4, there exists a unique $\bar{t} > 0$ which satisfies $\gamma_\Omega(\bar{t}\tilde{u}) = 0$. Using the condition $(\Phi 5)$, then

$$
\frac{1}{2}\int_\Omega \phi(\bar{t}^2|\nabla \tilde{u}|^2)\,\mathrm{d}x \leq \liminf_{n\to\infty}\frac{1}{2}\int_\Omega \phi(\bar{t}^2|\nabla \tilde{u}_n|^2)\,\mathrm{d}x.
$$

By the definition of $\tilde{c}$ and equation (2), we have

$$
\begin{aligned}
\tilde{c} \leq I_\Omega(\bar{t}\tilde{u}) &= \frac{1}{2}\int_\Omega \phi(\bar{t}^2|\nabla \tilde{u}|^2)\,\mathrm{d}x + \frac{\bar{t}^\alpha}{\alpha}\int_\Omega |\tilde{u}|^\alpha\,\mathrm{d}x - \int_\Omega F(\bar{t}\tilde{u})\,\mathrm{d}x \\
&\leq \liminf_{n\to\infty}\int_\Omega \Big[\frac{1}{2}\phi(\bar{t}^2|\nabla \tilde{u}_n|^2) + \frac{\bar{t}^\alpha}{\alpha}|\tilde{u}_n|^\alpha - F(\bar{t}\tilde{u}_n)\Big]\,\mathrm{d}x \\
&\leq \liminf_{n\to\infty} I_\Omega(\bar{t}\tilde{u}_n) \leq \liminf_{n\to\infty} I_\Omega(\tilde{u}_n) = \tilde{c}.
\end{aligned}
$$

Thus we get

$$I_\Omega(\bar{t}\tilde{u}) = \tilde{c},$$

and $\tilde{c}$ is attained by $\bar{t}\tilde{u}$.

Step 2. In the following, we prove that $\bar{t}\tilde{u}$ is a radial solution of equation (2.1), which is similar to the Lemma 2.7 of [14]. For simplicity, we denote $\tilde{u}$ to $\bar{t}\tilde{u}$. Suppose $\tilde{u} \in \mathrm{M}(\Omega)$, $I_\Omega(\tilde{u}) = \tilde{c}$, but the conclusion of the lemma is not true. Then we can find a function $\varphi \in \mathcal{W}_r'(\mathbb{R}^N)$ such that

$$\langle I_\Omega'(\tilde{u}), \varphi\rangle = \int_\Omega \phi'(|\nabla \tilde{u}|^2)\nabla \tilde{u}\nabla\varphi \,\mathrm{d}x + \int_\Omega |\tilde{u}|^{\alpha-2}\tilde{u}\varphi \,\mathrm{d}x - \int_\Omega f(\tilde{u})\varphi \,\mathrm{d}x \le -1. \tag{2.11}$$

Choosing $\varepsilon > 0$ small enough such that

$$\left\langle I_\Omega'(t\tilde{u} + \sigma\varphi), \varphi\right\rangle \le -\frac{1}{2}, \qquad \forall\ |t-1| + |\sigma| \le \varepsilon.$$

Let η be a cut-off function such that

$$\eta(t) = \begin{cases} 1, & |t-1| \le \frac{1}{2}\varepsilon, \\ 0, & |t-1| \ge \varepsilon. \end{cases}$$

We estimate

$$\sup_t I_\Omega(t\tilde{u} + \varepsilon\eta(t)\varphi).$$

If $|t-1| \le \varepsilon$, then

$$\begin{aligned} I_\Omega\left(t\tilde{u} + \varepsilon\eta(t)\varphi\right) &= I_\Omega(t\tilde{u}) + \int_0^1 \langle I_\Omega'(t\tilde{u} + \sigma\varepsilon\eta(t)\varphi), \varepsilon\eta(t)\varphi\rangle \,\mathrm{d}\sigma \\ &\le I_\Omega(t\tilde{u}) - \frac{1}{2}\varepsilon\eta(t). \end{aligned} \tag{2.12}$$

For $|t-1| \ge \varepsilon$, $\eta(t) = 0$, and the above estimate is trivial. Now, since $\tilde{u} \in \mathrm{M}(\Omega)$, for $t \ne 1$, we get $I_\Omega(t\tilde{u}) < I_\Omega(\tilde{u})$ by Lemma 2.5. Hence it follows from equation (2.12) that

$$I_\Omega\left(t\tilde{u} + \varepsilon\eta(t)\varphi\right) \le \begin{cases} I_\Omega(t\tilde{u}) < I_\Omega(\tilde{u}), & t \ne 1, \\ I_\Omega(\tilde{u}) - \frac{1}{2}\varepsilon\eta(1) = I_\Omega(\tilde{u}) - \frac{1}{2}\varepsilon, & t = 1. \end{cases}$$

In any case, we have $I_\Omega(t\tilde{u} + \varepsilon\eta(t)\varphi) < I_\Omega(\tilde{u}) = \tilde{c}$. In particular,

$$\sup_{0\le t\le 2} I_\Omega\left(t\tilde{u} + \varepsilon\eta(t)\varphi\right) < \tilde{c}.$$

Since $\tilde{u} \in \mathrm{M}(\Omega)$, we have

$$\int_\Omega \phi'(|\nabla\tilde{u}|^2)|\nabla\tilde{u}|^2 \,\mathrm{d}x + \int_\Omega |\tilde{u}|^\alpha \,\mathrm{d}x - \int_\Omega f(\tilde{u})\tilde{u} \,\mathrm{d}x = 0. \tag{2.13}$$

Let

$$\begin{aligned} h(t) = \int_\Omega \big[&\phi'(|\nabla(t\tilde{u} + \varepsilon\eta(t)\varphi)|^2)|\nabla(t\tilde{u} + \varepsilon\eta(t)\varphi)|^2 + |t\tilde{u} + \varepsilon\eta(t)\varphi|^\alpha \\ &- f(t\tilde{u} + \varepsilon\eta(t)\varphi)(t\tilde{u} + \varepsilon\eta(t)\varphi)\big] \,\mathrm{d}x. \end{aligned}$$

Without loss of generality, we assume $\varepsilon < \frac{1}{4}$. For $t = 2$, we have $\eta(2) = 0$, thus from (2.7)-(2.8) and (2.13)

$$
\begin{aligned}
h(2) &= \int_\Omega \left[4\phi'(4|\nabla \tilde{u}|^2)|\nabla \tilde{u}|^2 + 2^\alpha |\tilde{u}|^\alpha - f(2\tilde{u})2\tilde{u}\right] \mathrm{d}x \\
&= \int_\Omega \left[4\phi'(4|\nabla \tilde{u}|^2)|\nabla \tilde{u}|^2 - 2^\theta \phi'(|\nabla \tilde{u}|^2)|\nabla \tilde{u}|^2\right] \mathrm{d}x + \int_\Omega (2^\alpha - 2^\theta)|\tilde{u}|^\alpha \,\mathrm{d}x \\
&\qquad + \int_\Omega \left[2^\theta f(\tilde{u})\tilde{u} - f(2\tilde{u})2\tilde{u}\right] \mathrm{d}x \\
&< 0.
\end{aligned}
$$

For $t = \frac{1}{2}$, we have

$$
\begin{aligned}
h\left(\frac{1}{2}\right) &= \int_\Omega \left[\frac{1}{4}\phi'\left(\frac{1}{4}|\nabla \tilde{u}|^2\right)|\nabla \tilde{u}|^2 + \frac{1}{2^\alpha}|\tilde{u}|^\alpha - f\left(\frac{1}{2}\tilde{u}\right)\frac{1}{2}\tilde{u}\right] \mathrm{d}x \\
&= \int_\Omega \left[\frac{1}{4}\phi'\left(\frac{1}{4}|\nabla \tilde{u}|^2\right)|\nabla \tilde{u}|^2 - \frac{1}{2^\theta}\phi'(|\nabla \tilde{u}|^2)|\nabla \tilde{u}|^2\right] \mathrm{d}x + \int_\Omega \left(\frac{1}{2^\alpha} - \frac{1}{2^\theta}\right)|\tilde{u}|^\alpha \,\mathrm{d}x \\
&\quad + \int_\Omega \left[\frac{1}{2^\theta} f(\tilde{u})\tilde{u} - f\left(\frac{1}{2}\tilde{u}\right)\frac{1}{2}\tilde{u}\right] \mathrm{d}x \\
&> 0.
\end{aligned}
$$

Consequently, we can find $\tilde{t} \in (\frac{1}{2}, 2)$ such that $h(\tilde{t}) = 0$. It implies $\tilde{t}\tilde{u} + \varepsilon\eta(\tilde{t})\varphi \in \mathrm{M}(\Omega)$, which contradicts with (2.11). From this, $\tilde{u}$ is a solution for equation (2.1).

If $\alpha \geq q$, we infer that the solution $\tilde{u}$ is positive by Theorem 1 of [30]. Thus, we complete the proof. □

We shall show any $\mathcal{W}_r(\mathbb{R}^N)$-solution of the equation (1.7) is $C^{1,\gamma}_{loc}(\mathbb{R}^N)$-solution of the equation (1.7).

Lemma 2.7. *Assume u be a weak solution of (1.7), $1 < p < q < min\{N, p^*\}$, $q \leq \alpha \leq p^* q'/p'$, $u \in \mathcal{W}_r(\mathbb{R}^N)$, $(\Phi 1)$–$(\Phi 5)$ and (f_1)–(f_3) hold, then $u \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $0 < \gamma < 1$.*

Proof. We first prove by the Moser's iteration that $u \in L^\infty(\mathbb{R}^N)$, then belongs to $C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$. Since $u \in \mathcal{W}_r(\mathbb{R}^N)$, $u \in L^{p^*}(\mathbb{R}^N)$. For $r > 0$ to be determined later, taking $\varphi = |u^T|^{pr} u$ as a test function with

$$
u^T = \begin{cases} T, & u > T, \\ u, & |u| \leq T, \\ -T, & u < -T. \end{cases}
$$

Moreover, without any loss of generality, we shall assume that $T > 1$. Then $\nabla u \nabla \varphi = pr|u^T|^{pr}|\nabla u^T|^2 + |u^T|^{pr}|\nabla u|^2$, u is a weak solution of equation (1.7), i.e.,

$$
\int_{\mathbb{R}^N} \phi'(|\nabla u|^2)\nabla u \nabla \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{\alpha-2} u\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N} f(u)\varphi \,\mathrm{d}x.
$$

We have

$$
\begin{aligned}
(pr+1)&\int_{|u|\leq T} \phi'(|\nabla u|^2)|\nabla u|^2 |u^T|^{pr} \,\mathrm{d}x \\
&+ \int_{|u|>T} \phi'(|\nabla u|^2)|\nabla u|^2 |u^T|^{pr} \,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^\alpha |u^T|^{pr} \,\mathrm{d}x = \int_{\mathbb{R}^N} f(u)u|u^T|^{pr} \,\mathrm{d}x.
\end{aligned}
$$

Define $A = \{x \in \mathbb{R}^N : |u| \le T\} \cap \Lambda^c_{\nabla u}$ and $B = \{x \in \mathbb{R}^N : |u| \le T\} \cap \Lambda_{\nabla u}$, then

$$\begin{aligned}
\int_{\mathbb{R}^N} f(u)u|u^T|^{pr}\,\mathrm{d}x &\ge (pr+1)\int_{|u|\le T}\phi'(|\nabla u|^2)|\nabla u|^2|u^T|^{pr}\,\mathrm{d}x + \int_{|u|\le T}|u|^\alpha|u^T|^{pr}\,\mathrm{d}x\\
&\ge C(1+r)^{1-p}\min\Big\{\int_A |\nabla|u|^{1+r}|^p\,\mathrm{d}x, \int_B |\nabla|u|^{1+r}|^q\,\mathrm{d}x\Big\}\\
&\quad + \frac{1}{T^{(\alpha-p)r}}\int_{|u|\le T}||u|^{1+r}|^\alpha\,\mathrm{d}x\\
&\ge C(1+r)^{1-p}\Big[\|\nabla|u|^{1+r}\|^q_{L^p(|u|\le T)+L^q(|u|\le T)} + \||u|^{1+r}\|^\alpha_{L^\alpha(|u|\le T)}\Big]\\
&\ge C(1+r)^{1-p}\||u|^{1+r}\|^p_{L^{p^*}(|u|\le T)}\\
&\ge C(1+r)^{1-p}\Big(\int_{|u|\le T}|u|^{(1+r)p^*}\,\mathrm{d}x\Big)^{\frac{p}{p^*}}.
\end{aligned}$$

Set $d = 1+r = \frac{Np-(N-p)(s-p)}{(N-p)p} > 1$, $s \in (\alpha, p^*)$. Let $T \to +\infty$, by equation (2.3) and Hölder inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} f(u)u|u^T|^{pr}\,\mathrm{d}x &\le C_\varepsilon\int_{\mathbb{R}^N}|u|^{s-p}|u|^{pr+p}\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^N}|u|^{\alpha-p}|u|^{pr+p}\,\mathrm{d}x\\
&\le C_\varepsilon\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{p^*-pd}{p^*}}\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{pd}{p^*}}\\
&\quad+\varepsilon\Big(\int_{\mathbb{R}^N}|u|^{\bar\alpha}\,\mathrm{d}x\Big)^{\frac{p^*-pd}{p^*}}\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{pd}{p^*}}\\
&\le C\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{pd}{p^*}},
\end{aligned}$$

where $\alpha < \bar\alpha = \frac{(\alpha-p)(Np)}{(N-p)(s-p)} < p^*$. Then we get

$$\Big(\int_{\mathbb{R}^N}|u|^{p^*d}\,\mathrm{d}x\Big)^{\frac{p}{p^*}} \le C(1+r)^{p-1}\int_{\mathbb{R}^N}f(u)u|u^T|^{pr}\,\mathrm{d}x \le C(1+r)^{p-1}\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{pd}{p^*}}.$$

Hence

$$\Big(\int_{\mathbb{R}^N}|u|^{p^*d}\,\mathrm{d}x\Big)^{\frac{1}{p^*d}} \le C(1+r)^{\frac{p-1}{pd}}\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{1}{p^*}}.$$

Therefore

$$\Big(\int_{\mathbb{R}^N}|u|^{p^*d^k}\,\mathrm{d}x\Big)^{\frac{1}{p^*d^k}} \le (\Pi_{i=1}^k Cd^i)^{\frac{1}{d^i}}\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{1}{p^*}}.$$

Since $\Pi_{i=1}^\infty (Cd^i)^{\frac{1}{d^i}} \le C^*$ for some constant $C^* > 0$, we then deduce that $u \in L^\infty(\mathbb{R}^N)$. Suppose u is a weak solution of the equation (1.7) and $u \in \mathcal{W}_r(\mathbb{R}^N)$, we have that $u \in \mathcal{C}^{1,\gamma}_{loc}(\mathbb{R}^N)$ for some $\gamma > 0$ by Chapter 4 of [19] or [33]. □

3 Existence of sign-changing solutions

In this section, we construct infinitely many nodal solutions for equation (1.7). For any given k numbers r_j $(j = 1,\dots,k)$ such that $0 < r_1 < r_2 < \cdots < r_k < +\infty$, we denote $r_0 = 0, r_{k+1} = \infty$,

$$\Omega^1 = \{x \in \mathbb{R}^N : |x| < r_1\} \quad \text{and} \quad \Omega^j = \{x \in \mathbb{R}^N : r_{j-1} < |x| < r_j\}.$$

We will always extend $u_j \in \mathcal{W}_r(\Omega^j)$ to $\mathcal{W}_r(\mathbb{R}^N)$ by setting $u_j \equiv 0$ for $x \in \mathbb{R}^N \backslash \Omega^j$ for every u_j, $j = 1, 2, \ldots, k+1$. For convenience, we use $I(u_j)$ to replace $I_{\Omega^j}(u_j)$ and $\gamma(u_j)$ to replace $\gamma_{\Omega^j}(u_j)$. Define

$$Y_k^{\pm}(r_1, r_2, \ldots, r_{k+1}) = \Big\{ u \in \mathcal{W}_r(\mathbb{R}^N) \mid u = \pm \sum_{j=1}^{k+1} (-1)^{j-1} u_j,\ u_j \geq 0,$$
$$u_j \not\equiv 0,\ u_j \in \mathcal{W}_r(\Omega^j),\ j = 1, 2, \ldots, k+1 \Big\},$$

$$\mathrm{M}_k^{\pm} = \{ u \in \mathcal{W}_r(\mathbb{R}^N) \mid \exists\, 0 = r_0 < r_1 < r_2 < \cdots < r_k < r_{k+1} = +\infty,$$
$$\text{such that } u \in Y_k^{\pm}(r_1, r_2, \ldots, r_{k+1}) \text{ and } u_j \in \mathrm{M}(\Omega^j),\ j = 1, 2, \ldots, k+1 \}.$$

Note that $\mathrm{M}_k^{\pm} \neq \varnothing$, $k = 1, 2, \ldots$ In order to prove the existence of non-negative critical points of energy functional I, similar to [6] or [10], we only need to extend $f(u)$ as follows

$$f^+(u) := \begin{cases} f(u), & \text{if } u \geq 0, \\ -f(-u), & \text{if } u < 0, \end{cases}$$

thus the oddness assumption on nonlinear term is actually unnecessary. The function $I^+(u)$ is defined on $\mathcal{W}_r(\mathbb{R}^N)$ by

$$I^+(u) = \frac{1}{2} \int_{\mathbb{R}^N} \phi(|\nabla u|^2)\, \mathrm{d}x + \frac{1}{\alpha} \int_{\mathbb{R}^N} |u|^{\alpha}\, \mathrm{d}x - \int_{\mathbb{R}^N} F^+(u)\, \mathrm{d}x,$$

$c_k^+ = \inf_{u \in \mathrm{M}_k^+} I^+(u)$ in the same way as those in [10]. For M_k^-, we can complete the proof in the same way. By the arguments of the Section 2, it is not difficult to verify that

$$\forall\, u = \sum_{j=1}^{k+1} (-1)^{j-1} u_j \in \mathrm{M}_k^+ \Leftrightarrow I(u) = \max_{\substack{\alpha_j > 0 \\ 1 \leq j \leq k+1}} I\Big(\sum_{j=1}^{k+1} \alpha_j \check{u}_j \Big),$$

where $\check{u}_j = (-1)^{j-1} u_j$.

Set

$$c_k = \inf_{u \in \mathrm{M}_k^+} I(u), \qquad k = 1, 2, \ldots$$

Lemma 3.1. *c_k is attained provided that $1 < p < q < \min\{N, p^*\}$, $q \leq \alpha \leq p^* q'/p'$, $(\Phi 1)$–$(\Phi 5)$ and (f_1)–(f_3) hold.*

Proof. We prove by induction that for each k there exists $\bar{u}_k \in \mathrm{M}_k^+$ such that

$$I(\bar{u}_k) = c_k.$$

For $k = 0$ or $\Omega = \mathbb{R}^N$, we can directly derive from Lemma 2.6. We discuss the case $k \geq 1$ in the following.

First, we prove I is bounded from below on M_k^+ by a positive constant. Let $\bar{u} \in \mathrm{M}_k^+$, then $\bar{u} = \sum_{j=1}^{k+1} (-1)^{j-1} \bar{u}_j$ and $\bar{u}_j \in \mathrm{M}(\Omega^j)$, $j = 1, 2, \ldots, k+1$. By the similar arguments of inequality (2.10), we have $\|\bar{u}_j\|_{\Omega^j} \geq C_j$. It follows from the same computations in (2.9) that

$$I(\bar{u}) = I\Big(\sum_{j=1}^{k+1} (-1)^{j-1} \bar{u}_j \Big) = \sum_{j=1}^{k+1} I(\bar{u}_j) \geq C \sum_{j=1}^{k+1} \|\bar{u}_j\|_{\Omega_j}^{\alpha} \geq C \sum_{j=1}^{k+1} C_j^{\alpha} = \bar{C}. \tag{3.1}$$

There exists a positive constant $\bar{C} > 0$ such that $I(\bar{u}) \geq \bar{C}$, for all $\bar{u} \in \mathrm{M}_k^+$.

Second, we suppose the conclusion is true for $k-1$ and let $\{\bar{u}_m\}_{m\geq 1}$ be a minimizing sequence of c_k in M_k^+, that is

$$\lim_{m\to\infty} I(\bar{u}_m) = c_k, \qquad \bar{u}_m \in \mathrm{M}_k^+, \qquad m = 1,2,\ldots$$

$\bar{u}_m$ corresponding to k nodes, $r_m^1, r_m^2, \ldots, r_m^k$, with $0 < r_m^1 < r_m^2 < \cdots < r_m^k < \infty$, set

$$\Omega_m^i = \{x \in \mathbb{R}^N : r_m^{i-1} < |x| < r_m^i\},$$

and

$$\bar{u}_m^i = \begin{cases} \bar{u}_m, & \text{if } x \in \Omega_m^i, \\ 0, & \text{if } x \notin \Omega_m^i. \end{cases}$$

We can select a subsequence $\{r_m^i\}$ such that $\lim_{m\to\infty} r_m^i = r_i$, and $0 \leq r_1 \leq r_2 \leq \cdots \leq r_k \leq +\infty$.

Now we give the following claims.

Claim 1: Under the assumptions of Lemma 3.1, $r_i \neq r_{i-1}$, $i = 1,2,\ldots,k$. Here we denote $r_0 = 0$.

If $r_i = r_{i-1}$ for some $i \in \{1,\ldots,k\}$. Suppose there exists $i_0 \in \{1,\ldots,k\}$ such that $r_{i_0} = r_{i_0-1}$, then $\lim_{m\to\infty} r_m^{i_0} = \lim_{m\to\infty} r_m^{i_0-1}$. We denote the measure of $\Omega_m^{i_0}$ by $|\Omega_m^{i_0}|$, so that $|\Omega_m^{i_0}| \to 0$ as $m \to \infty$. Since $\bar{u}_m^{i_0} \in \mathrm{M}(\Omega_m^{i_0})$, by Proposition 2.2 of [1] and Lemma 2.1, we have

$$\begin{aligned}
\|\bar{u}_m^{i_0}\|_{\Omega_m^{i_0}}^{\alpha} &\leq C\Big\{ \|\nabla \bar{u}_m^{i_0}\|^q_{L^p(\Omega_m^{i_0})+L^q(\Omega_m^{i_0})} + \|\bar{u}_m^{i_0}\|^{\alpha}_{\Omega_m^{i_0}} \Big\} \\
&\leq C\Big\{ \max\Big\{ \int_{\{x\in\Omega_m^{i_0}:|\nabla \bar{u}_m^{i_0}|\leq 1\}} |\nabla \bar{u}_m^{i_0}|^q \,\mathrm{d}x, \int_{\{x\in\Omega_m^{i_0}:|\nabla \bar{u}_m^{i_0}|>1\}} |\nabla \bar{u}_m^{i_0}|^p \,\mathrm{d}x \Big\} + \int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^{\alpha} \,\mathrm{d}x \Big\} \\
&\leq C\Big\{ \int_{\Omega_m^{i_0}} \phi'(|\nabla \bar{u}_m^{i_0}|^2)|\nabla \bar{u}_m^{i_0}|^2 \,\mathrm{d}x + \int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^{\alpha} \,\mathrm{d}x \Big\} \\
&\leq C \int_{\Omega_m^{i_0}} f(\bar{u}_m^{i_0})\bar{u}_m^{i_0} \,\mathrm{d}x \\
&\leq C_\varepsilon \int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^s \,\mathrm{d}x + \varepsilon \int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^{\alpha} \,\mathrm{d}x \\
&\leq C_\varepsilon \int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^s \,\mathrm{d}x + \varepsilon \|\bar{u}_m^{i_0}\|^{\alpha}.
\end{aligned}$$

Let $\varepsilon = \frac{1}{2}$, then

$$\|\bar{u}_m^{i_0}\|^{\alpha}_{\Omega_m^{i_0}} \leq C\left(\int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^{p^*} dx \right)^{\frac{s}{p^*}} |\Omega_m^{i_0}|^{1-\frac{s}{p^*}} \leq C\|\bar{u}_m^{i_0}\|^s_{\Omega_m^{i_0}} |\Omega_m^{i_0}|^{1-\frac{s}{p^*}}.$$

Since C is positive constants and $\alpha < s < p^*$, we deduce that

$$\|\bar{u}_m^{i_0}\|_{\Omega_m^{i_0}} \to \infty, \quad \text{as } m \to \infty.$$

By inequality (3.1),

$$I(\bar{u}_m^{i_0}) \to \infty, \quad \text{as } m \to \infty. \tag{3.2}$$

From the inductive assumption and equation (3.2), for $\varepsilon > 0$ fixed we can choose $L > 0$ such that

$$I(\bar{u}_m^{i_0}) > c_k - c_{k-1} + \varepsilon, \qquad |I(\bar{u}_m) - c_k| < \varepsilon, \quad \text{as } m \geq L.$$

Then we define $\bar{v}(x) \in \mathrm{M}_{k-1}^{+}$ by

$$\bar{v}(x) = \begin{cases} \bar{u}_m^l(x), & \text{if } x \in \Omega_m^l \ \text{ as } l < i, \\ 0, & \text{if } x \in \Omega_m^{i_0}, \\ \bar{u}_m^l(x), & \text{if } x \in \Omega_m^l \ \text{ as } l > i. \end{cases}$$

Hence

$$I(\bar{v}(x)) = I(\bar{u}_m) - I(\bar{u}_m^{i_0}) < c_k + \varepsilon - (c_k - c_{k-1} + \varepsilon) = c_{k-1}, \quad \text{as } m \geq L,$$

which contradicts with $c_{k-1} = \inf_{u \in \mathrm{M}_{k-1}^{+}} I(u)$. Thus $r_i \neq r_{i-1}$, $i = 1, 2, \ldots, k$. Then the proof of Claim 1 is completed.

Claim 2: Under the assumptions of Lemma 3.1, $r^k < \infty$.

If $r^k \to \infty$, then $r_m^k \to \infty$. It follows from Claim 1 and $\bar{u}_m^k \in \mathrm{M}(\Omega_m^k)$ that

$$\begin{aligned} \|\bar{u}_m^k\|_{\Omega_m^k}^{\alpha} &\leq C\Big\{ \max\Big\{ \int_{\{x\in\Omega_m^k : |\nabla \bar{u}_m^k| \leq 1\}} |\nabla \bar{u}_m^k|^q \,\mathrm{d}x, \int_{\{x\in\Omega_m^k : |\nabla \bar{u}_m^k| > 1\}} |\nabla \bar{u}_m^k|^p \,\mathrm{d}x \Big\} + \int_{\Omega_m^k} |\bar{u}_m^k|^{\alpha} \,\mathrm{d}x \Big\} \\ &\leq C\Big\{ \int_{\Omega_m^k} \phi'(|\nabla \bar{u}_m^k|^2) |\nabla \bar{u}_m^k|^2 \,\mathrm{d}x + \int_{\Omega_m^k} |\bar{u}_m^k|^{\alpha} \,\mathrm{d}x \Big\} \\ &\leq C_\varepsilon \int_{\Omega_m^k} |\bar{u}_m^k|^{s} \,\mathrm{d}x + \varepsilon \int_{\Omega_m^k} |\bar{u}_m^k|^{\alpha} \,\mathrm{d}x. \end{aligned}$$

Using Lemma 2.3, we deduce that

$$\|\bar{u}_m^k\|_{\Omega_m^k}^{\alpha} \leq C \|\bar{u}_m^k\|_{\Omega_m^k}^{s-\alpha} \int_{\Omega_m^k} |\bar{u}_m^k|^{\alpha} |x|^{\frac{(q-N)(s-\alpha)}{q}} \,\mathrm{d}x \leq C \|\bar{u}_m^k\|_{\Omega_m^k}^{s} |r_m^k|^{\frac{(q-N)(s-\alpha)}{q}},$$

so $\|\bar{u}_m^k\|_{\Omega_m^k} \geq C |r_m^k|^{\frac{N-q}{q}}$. By inequality (3.1) we find

$$I(\bar{u}_m^k) \to \infty, \quad \text{as} \quad m \to \infty. \tag{3.3}$$

Similar to the proof of Claim 1, we can obtain $r^k < \infty$. Claim 2 is therefore proved.

From the above two claims, by selecting a subsequence, we may assume that $\lim_{m\to\infty} r_m^i = r^i$, and clearly $0 < r^1 < r^2 < \cdots < r^k < \infty$. Define $\Omega^i = \{x \in \mathbb{R}^N \mid r^{i-1} < |x| < r^i\}$, for all $i = 1, 2, \ldots, k+1$, $r^0 = 0$, $r^{k+1} = +\infty$. Lemma 2.6 implies that $\bar{c} = \inf_{u \in \mathrm{M}(\Omega^i)} I(u)$ is attained by some positive function $\hat{u}^i$ which satisfies the following boundary value problem

$$\begin{cases} -\nabla \cdot [\phi'(|\nabla u|^2) \nabla u] + |u|^{\alpha-2} u = f(u), & x \in \Omega^i, \\ u|_{\partial \Omega^i} = 0. \end{cases}$$

Define $\bar{u}_k = \sum_{i=1}^{k+1} (-1)^{i-1} \hat{u}^i(x)$, $(\hat{u}^i = 0, x \notin \Omega^i)$. Thus $\bar{u}_k \in \mathrm{M}_k^{+}$.

We define functions $v_m^i : [r^{i-1}, r^i] \to \mathbb{R}$ such that

$$\begin{cases} v_m^i := a_m^i \bar{u}_m^i \Big(\frac{r_m^i - r_m^{i-1}}{r^i - r^{i-1}} (t - r^{i-1}) + r_m^{i-1} \Big), & \text{for } i = 1, \ldots, k, \\ v_m^{k+1} := a_m^{k+1} \bar{u}_m^{k+1} \Big(\frac{r_m^k}{r^k} t \Big), \end{cases}$$

where $r_m^0 = 0$, $r_m^{k+1} = \infty$ and a_m^i is a unique positive real number such that $v_m^i \in \mathrm{M}(\Omega^i)$, for all $i = 1, 2, \ldots, k+1$. For m large enough, we can compute that

$$\int_{\Omega^i} \phi'(|\nabla v_m^i|^2) |\nabla v_m^i|^2 \,\mathrm{d}t = \int_{\Omega_m^i} \phi'(|a_m^i|^2 |\nabla \bar{u}_m^i|^2) |a_m^i|^2 |\nabla \bar{u}_m^i|^2 \,\mathrm{d}x + o(1),$$

$$\int_{\Omega^i} |v_m^i|^\alpha \, dt = |a_m^i|^\alpha \int_{\Omega_m^i} |\bar{u}_m^i|^\alpha \, dx + o(1),$$

$$\int_{\Omega^i} f(v_m^i) v_m^i \, dt = \int_{\Omega_m^i} f(a_m^i \bar{u}_m^i) a_m^i \bar{u}_m^i \, dx + o(1).$$

Since $v_m^i \in \mathrm{M}(\Omega^i)$, it follows

$$\int_{\Omega_m^i} \phi'(|a_m^i|^2 |\nabla \bar{u}_m^i|^2) |a_m^i|^2 |\nabla \bar{u}_m^i|^2 \, dx + |a_m^i|^\alpha \int_{\Omega_m^i} |\bar{u}_m^i|^\alpha \, dx - \int_{\Omega_m^i} f(a_m^i \bar{u}_m^i) a_m^i \bar{u}_m^i \, dx = o(1), \tag{3.4}$$

for all $i = 1, 2, \ldots, k+1$. Note that it also holds

$$\int_{\Omega_m^i} \phi'(|\nabla \bar{u}_m^i|^2) |\nabla \bar{u}_m^i|^2 \, dx + \int_{\Omega_m^i} |\bar{u}_m^i|^\alpha \, dx - \int_{\Omega_m^i} f(\bar{u}_m^i) \bar{u}_m^i \, dx = 0, \tag{3.5}$$

for each i. Using an argument similar to that in the proof of Lemma 2.4, by (3.4) and (3.5), we can obtain that $\lim_{m\to\infty} a_m^i = 1$ for all i. Therefore we deduce that

$$\lim_{m\to\infty} I\left(a_m^i \bar{u}_m^i(x)\right) = \lim_{m\to\infty} I\left(\bar{u}_m^i(x)\right).$$

On the other hand, since $I(\hat{u}^i) = \inf_{u\in \mathrm{M}(\Omega^i)} I(u)$ and $a_m^i \bar{u}_m^i(x) \in \mathrm{M}(\Omega^i)$, we have

$$I(\hat{u}^i) \le I\left(a_m^i \bar{u}_m^i(x)\right).$$

Thus

$$\lim_{m\to\infty} I\left(\bar{u}_m^i(x)\right) \ge I(\hat{u}^i),$$

and

$$c_k = \lim_{m\to\infty} I(\bar{u}_m(x)) = \lim_{m\to\infty} \sum_{i=1}^{k+1} I(\bar{u}_m^i(x)) \ge \sum_{i=1}^{k+1} I(\hat{u}^i) = I(\bar{u}_k).$$

Since $\bar{u}_k \in \mathrm{M}_k^+$, which means that c_k is attained. □

Now, we begin to prove Theorem 1.3. Because the weak solutions of (1.7) are of class $C_{loc}^{1,\gamma}(\mathbb{R}^N)$, as stated in Lemma 2.7. We apply some ideas of in [21,22,35] to prove the minimizer of c_k is the weak solution of (1.7) instead of glue the function in each annuli by matching the normal derivative at each junction point.

Proof of Theorem 1.1. By Lemma 3.1, there exists $\bar{u}_k \in \mathrm{M}_k^+$ which attains c_k. Thus we get k nodes:

$$r_1, r_2, \cdots, r_k, \qquad 0 < r_1 < r_2 < \cdots < r_k < +\infty, \qquad \Omega^i = \{x \in \mathbb{R}^N : r_{i-1} < |x| < r_i\}$$

and

$$(\bar{u}_k)^i = \begin{cases} \bar{u}_k(x), & x \in \Omega^i, \\ 0, & x \notin \Omega^i. \end{cases}$$

For convenience, $u := \bar{u}_k$, and u satisfies equation (1.7) in $\{x \in \mathbb{R}^N : |x| \ne r_i, i = 1, 2, \ldots, k\}$.

In order to show that u is a critical point of I. We assume by contradiction that there exists $\psi \in \mathcal{W}_r'(\mathbb{R}^N)$ such that

$$\langle I'(u), \psi \rangle = -2.$$

Similarly to the proof of Step 2 in Lemma 2.6 we choose $\delta \in (0,1)$ such that if $s = (s_1, s_2, \dots, s_{k+1}) \in D$ and $0 \le \epsilon \le \delta$, then

$$\left\langle I'\left(\sum_{i=1}^{k+1} s_i u^i + \epsilon\psi\right), \psi\right\rangle < -1,$$

where

$$D = \{(s_1, \dots, s_{k+1}) \in \mathbb{R}^{k+1} \ : \ |s_i - 1| \le \delta, \text{ for all } i \in \{1, \dots, k+1\}\}.$$

There is a sufficiently small ϵ such that $\sum_{i=1}^{k+1} s_i u^i + \epsilon\psi$ changes sign exactly k times with k nodes $0 < r_1(s,\epsilon) < \cdots < r_k(s,\epsilon) < \infty$. Here $r_j(s,\epsilon)$ denotes that r_j depends on s, ϵ for all $j = 1, \cdots, k$. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function which satisfies $\eta(s) = 0$ in a neighborhood of ∂D, $\eta(1, \dots, 1) = 1$ and $0 \le \eta(s) \le 1$ for all $s \in D$. If δ is small enough, we see that $\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi$ also has exactly k nodes $0 < r_1(s) < \cdots < r_k(s) < \infty$ for all $s \in D$, $r_j(s)$ is continuous about s for every $j = 1, \dots, k$, and

$$\left\langle I'\left(\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi\right), \psi\right\rangle < -1. \tag{3.6}$$

We claim that there exists $s \in D$ such that $\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi \in \mathrm{M}_k^+$. Let

$$H_i(s) = \int_{\mathbb{R}^N} \left[\phi'(|\nabla g_i(s)|^2)|\nabla g_i(s)|^2 + |g_i(s)|^\alpha - f(g_i(s))g_i(s)\right] \mathrm{d}x, \qquad \forall\, 1 \le i \le k+1,$$

and

$$g_i(s) = \left(\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi\right)\Bigg|_{\Omega_s^i},$$

where $\Omega_s^i = \{x \in \mathbb{R}^N : r_{i-1}(s) < |x| < r_i(s)\}$ for all $1 \le i \le k+1$, $r_0(s) = 0$ and $r_{k+1}(s) = \infty$. Suppose that $s \in \partial D$, then $\eta(s) = 0$, $g_i(s) = s_i u^i$. For $s_i = 1 + \delta$, by (2.7)–(2.8), we have

$$\begin{aligned}
H_i(1+\delta) &= \int_{\Omega^i} \left[(1+\delta)^2\phi'((1+\delta)^2|\nabla u^i|^2)|\nabla u^i|^2 + (1+\delta)^\alpha|u^i|^\alpha - f((1+\delta)u^i)(1+\delta)u^i\right] \mathrm{d}x \\
&= \int_{\Omega^i} \left[(1+\delta)^2\phi'((1+\delta)^2|\nabla u^i|^2)|\nabla u^i|^2 - (1+\delta)^\theta\phi'(|\nabla u^i|^2)|\nabla u^i|^2\right] \mathrm{d}x \\
&\quad + \int_{\Omega^i} ((1+\delta)^\alpha - (1+\delta)^\theta)|u^i|^\alpha \,\mathrm{d}x + \int_{\Omega^i} \left[(1+\delta)^\theta f(u^i)u^i - f((1+\delta)u^i)(1+\delta)u^i\right] \mathrm{d}x \\
&< 0.
\end{aligned}$$

For $s_i = 1 - \delta$, we get

$$\begin{aligned}
H_i(1-\delta) &= \int_{\Omega^i} \left[(1-\delta)^2\phi'((1-\delta)^2|\nabla u^i|^2)|\nabla u^i|^2 + (1-\delta)^\alpha|u^i|^\alpha - f((1-\delta)u^i)(1-\delta)u^i\right] \mathrm{d}x \\
&= \int_{\Omega^i} \left[(1-\delta)^2\phi'((1-\delta)^2|\nabla u^i|^2)|\nabla u^i|^2 - (1-\delta)^\theta\phi'(|\nabla u^i|^2)|\nabla u^i|^2\right] \mathrm{d}x \\
&\quad + \int_{\Omega^i} ((1-\delta)^\alpha - (1-\delta)^\theta)|u^i|^\alpha \,\mathrm{d}x + \int_{\Omega^i} \left[(1-\delta)^\theta f(u^i)u^i - f((1-\delta)u^i)(1-\delta)u^i\right] \mathrm{d}x \\
&> 0.
\end{aligned}$$

By the homotopy invariance of the topological degree (or Miranda's Theorem [23]), we see that there exists $s \in D$ such that $H_i(s) = 0$ for all $1 \le i \le k+1$. That is $\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi \in \mathrm{M}_k^+$.

From the claim, we get $I\big(\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi\big) \geq c_k$. On the other hand, by (3.6), there holds that

$$\begin{aligned} I\Big(\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi\Big) &= I\Big(\sum_{i=1}^{k+1} s_i u^i\Big) + \int_0^1 \Big\langle I'\Big(\sum_{i=1}^{k+1} s_i u^i + \sigma\delta\eta(s)\psi\Big), \delta\eta(s)\psi\Big\rangle \,\mathrm{d}\sigma \\ &\leq I\Big(\sum_{i=1}^{k+1} s_i u^i\Big) - \delta\eta(s). \end{aligned}$$

If $s_i = 1$ for all $1 \leq i \leq k+1$, then we have

$$c_k \leq I\Big(\sum_{i=1}^{k+1} u^i\Big) - \delta\eta(1,\dots,1) = c_k - \delta,$$

which is impossible. If $s_i \neq 1$ for some $1 \leq i \leq k+1$, then we obtain

$$\begin{aligned} c_k \leq I\Big(\sum_{i=1}^{k+1} s_i u^i\Big) &= \sum_{i=1}^{k+1} \int_{\Omega_i} \Big[\phi'\Big(s_i^2|\nabla u^i|^2\Big)s_i^2|\nabla u^i|^2 + s_i^\alpha|u^i|^\alpha - f(s_i u^i)(s_i u^i)\Big]\,\mathrm{d}x \\ &= \sum_{i=1}^{k+1} I_{\Omega_i}(s_i u^i) < \sum_{i=1}^{k+1} I_{\Omega_i}(u^i) = I\Big(\sum_{i=1}^{k+1} u^i\Big) = c_k, \end{aligned}$$

which is also a contradiction.

Therefore, the function u is indeed a radial solution of (1.7), which changes sign exactly k times. We complete the proof. □

Acknowledgements

The author would like to thank Professor Yinbin Deng very much for stimulating discussions and helpful suggestions on the present paper. The research of X. L. Yang was supported by the Natural Science Foundation of China (No. 11771170) and the graduate education innovation funding [grant number 2019CXZZ082] from Central China Normal University.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 33, 1–19; https://doi.org/10.14232/ejqtde.2021.1.33 www.math.u-szeged.hu/ejqtde/

Infinitely many radial positive solutions for nonlocal problems with lack of compactness

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Received 14 January 2021, appeared 11 April 2021
Communicated by Dimitri Mugnai

Abstract. We are concerned with the qualitative and asymptotic analysis of solutions to the nonlocal equation

$$(-\Delta)^s u + V(|z|)u = Q(|z|)u^p \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $0 < s < 1$, and $1 < p < \frac{2N}{N-2s}$. As $r \to \infty$, we assume that the potentials $V(r)$ and $Q(r)$ behave as

$$V(r) = V_0 + \frac{a_1}{r^\alpha} + O\left(\frac{1}{r^{\alpha+\theta_1}}\right)$$
$$Q(r) = Q_0 + \frac{a_2}{r^\beta} + O\left(\frac{1}{r^{\beta+\theta_2}}\right)$$

where $a_1,\ a_2 \in \mathbb{R}$, $\alpha,\ \beta > \frac{N+2s}{N+2s+1}$, and $\theta_1,\ \theta_2 > 0,\ V_0,\ Q_0 > 0$. Under various hypotheses on a_1, a_2, α, β, we establish the existence of infinitely many radial solutions. A key role in our arguments is played by the Lyapunov–Schmidt reduction method.

Keywords: fractional Laplacian, radial solution, lack of compactness, Lyapunov–Schmidt reduction method.

2020 Mathematics Subject Classification: 35R11, 35A15, 35B40, 47G20.

1 Introduction and the main result

We consider the following nonlocal equation driven by the fractional Laplace operator

$$(-\Delta)^s u + V(|z|)u = Q(|z|)u^p, \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

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Fractional powers of the Laplacian arise in various equations in mathematical physics and related fields; see, e.g., [1], [9], and [14]. Numerous results related to equations with fractional Laplace operator sprout in literature. A characterization of the fractional Laplacian through Dirichlet–Neumann maps was given in [3]. Regularity for fractional elliptic equations was investigated in [4] and [17]. Existence of solutions was studied in many papers; see, e.g., [2,7,12].

Along with different results, there are various enlightening approaches. In [10], the author obtained some symmetry results for equations involving the fractional Laplacian in $\mathbb{R}^N$ by the method of moving planes. In [2], symmetry results for nonlinear equations with fractional Laplacian were achieved by the sliding method. Geometric inequality was applied to investigate symmetry properties for a boundary reaction problem in [18]. The method of moving planes and ABP (Aleksandrov–Bakelman–Pucci) estimates for fractional Laplacian were employed in [6] to study radial symmetry and monotonicity properties for positive solutions of fractional Laplacian. We refer the readers to [7] and [11] for very recent new approaches dealing with fractional Laplacian equations, and to [15] for a comprehensive overview of variational methods for nonlocal fractional problems.

Inspired by [19], we obtain the existence of radial positive solutions to (1.1) by Lyapunov–Schmidt reduction. To the best of our knowledge, this method has never been employed in investigating radial solutions to equations as (1.1).

We will use the radial solution of

$$(-\Delta)^s u + u = u^p \quad \text{in } \mathbb{R}^N \tag{1.2}$$

to build up the approximate solutions of problem (1.1). The uniqueness and nondegeneracy of the radial positive solution to problem (1.2) are established in [8].

Our result is based on the following growth assumptions for $V(|z|)$ and $Q(|z|)$ near infinity:

(V): there exist constants $a_1 \in R$, $\alpha > 1$, and $\theta_1 > 0$, such that $V(r) = V_0 + \frac{a_1}{r^\alpha} + O(\frac{1}{r^{\alpha+\theta_1}})$ as $r \to \infty$;

(Q): there exist constants $a_2 \in R$, $\beta > 1$, and $\theta_2 > 0$, such that $Q(r) = Q_0 + \frac{a_2}{r^\beta} + O(\frac{1}{r^{\beta+\theta_2}})$ as $r \to \infty$.

We assume throughout this paper that $V_0 = 1$ and $Q_0 = 1$.

Let

$$x_j = \left(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\right), \qquad j = 1, \dots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, $r \in \left[r_0 k^{\frac{N+2s}{N+2s-\tau}}, r_1 k^{\frac{N+2s}{N+2s-\tau}}\right]$, $\tau = \min\{\alpha, \beta\}$, $0 < r_0 < r_1$, and k is the number of the bumps of the solution.

Set $z = (z', z'')$, $z' \in \mathbb{R}^2$, $z'' \in \mathbb{R}^{N-2}$ and define

$$H_{rs} = \Big\{ u : u \in H^s(\mathbb{R}^N), u \text{ is even in } z_h,\ h = 2, \dots, N,$$
$$u(r\cos\theta, r\sin\theta, z'') = u\left(r\cos\left(\theta + \frac{2\pi j}{k}\right), r\sin\left(\theta + \frac{2\pi j}{k}\right), z''\right)\Big\},$$

where $H^s(\mathbb{R}^N)$ represents the fractional Sobolev space

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \frac{u(x) - u(y)}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}, \qquad 0 < s < 1.$$

Let W be the unique nondegenerate radial positive solution of problem (1.2), Then the result in [8] shows that there exist constants $B_1 > B_2 > 0$, such that

$$\frac{B_2}{1+|z-x_j|^{N+2s}} \le W_{x_j}(z) \le \frac{B_1}{1+|z-x_j|^{N+2s}},$$

where $W_{x_j}(z) = W(z-x_j)$.

Set

$$U_r(z) = \sum_{j=1}^{k} W_{x_j}(z).$$

The main result of this paper establishes the following multiplicity property.

Theorem 1.1. *Assume that $V(r)$, $Q(r)$ satisfy (V) and (Q), while a_1, a_2, α, β satisfy one of the following conditions:*

(i) $a_1 > 0$, $a_2 = 0$, $\alpha < N+2s$, *and* $\alpha \le \beta$;

(ii) $a_1 > 0$, $a_2 > 0$, $\alpha < N+2s$, *and* $\beta \ge N+2s$;

(iii) $a_1 > 0$, $a_2 < 0$, $\alpha < N+2s$, *and* $\alpha > \beta$;

(iv) $a_1 = 0$, $a_2 < 0$, $\alpha \ge \beta$, *and* $\beta < N+2s$;

(v) $a_1 < 0$, $a_2 < 0$, $\alpha \ge N+2s$, *and* $\beta < N+2s$.

Then there exists a positive integer k_0 such that for any $k \ge k_0$, problem (1.1) has a solution U_k of the form

$$U_k(z) = U_{r_k}(z) + w_k,$$

where $w_k \in H_{rs}$, $r_k \in \left[r_0 k^{\frac{N+2s}{N+2s-\tau}}, r_1 k^{\frac{N+2s}{N+2s-\tau}}\right]$, and as $k \to +\infty$,

$$\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} w_k|^2 + w_k^2\right) \to 0.$$

For some of the abstract methods used in this paper, we refer to the monographs by Molica Bisci and Pucci [13] and Papageorgiou, Rădulescu and Repovš [16].

2 Reduction

Let

$$P_j = \frac{\partial W_{x_j}}{\partial r}, \qquad j = 1, \ldots, k,$$

where

$$x_j = \left(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\right), \qquad j = 1, \ldots, k$$

and

$$r \in S := \left[\left(\frac{N+2s}{\tau} - \epsilon\right)^{\frac{1}{N+2s-\tau}} k^{\frac{N+2s}{N+2s-\tau}}, \left(\frac{N+2s}{\tau} + \epsilon\right)^{\frac{1}{N+2s-\tau}} k^{\frac{N+2s}{N+2s-\tau}}\right].$$

We have denoted $\tau := \min\{\alpha, \beta\}$, where α and β are the constants in the expansions of V and Q, and $\epsilon > 0$ is a small constant.

Define

$$H := \left\{ u : u \in H_{rs}, \int_{\mathbb{R}^N} W_{x_j}^{P-1} P_j u = 0,\ j = 1, \dots, k. \right\}.$$

The norm and the inner product in $H^s(\mathbb{R}^N)$ are defined as

$$\|u\| = \langle u, u \rangle^{\frac{1}{2}}, \qquad u \in H^s(\mathbb{R}^N),$$

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(|z|) uv \right), \qquad u, v \in H^s(\mathbb{R}^N).$$

We can easily check that

$$\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(|z|) uv - pQ(|z|) U_r^{p-1} uv \right), \qquad u, v \in H$$

is a bounded bilinear functional in H. Thus, there exists a bounded linear operator M from H to H satisfying

$$\langle Mu, v \rangle = \int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(|z|) uv - pQ(|z|) U_r^{p-1} uv \right), \qquad u, v \in H. \tag{2.1}$$

We now establish that M is invertible in H.

Lemma 2.1. *There exists a constant $\rho > 0$, independent of k, such that for any $r \in S$,*

$$\|Mu\| \geq \rho \|u\|, \qquad u \in H.$$

Proof. We argue by contradiction. If the thesis does not hold, then for any $\rho_k = \frac{1}{k}(k \to +\infty)$, there exists $r_k \in S$, $u_k \in H$, such that

$$\|Mu_k\| < \rho_k \|u_k\|.$$

It follows that

$$\|Mu_k\| = o(1) \|u_k\|.$$

Then,

$$\langle Mu_k, \varphi \rangle = o(1) \|u_k\| \|\varphi\|, \qquad \forall \varphi \in H. \tag{2.2}$$

We can assume $\|u_k\|^2 = k$.

Let

$$\Omega_j = \left\{ z = (z', z'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{z'}{|z'|}, \frac{x_j}{|x_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.$$

By symmetry and the definition of M, we conclude from (2.2) that for all $\varphi \in H$,

$$\int_{\Omega_1} \left((-\Delta)^{\frac{s}{2}} u_k (-\Delta)^{\frac{s}{2}} \varphi + V(|z|) u_k \varphi - pQ(|z|) U_{r_k}^{p-1} u_k \varphi \right) = \frac{1}{k} \langle Mu_k, \varphi \rangle = o\left(\frac{1}{\sqrt{k}} \right) \|\varphi\|. \tag{2.3}$$

Particularly,

$$\int_{\Omega_1} \left(|(-\Delta)^{\frac{s}{2}} u_k|^2 + V(|z|) {u_k}^2 - pQ(|z|) U_{r_k}^{p-1} {u_k}^2 \right) = o(1)$$

and

$$\int_{\Omega_1} \left(|(-\Delta)^{\frac{s}{2}} u_k|^2 + V(|z|) {u_k}^2 \right) = 1. \tag{2.4}$$

Let $\widetilde{u_k}(z) = u_k(z + x_1)$. Since

$$|x_2 - x_1| = 2r \sin\frac{\pi}{k} \geq 2r\frac{\pi}{2k} \geq \left(\frac{N+2s}{2\tau}\right)^{\frac{1}{N+2s-\tau}} k^{\frac{\tau}{N+2s-\tau}} \pi,$$

it follows that for any $R > 0$, $B_R(x_1) \subset \Omega_1$. Then from (2.4), we have for all $R > 0$,

$$\int_{B_R(0)} \left(|(-\Delta)^{\frac{s}{2}} \widetilde{u_k}|^2 + V(|z|)\widetilde{u_k}^2\right) \leq 1.$$

So, we can assume that there exists $u \in H^s(\mathbb{R}^N)$, such that as $k \to +\infty$,

$$\widetilde{u_k} \rightharpoonup u, \quad \text{in } H^s_{\text{loc}}(\mathbb{R}^N),$$

and

$$\widetilde{u_k} \to u, \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N).$$

Since $\widetilde{u_k}$ is even in z_h, $h = 2, \ldots, N$, then u is even in z_h, $h = 2, \ldots, N$.

Besides, by

$$\int_{\mathbb{R}^N} W^{p-1}_{x_1} P_1 u_k = 0,$$

we know that

$$\int_{\mathbb{R}^N} W^{p-1}\frac{\partial W}{\partial x_1}\widetilde{u_k} = 0.$$

So, u satisfies

$$\int_{\mathbb{R}^N} W^{p-1}\frac{\partial W}{\partial x_1} u = 0. \tag{2.5}$$

We prove in what follows that u satisfies

$$(-\Delta)^s u + u - pW^{p-1}u = 0, \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

Define

$$\widetilde{H} = \left\{\varphi : \varphi \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} W^{p-1}\frac{\partial W}{\partial x_1}\varphi = 0\right\}.$$

For any $R > 0$, let $\varphi \in C_0^\infty(B_R(0)) \cap \widetilde{H}$ be any function which is even in z_h, $h = 2, \ldots, N$. Then $\varphi_k(z) := \varphi(z - x_1) \in C_0^\infty(B_R(x_1))$. Substituting φ in (2.3) with φ_k, then by Lemma A.1, we get

$$\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}\varphi + u\varphi - pW^{p-1}u\varphi\right) = 0. \tag{2.7}$$

In addition, since u is even in z_h, $h = 2, \ldots, N$, relation (2.7) holds for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, which is odd in z_h, $h = 2, \ldots, N$. Thus, relation (2.7) is true for any $\varphi \in C_0^\infty(B_R(0)) \cap \widetilde{H}$. By the density of $C_0^\infty(\mathbb{R}^N)$ in $H^s(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}\varphi + u\varphi - pW^{p-1}u\varphi\right) = 0, \qquad \forall \varphi \in \widetilde{H} \tag{2.8}$$

Meanwhile, relation (2.8) holds for $\varphi = \frac{\partial W}{\partial x_1}$. Therefore, (2.8) holds for any $\varphi \in H^s(\mathbb{R}^N)$. Substituting φ in (2.8) with u yields (2.6).

Since W is non-degenerate, we have $u = C\frac{\partial W}{\partial x_1}$ because u is even in z_h, $h = 2, \ldots, N$. By (2.5), we know that

$$u = 0,$$

which implies

$$\int_{B_R(x_1)} {u_k}^2 = o(1), \qquad \forall R > 0.$$

Besides, it follows from Lemma A.1 that there exists $C' > 0$ such that

$$U_{r_k}(x) \le C', \quad \text{for all } x \in \Omega_1.$$

It follows that

$$\begin{aligned} o(1) &= \int_{\Omega_1} \left(|(-\Delta)^{\frac{s}{2}} u_k|^2 + V(|z|){u_k}^2 - pQ(|z|)U_{r_k}^{p-1}{u_k}^2\right) \\ &= \int_{\Omega_1} \left(|(-\Delta)^{\frac{s}{2}} u_k|^2 + V(|z|){u_k}^2\right) + o(1) - C\int_{\Omega_1} {u_k}^2 \\ &\ge \frac{1}{2}\int_{\Omega_1} \left(|(-\Delta)^{\frac{s}{2}} u_k|^2 + V(|z|){u_k}^2\right) + o(1), \end{aligned}$$

which contradicts (2.4). The proof is now complete. □

Define

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + V(|z|)u^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(|z|)|u|^{p+1}. \tag{2.9}$$

Let

$$J(\phi) = I(U_r + \phi), \qquad \phi \in H.$$

Then,

$$\begin{aligned} J(0) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} U_r|^2 + V(|z|){U_r}^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(|z|)|U_r|^{p+1}. \\ &= \frac{1}{2}\int_{\mathbb{R}^N} U_r \sum_{j=1}^{k} W_{x_j}^p + \frac{1}{2}\int_{\mathbb{R}^N} (V(|z|)-1){U_r}^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} (Q(|z|)-1)U_r^{p+1} - \frac{1}{p+1}\int_{\mathbb{R}^N} U_r^{p+1} \end{aligned}$$

because W_{x_j} solves (1.2).

Lemma 2.2. *There exists a positive integer k_0 such that for each $k \ge k_0$, there is a C^1 map from S to H_{rs}: $\phi_k = \phi_k(r)$, $r = |x_1|$, satisfying $\phi_k \in H$, and*

$$J'(\phi_k)\big|_H = 0.$$

Moreover, there exists a constant $C > 0$, independent of k, such that

$$\|\phi_k\| \le \frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}+\delta}}, \tag{2.10}$$

where $\delta > 0$ is a small constant.

Proof. We expand $J(\phi_k)$ as

$$J(\phi_k) = J(0) + l(\phi_k) + \frac{1}{2}\langle M\phi_k, \phi_k\rangle + R(\phi_k), \qquad \phi_k \in H,$$

where

$$\begin{aligned} l(\phi_k) &= \langle I'(U_r), \phi_k\rangle \\ &= \int_{\mathbb{R}^N} (V(|z|)-1)U_r\phi_k + \int_{\mathbb{R}^N}\left(\sum_{j=1}^{k} W_{x_j}^p - U_r^p\right)\phi_k - \int_{\mathbb{R}^N}(Q(|z|)-1)U_r^p\phi_k. \end{aligned}$$

M is the bounded linear map defined in (2.1) and

$$R(\phi_k) = -\frac{1}{p+1}\int_{\mathbb{R}^N} Q(|z|)\Big(|U_r+\phi_k|^{p+1} - U_r^{p+1} - (p+1)U_r^p\phi_k - \frac{1}{2}p(p+1)U_r^{p-1}{\phi_k}^2\Big).$$

Since $l(\phi_k)$ is a bounded linear functional in H, there exists $l_k \in H$, such that

$$l(\phi_k) = \langle l_k, \phi_k\rangle.$$

Then, ϕ_k being a critical point of J is equivalent to

$$l_k + M\phi_k + R'(\phi_k) = 0. \tag{2.11}$$

Since M is invertible, we can infer from (2.11) that

$$\phi_k = T(\phi_k) := -M^{-1}(l_k + R'(\phi_k)).$$

Define

$$E = \left\{\phi_k : \phi_k \in H,\ \|\phi_k\| \le \frac{1}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}}}\right\}.$$

Next, we check that T is a contraction map from E to E.

Case 1: $p \le 2$. It is easy to verify that

$$\|R'(\phi_k)\| \le C\|\phi_k\||^p.$$

In fact,

$$\begin{aligned} |\langle R'(\phi_k), v\rangle| &= \left|\int_{\mathbb{R}^N} Q(|z|)(pU_r^{p-1}\phi_k v + U_r^p v - |U_r+\phi_k|^{p-1}(U_r+\phi_k)v)\right| \\ &= \left|\int_{\mathbb{R}^N} Q(|z|)p(|U_r+\theta\phi_k|^{p-1} - U_r^{p-1})\phi_k v\right| \\ &\le C\int_{\mathbb{R}^N} |(|U_r+\theta\phi_k|^{p-1} - U_r^{p-1})|\,|\phi_k|\,|v| \\ &\le C\int_{\mathbb{R}^N} |\theta\phi_k|^{p-1}|\phi_k|\,|v| \\ &\le C\|\phi_k\|^p\|v\| \end{aligned}$$

where $0 < \theta < 1$.

Then, by the boundedness of M and Lemma 2.3,

$$\|T(\phi_k)\| \leq C(\|l_k\| + \|\phi_k\|^p) \leq \frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}+\delta}} + \frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}p}} \leq \frac{1}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}}} \tag{2.12}$$

which implies that T maps E to E.

In addition,

$$\|R''(\phi_k)\| \leq C\|\phi_k\|^{p-1}.$$

In fact,

$$\begin{aligned}
|\langle R''(\phi_k)v, h\rangle| &= \left| \int_{\mathbb{R}^N} Q(|z|)(pU_r^{p-1}vh - p|U_r + \phi_k|^{p-1}vh) \right| \\
&= p\left| \int_{\mathbb{R}^N} Q(|z|)(U_r^{p-1}vh - |U_r + \phi_k|^{p-1})vh \right| \\
&\leq C \int_{\mathbb{R}^N} |\phi_k|^{p-1}|v||h| \\
&\leq C\|\phi_k\|^{p-1}\|v\| \, \|h\|.
\end{aligned}$$

Thus, for $\phi_{k_1}, \phi_{k_2} \in E$,

$$\begin{aligned}
\|T(\phi_{k_1}) - T(\phi_{k_2})\| &= \|M^{-1}R'(\phi_{k_1}) - M^{-1}R'(\phi_{k_2})\| \\
&\leq \|M^{-1}\| \, \|R'(\phi_{k_1}) - R'(\phi_{k_2})\| \leq \|M^{-1}\| \, \|R''(\phi_{k_1} + \theta(\phi_{k_2} - \phi_{k_1})\| \, \|\phi_{k_2} - \phi_{k_1}\| \\
&\leq C(\|\phi_{k_1}\|^{p-1} + \|\phi_{k_2}\|^{p-1}\|)\|\phi_{k_1} - \phi_{k_2}\| \leq \frac{1}{2}\|\phi_{k_1} - \phi_{k_2}\|.
\end{aligned}$$

Note that the last inequality holds only when k is large enough, which implies the existence of k_0 in Lemma 2.2. Therefore, T is a contraction map from E to E. Then the contraction mapping theorem implies the existence of ϕ_k as a critical point of J restricted to H.

Case 2: $p > 2$.

Setting $h(t) = |U_r + t\phi_k|^{p-1}(U_r + t\phi_k)v$, then by Taylor's formula,

$$\begin{aligned}
|\langle R'(\phi_k), v\rangle| &= \left| \int_{\mathbb{R}^N} Q(|z|)(pU_r^{p-1}\phi_k v + U_r^p v - |U_r + \phi_k|^{p-1}(U_r + \phi_k)v) \right| \\
&= \left| \int_{\mathbb{R}^N} Q(|z|)(-\frac{1}{2}h''(\theta)) \right| \\
&\leq C\left| \int_{\mathbb{R}^N} p(p-1)|U_r + \theta\phi_k|^{p-2} \frac{U_r + \theta\phi_k}{|U_r + \theta\phi_k|}\phi_k^2 v \right| \\
&\leq C \int_{\mathbb{R}^N} (\|\phi_k\|^2 + \|\phi_k\|^p)\|v\| \\
&\leq C\|\phi_k\|^2\|v\|
\end{aligned}$$

which implies that $\|R'(\phi_k)\| \leq C\|\phi_k\|^2$.

By the mean value theorem we obtain

$$\begin{aligned}
|\langle R''(\phi_k)v, h\rangle| &= p\left| \int_{\mathbb{R}^N} Q(|z|)(U_r^{p-1}vh - |U_r + \phi_k|^{p-1})vh \right| \\
&= p(p-1)\left| \int_{\mathbb{R}^N} Q(|z|)|U_r + \theta\phi_k|^{p-3}(U_r + \theta\phi_k)\phi_k vh \right| \\
&\leq C \int_{\mathbb{R}^N} (U_r^{p-2} + |\phi_k|^{p-2})|\phi_k||v||h|
\end{aligned}$$

where $0 < \theta < 1$.

By Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} U_r^{p-2}|\phi_k||v||h| &\le C\left(\int_{\mathbb{R}^N} U_r^{p+1}\right)^{\frac{p-2}{p+1}}\left(\int_{\mathbb{R}^N}|\phi_k|^{p+1}\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^N}|v|^{p+1}\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^N}|h|^{p+1}\right)^{\frac{1}{p+1}}\\ &\le C\|\phi_k\|\,\|v\|\,\|h\|\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}|\phi_k|^{p-1}|v||h| &\le \left(\int_{\mathbb{R}^N}|\phi_k|^{p+1}\right)^{\frac{p-1}{p+1}}\left(\int_{\mathbb{R}^N}|v|^{p+1}\right)^{\frac{1}{p+1}}\left(\int_{\mathbb{R}^N}|h|^{p+1}\right)^{\frac{1}{p+1}}\\ &\le C\|\phi_k\|^{p-1}\|v\|\|h\|\,.\end{aligned}$$

Therefore, $\|R''(\phi_k)\| \le C\|\phi_k\|$.

Arguing similarly as in case 1, we have,

$$\|T(\phi_k)\| \le C(\|l_k\| + \|\phi_k\|^2) \le \frac{1}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}}}. \tag{2.13}$$

and T is a contraction map from E to E. The existence of ϕ_k follows from the contraction mapping theorem, and (2.10) follows from (2.12) and (2.13).

Following the argument employed in [5] to prove Lemma 4.4, we conclude that $\phi_k(r)$ is continuously differentiable in r. □

Lemma 2.3. *If $\tau = \min\{\alpha, \beta\} < N + 2s$, there exists a small constant $\delta > 0$ such that*

$$\|l_k\| \le \frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}+\delta}}.$$

Proof. We have

$$\begin{aligned}\langle l_k, \phi_k\rangle &= l(\phi_k)\\ &= \int_{\mathbb{R}^N}(V(|z|)-1)U_r\phi_k + \int_{\mathbb{R}^N}\left(\sum_{j=1}^k W_{x_j}^p - U_r^p\right)\phi_k - \int_{\mathbb{R}^N}(Q(|z|-1)U_r^p\phi_k.\end{aligned} \tag{2.14}$$

By symmetry,

$$\begin{aligned}\int_{\mathbb{R}^N}(V(|z|)-1)U_r\phi_k &= \int_{\mathbb{R}^N}(V(|z|)-1)\left(\sum_{j=1}^k W_{x_j}\right)\phi_k\\ &= \sum_{j=1}^k\int_{\mathbb{R}^N}(V(|z|)-1)W_{x_j}\phi_k = k\int_{\mathbb{R}^N}(V(|z|)-1)W_{x_1}\phi_k\end{aligned} \tag{2.15}$$

and

$$\int_{\mathbb{R}^N}(V(|z|)-1)W_{x_1}\phi_k = \int_{B_{\frac{r}{2}}(0)}(V(|z|)-1)W_{x_1}\phi_k + \int_{\mathbb{R}^N\setminus B_{\frac{r}{2}}(0)}(V(|z|)-1)W_{x_1}\phi_k.$$

For $z \in \mathbb{R}^N\setminus B_{\frac{r}{2}}(0)$,

$$V(|z|)-1 = \frac{a_1}{|z|^\alpha} + O\left(\frac{1}{|z|^{\alpha+\theta_1}}\right) \le \frac{C}{|z|^\alpha} \le \frac{2^\alpha C}{|r|^\alpha},$$

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_{\frac{r}{2}}(0)}(V(|z|)-1)W_{x_1}\phi_k &= \frac{2^\alpha C}{|r|^\alpha}\int_{\mathbb{R}^N\setminus B_{\frac{r}{2}}(0)}W_{x_1}\phi_k\\ &\le \frac{2^\alpha C}{|r|^\alpha}\Big(\int_{\mathbb{R}^N\setminus B_{\frac{r}{2}}(0)}W_{x_1}^2\Big)^{\frac12}\Big(\int_{\mathbb{R}^N\setminus B_{\frac{r}{2}}(0)}\phi_k^2\Big)^{\frac12}=O\Big(\frac{1}{r^\alpha}\Big)\|\phi_k\|,\\ \int_{B_{\frac{r}{2}}(0)}(V(|z|)-1)W_{x_1}\phi_k\le C\Big(\int_{B_{\frac{r}{2}}(0)}W_{x_1}^2\Big)^{\frac12}&\Big(\int_{B_{\frac{r}{2}}(0)}\phi_k^2\Big)^{\frac12}\le C\frac{1}{r^{\frac{N}{2}+2s}}\|\phi_k\|.\end{aligned}$$

Then we conclude that

$$\int_{\mathbb{R}^N}(V(|z|)-1)W_{x_1}\phi_k\le O\Big(\frac{1}{r^\alpha}\Big)\|\phi_k\|+C\frac{1}{r^{\frac{N}{2}+2s}}\|\phi_k\|. \tag{2.16}$$

By the mean value theorem and Lemma A.1,

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}\Big(\sum_{j=1}^k W_{x_j}^p-U_r^p\Big)\phi_k\Big| &= k\Big|\int_{\Omega_1}\Big(\sum_{j=1}^k W_{x_j}^p-U_r^p\Big)\phi_k\Big|\\ &\le Ck\Big|\int_{\Omega_1}W_{x_1}^{p-1}\Big(\sum_{j=2}^k W_{x_j}\Big)\phi_k\Big|\le Ck\frac{1}{(k^{-1}r)^{N+2s}}\Big(\int_{\Omega_1}W_{x_1}^p\Big)^{\frac{p-1}{p}}\Big(\int_{\Omega_1}|\phi_k|^p\Big)^{\frac1p}\\ &\le Ck\frac{1}{(k^{-1}r)^{N+2s}}\|\phi_k\|.\end{aligned} \tag{2.17}$$

By the boundedness of U_r we have

$$\begin{aligned}\int_{\mathbb{R}^N}(Q(|z|-1)U_r^p\phi_k &= \int_{\mathbb{R}^N}(Q(|z|-1)U_r^{p-1}U_r\phi_k\\ &\le C\int_{\mathbb{R}^N}(Q(|z|-1)U_r\phi_k\le Ck\int_{\mathbb{R}^N}(Q(|z|-1)W_{x_1}\phi_k\\ &\le Ck\Big(O\Big(\frac{1}{r^\beta}\Big)+\frac{1}{r^{\frac{N}{2}+2s}}\Big)\|\phi_k\|.\end{aligned} \tag{2.18}$$

Combining relations (2.14)–(2.18), we obtain

$$\begin{aligned}\langle l_k,\phi_k\rangle &= \int_{\mathbb{R}^N}(V(|z|)-1)U_r\phi_k+\int_{\mathbb{R}^N}\Big(\sum_{i=1}^k W_{x_j}^p-U_r^p\Big)\phi_k-\int_{\mathbb{R}^N}(Q(|z|-1)U_r^p\phi_k\\ &\le k\Big(O\Big(\frac{1}{r^\alpha}\Big)+O\Big(\frac{1}{r^\beta}\Big)+C\frac{1}{r^{\frac{N}{2}+2s}}+\frac{C}{(k^{-1}r)^{N+2s}}\Big)\|\phi_k\|\\ &\le k\Big(O\Big(\frac{1}{r^\tau}\Big)+\frac{C}{r^{\frac{N}{2}+2s}}+\frac{C}{(k^{-1}r)^{N+2s}}\Big)\|\phi_k\|.\end{aligned} \tag{2.19}$$

By $r\in S$, it holds that $r\sim k^{\frac{N+2s}{N+2s-\tau}}$, $\frac{C}{(k^{-1}r)^{N+2s}}\sim O(\frac{1}{r^\tau})$,

$$kO\Big(\frac{1}{r^\tau}\Big)<C\frac{1}{k^{\frac{(N+2s)\tau}{N+2s-\tau}-1}}\le\frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}+\delta}},$$

and $k\frac{1}{r^{\frac{N}{2}+2s}}<\frac{1}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}}}$, for $\tau<N+2s$.

We conclude that if $\tau < N + 2s$, then

$$\|l_k\| \leq \frac{C}{k^{\frac{(N+2s)(\tau-1)+\tau}{2(N+2s-\tau)}+\delta}}.$$

The proof is now complete. □

3 Proof of the main result

Define

$$G(r) = I(U_r + \phi_k), \qquad \forall r \in S,$$

where $\phi_k = \phi_k(r)$ is the map obtained in Lemma 2.2.

According to Lemma 6.1 in [5], if r is a critical point of $G(r)$, then $U_r + \phi_k(r)$ is a solution of (1.1).

From the energy expansion in the Appendix, we have

$$\begin{aligned} J(0) &= I(U_r) \\ &= k\left(D + \frac{a_1 A_1}{r^\alpha} - \frac{a_2 A_2}{r^\beta} - \frac{B}{(k^{-1}r)^{N+2s}} + O\left(\frac{1}{r^{\alpha+\tau_1}}\right) + O\left(\frac{1}{r^{\beta+\tau_2}}\right) + O\left(\frac{1}{(k^{-1}r)^{N+2s+\sigma}}\right)\right). \end{aligned}$$

Set

$$H(r) = \frac{a_1 A_1}{r^\alpha} - \frac{a_2 A_2}{r^\beta} - \frac{B}{(k^{-1}r)^{N+2s}}.$$

We prove in what follows that in any of the cases in Theorem 1.1, $H(r)$ has a maximum point r_k.

For case (i): if $a_1 > 0$, $a_2 = 0$, $\alpha < N + 2s$, and $\alpha \leq \beta$ then

$$H'(r) = -\frac{\alpha a_1 A_1}{r^{\alpha+1}} + \frac{B(N+2s)k^{N+2s}}{r^{N+2s+1}}$$

and r_k satisfies

$$\frac{\alpha a_1 A_1}{r_k^{\alpha+1}} = \frac{B(N+2s)k^{N+2s}}{r_k^{N+2s+1}}.$$

Actually, calculating the maximum points in these cases can be summed up as

$$\frac{\tau C}{r_k^{\tau+1}} = \frac{C'(N+2s)k^{N+2s}}{r_k^{N+2s+1}},$$

where $\tau = \min\{\alpha, \beta\}$. Then $H(r)$ has a maximum point

$$r_k = \left(\frac{(N+2s)C'}{\tau C}\right)^{\frac{1}{N+2s-\tau}} k^{\frac{N+2s}{N+2s-\tau}},$$

which is an interior point of S. Then, there exists a small constant δ such that

$$J(0) = k\left(D + \frac{a_1 A_1}{r^\alpha} - \frac{a_2 A_2}{r^\beta} - \frac{B}{(k^{-1}r)^{N+2s}} + O\left(\frac{1}{k^{\frac{(N+2s)\tau}{N+2s-\tau}+\delta}}\right)\right).$$

Consequently,

$$\begin{aligned}G(r) &= I(W_r+\phi_k) = I(W_r) + l(\phi_k) + \frac{1}{2}\langle M\phi_k, \phi_k\rangle + R(\phi_k)\\ &= J(0) + O(\|l_k\|\|\phi_k\| + \|\phi_k\|^2)\\ &= J(0) + O\left(\frac{1}{k^{\frac{(N+2s)\tau}{N+2s-\tau}-1+\delta}}\right)\\ &= k\left(D + \frac{a_1A_1}{r^{\alpha}} - \frac{a_2A_2}{r^{\beta}} - \frac{B}{(k^{-1}r)^{N+2s}} + O\left(\frac{1}{k^{\frac{(N+2s)\tau}{N+2s-\tau}+\delta}}\right)\right).\end{aligned}$$

Since $H(r)$ has a maximum point r_k which is an interior point of S in any of the cases listed, then $G(r)$ has a critical point $\widetilde{r_k}$ in the interior of S. This means that the function

$$U_{\widetilde{r_k}} + \phi_k(\widetilde{r_k})$$

is a solution of problem (1.1). The proof is now complete. □

A Appendix. Energy expansions

In this section, we obtain some energy estimates for the approximate solutions. Recall that

$$\Omega_j = \left\{ z = (z', z'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{z'}{|z'|}, \frac{x_j}{|x_j|} \right\rangle \geq \cos\frac{\pi}{k} \right\},$$

$$x_j = \left(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\right), \qquad j = 1, \ldots, k,$$

$$r \in S := \left[\left(\frac{N+2s}{\tau} - \epsilon\right)^{\frac{1}{N+2s-\tau}} k^{\frac{N+2s}{N+2s-\tau}}, \left(\frac{N+2s}{\tau} + \epsilon\right)^{\frac{1}{N+2s-\tau}} k^{\frac{N+2s}{N+2s-\tau}} \right],$$

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}}u|^2 + V(|z|)u^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(|z|)|u|^{p+1},$$

and

$$\begin{aligned}I(U_r) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}}U_r|^2 + V(|z|)U_r{}^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(|z|)|U_r|^{p+1}\\ &= \frac{1}{2}\int_{\mathbb{R}^N} U_r \sum_{j=1}^{k} W_{x_j}^p + \frac{1}{2}\int_{\mathbb{R}^N} (V(|z|)-1)U_r{}^2\\ &\quad - \frac{1}{p+1}\int_{\mathbb{R}^N} (Q(|z|)-1)U_r^{p+1} - \frac{1}{p+1}\int_{\mathbb{R}^N} U_r^{p+1}.\end{aligned}$$

Lemma A.1. *For any $z \in \Omega_1$, there exists $C > 0$, such that*

$$\sum_{j=2}^{k} W_{x_j}(z) \leq \frac{C}{(k^{-1}r)^{N+2s}}$$

Proof. By the definition of Ω_j, for any $z \in \Omega_1$,

$$|z - x_j| \geq \frac{1}{2}|x_j - x_1| = r\sin\frac{(j-1)\pi}{k} > 0 \qquad (j \geq 2).$$

Then

$$\begin{aligned}\sum_{j=2}^{k} W_{x_j}(z) &\leq C\sum_{j=2}^{k}\frac{1}{(r\sin\frac{(j-1)\pi}{k})^{N+2s}} = C\sum_{i=1}^{k-1}\frac{1}{(r\sin\frac{i\pi}{k})^{N+2s}}\\ &= \begin{cases} 2C\sum_{i=1}^{\frac{k-1}{2}}\frac{1}{(r\sin\frac{i\pi}{k})^{N+2s}}, & k \text{ is odd,}\\ 2C\Big(\sum_{i=1}^{\frac{k-2}{2}}\frac{1}{(r\sin\frac{i\pi}{k})^{N+2s}} + \frac{1}{(r\sin\frac{k\pi}{2k})^{N+2s}}\Big), & k \text{ is even.}\end{cases}\end{aligned}$$

When k is even,

$$\begin{aligned}\sum_{j=2}^{k} W_{x_j}(z) &\leq 2C\Big(\sum_{i=1}^{\frac{k-2}{2}}\frac{1}{(r\sin\frac{i\pi}{k})^{N+2s}} + \frac{1}{(r\sin\frac{k\pi}{2k})^{N+2s}}\Big)\\ &\leq 2C\Big(\sum_{i=1}^{\frac{k-2}{2}}\frac{1}{(r\frac{2i}{k})^{N+2s}} + \frac{1}{r^{N+2s}}\Big)\\ &\leq 2C\Big(\frac{1}{(k^{-1}r)^{N+2s}}\sum_{i=1}^{\frac{k-2}{2}}\frac{1}{i^{N+2s}} + \frac{1}{r^{N+2s}}\Big)\\ &\leq \frac{C}{(k^{-1}r)^{N+2s}}\end{aligned}$$

since $\sum_{i=1}^{+\infty}\frac{1}{i^{N+2s}}$ converges.

The proof of the case where k is odd follows with similar arguments. □

Lemma A.2. *We have*

$$\int_{\mathbb{R}^N}(V(|z|)-1)U_r^{\,2} = k\Big(\frac{a_1}{r^\alpha}\int_{\mathbb{R}^N}W^2 + O\Big(\frac{1}{r^{\alpha+\tau_1}}\Big) + O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma_1}}\Big)\Big)$$

where $\tau_1 > 0$, and $\sigma_1 > 0$ are small constants.

Proof. By symmetry,

$$\begin{aligned}\int_{\mathbb{R}^N}(V(|z|)-1)U_r^{\,2} &= k\int_{\Omega_1}(V(|z|)-1)\Big(W_{x_1} + \sum_{j=2}^{k}W_{x_j}\Big)^2\\ &= k\int_{\Omega_1}(V(|z|)-1)\Big(W_{x_1}^2 + 2W_{x_1}\sum_{j=2}^{k}W_{x_j} + \Big(\sum_{j=2}^{k}W_{x_j}\Big)^2\Big)\end{aligned} \tag{A.1}$$

and

$$\int_{\Omega_1}(V(|z|)-1)W_{x_1}^2 = \int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}^2 + \int_{B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}^2. \tag{A.2}$$

On the one hand,

$$\int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}^2 \le C\int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)} W_{x_1}^2 \le C\int_{|z-x_1|>\frac{r}{2}}\left(\frac{1}{|z-x_1|^{N+2s}}\right)^2$$
$$= C\int_{\frac{r}{2}}^{+\infty}\frac{t^{N-1}}{t^{2N+4s}}dt = C\int_{\frac{r}{2}}^{+\infty}\frac{1}{t^{N+4s+1}}dt = C\frac{1}{r^{N+4s}} = O\left(\frac{1}{r^{N+4s}}\right). \tag{A.3}$$

On the other hand,

$$\begin{aligned}&\int_{B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}^2\\ &= \int_{B_{\frac{r}{2}}(x_1)}\left(\frac{a_1}{|z|^\alpha}+O\left(\frac{1}{|z|^{\alpha+\theta_1}}\right)\right)W_{x_1}^2\\ &= \int_{B_{\frac{r}{2}}(x_1)}\left(\frac{a_1}{r^\alpha}+\frac{a_1}{r^{\alpha+1}}O(|z-x_1|)+\frac{C}{r^{\alpha+\theta_1}}+\frac{C}{r^{\alpha+\theta_1+1}}O(|z-x_1|)\right)W_{x_1}^2\\ &= \frac{a_1}{r^\alpha}\int_{B_{\frac{r}{2}}(x_1)}W_{x_1}^2+O\left(\frac{1}{r^{\alpha+\tau_1}}\right) = \frac{a_1}{r^\alpha}\int_{\mathbb{R}^N}W^2+O\left(\frac{1}{r^{\alpha+\tau_1}}\right)+O\left(\frac{1}{r^{N+4s}}\right),\end{aligned} \tag{A.4}$$

where $\tau_1 = \min\{1,\theta_1\}$.

By (A.2)–(A.4),

$$\int_{\Omega_1}(V(|z|)-1)W_{x_1}^2 = \frac{a_1}{r^\alpha}\int_{\mathbb{R}^N}W^2+O\left(\frac{1}{r^{N+4s}}\right)+O\left(\frac{1}{r^{\alpha+\tau_1}}\right), \tag{A.5}$$

$$\begin{aligned}\int_{\Omega_1}(V(|z|)-1)W_{x_1}\sum_{j=2}^k W_{x_j} &= \int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}\sum_{j=2}^k W_{x_j}\\ &\quad+\int_{B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}\sum_{j=2}^k W_{x_j}\end{aligned} \tag{A.6}$$

By the boundedness of $V(|z|)$ and Lemma A.1,

$$\begin{aligned}\int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}\sum_{j=2}^k W_{x_j} &\le \frac{C}{(k^{-1}r)^{N+2s}}\int_{\Omega_1\setminus B_{\frac{r}{2}}(x_1)}W_{x_1}\\ &\le \frac{C}{(k^{-1}r)^{N+2s}}\int_{|z-x_1|>\frac{r}{2}}\frac{1}{|z-x_1|^{N+2s}} = \frac{C}{(k^{-1}r)^{N+2s}}\int_{\frac{r}{2}}^{+\infty}\frac{t^{N-1}}{t^{N+2s}}dt\\ &= \frac{C}{(k^{-1}r)^{N+2s}}\int_{\frac{r}{2}}^{+\infty}\frac{1}{t^{2s+1}}dt = \frac{C}{(k^{-1}r)^{N+2s}}\frac{1}{r^{2s}}\\ &= O\left(\frac{1}{(k^{-1}r)^{N+4s}}\right).\end{aligned} \tag{A.7}$$

Similarly to (A.4),

$$\begin{aligned}\int_{B_{\frac{r}{2}}(x_1)}(V(|z|)-1)W_{x_1}\sum_{j=2}^k W_{x_j} &\le \frac{C}{(k^{-1}r)^{N+2s}}\int_{B_{\frac{r}{2}}(x_1)}\left(\frac{a_1}{|z|^\alpha}+O\left(\frac{1}{|z|^{\alpha+\theta_1}}\right)\right)W_{x_1}\\ &= \frac{C}{(k^{-1}r)^{N+2s}}\left(\frac{a_1}{r^\alpha}\int_{B_{\frac{r}{2}}(x_1)}W_{x_1}+O\left(\frac{1}{r^{\alpha+1}}\right)+O\left(\frac{1}{r^{\alpha+\theta_1}}\right)\right)\\ &\le \frac{C}{(k^{-1}r)^{N+2s}}\frac{C'}{r^\alpha} = O\left(\frac{1}{(k^{-1}r)^{N+2s+\alpha}}\right).\end{aligned} \tag{A.8}$$

By relations (A.6)–(A.8) we get

$$\int_{\Omega_1}(V(|z|)-1)W_{x_1}\sum_{j=2}^{k}W_{x_j}=O\Big(\frac{1}{(k^{-1}r)^{N+4s}}\Big)+O\Big(\frac{1}{(k^{-1}r)^{N+2s+\alpha}}\Big), \tag{A.9}$$

$$\begin{aligned}\int_{\Omega_1}(V(|z|)-1)\Big(\sum_{j=2}^{k}W_{x_j}\Big)^2 &\le \frac{C}{(k^{-1}r)^{N+2s}}\sum_{j=2}^{k}\int_{\Omega_1}W_{x_j}\\ &\le \frac{C}{(k^{-1}r)^{N+2s}}\frac{C_1}{(k^{-1}r)^{2s}}=O\Big(\frac{1}{(k^{-1}r)^{N+4s}}\Big).\end{aligned} \tag{A.10}$$

By (A.1), (A.5), (A.9), (A.10), We have

$$\begin{aligned}&\int_{\mathbb{R}^N}(V(|z|)-1)U_r{}^2\\ &=k\Big(\frac{a_1}{r^\alpha}\int_{\mathbb{R}^N}W^2+O\Big(\frac{1}{r^{N+4s}}\Big)+O\Big(\frac{1}{r^{\alpha+\tau_1}}\Big)+O\Big(\frac{1}{(k^{-1}r)^{N+4s}}\Big)+O\Big(\frac{1}{(k^{-1}r)^{N+2s+\alpha}}\Big)\Big)\\ &=k\Big(\frac{a_1}{r^\alpha}\int_{\mathbb{R}^N}W^2+O\Big(\frac{1}{r^{\alpha+\tau 1}}\Big)+O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma_1}}\Big)\Big),\end{aligned} \tag{A.11}$$

where $\sigma_1=\min\{2s,\alpha\}$. The proof is now complete. □

Remark A.3. Arguing similarly as in Lemma A.2, we have

$$\int_{\mathbb{R}^N}(Q(|z|)-1)U_r{}^{p+1}=k\Big(\frac{a_2}{r^\beta}\int_{\mathbb{R}^N}W^{p+1}+O\Big(\frac{1}{r^{\beta+\tau_2}}\Big)+O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma_2}}\Big)\Big), \tag{A.12}$$

where $\tau_2>0$ and $\sigma_2>0$ are small constants.

Lemma A.4. *There is a small constant $\delta>0$, such that*

$$I(U_r)=k\Big(D+\frac{a_1}{r^\alpha}A_1-\frac{a_2}{r^\beta}A_2-\frac{B}{(k^{-1}r)^{N+2s}}+O\Big(\frac{1}{k^{\frac{(N+2s)\tau}{(N+2s-\tau)}+\delta}}\Big)\Big),$$

where

$$D=\Big(\frac12-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N}W^{p+1},\qquad A_1=\frac12\int_{\mathbb{R}^N}W^2,\qquad A_2=\frac{1}{p+1}\int_{\mathbb{R}^N}W^{p+1},$$

and B satisfies

$$\frac12\int_{\Omega_1}W_{x_1}^p\sum_{j=2}^{k}W_{x_j}=\frac{B}{(k^{-1}r)^{N+2s}}.$$

Proof. We first prove that there exists small constant $\sigma_3>0$ such that

$$\begin{aligned}&\frac12\int_{\mathbb{R}^N}U_r\sum_{j=1}^{k}W_{x_j}^p-\frac{1}{p+1}\int_{\mathbb{R}^N}U_r^{p+1}\\ &=k\Big(\Big(\frac12-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N}W^{p+1}-\frac12\int_{\Omega_1}W_{x_1}^p\sum_{j=2}^{k}W_{x_j}+O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma_3}}\Big)\Big).\end{aligned} \tag{A.13}$$

By symmetry,

$$
\begin{aligned}
\int_{\mathbb{R}^N} U_r \sum_{j=1}^{k} W_{x_j}^p &= k \int_{\Omega_1} U_r \sum_{j=1}^{k} W_{x_j}^p \\
&= k \int_{\Omega_1} \Big(W_{x_1} + \sum_{i=2}^{k} W_{x_i} \Big) \Big(W_{x_1}^p + \sum_{j=2}^{k} W_{x_j}^p \Big) \\
&= k \Big(\int_{\Omega_1} W_{x_1}^{p+1} + W_{x_1}^p \sum_{i=2}^{k} W_{x_i} + W_{x_1} \sum_{j=2}^{k} W_{x_j}^p + \sum_{i=2}^{k} W_{x_i} \sum_{j=2}^{k} W_{x_j}^p \Big). \qquad \text{(A.14)}
\end{aligned}
$$

By Lemma A.1,

$$
\begin{aligned}
\int_{\Omega_1} W_{x_1} \sum_{j=2}^{k} W_{x_j}^p &\leq \int_{\Omega_1} W_{x_1} \sum_{j=2}^{k} \Big(\frac{C}{|z - x_j|^{N+2s}} \Big)^p \\
&\leq \int_{\Omega_1} W_{x_1} \sum_{j=2}^{k} \frac{C^p}{(\frac{1}{2}|x_j - x_1|)^{(N+2s)p}} = \sum_{j=2}^{k} \frac{C^p}{(\frac{1}{2}|x_j - x_1|)^{(N+2s)p}} \int_{\Omega_1} W_{x_1} \\
&\leq \frac{C'}{(k^{-1}r)^{(N+2s)p}} \int_{\Omega_1} W_{x_1} = O\Big(\frac{1}{(k^{-1}r)^{(N+2s)p}} \Big) \qquad \text{(A.15)}
\end{aligned}
$$

and

$$
\int_{\Omega_1} \sum_{i=2}^{k} W_{x_i} \sum_{j=2}^{k} W_{x_j}^p \leq \frac{C}{(k^{-1}r)^{N+2s}} \sum_{j=2}^{k} \int_{\Omega_1} W_{x_j}^p = O\Big(\frac{1}{(k^{-1}r)^{(N+2s)p+2s}} \Big). \qquad \text{(A.16)}
$$

By Taylor's formula,

$$
\begin{aligned}
&\int_{\Omega_1} U_r^{p+1} - \int_{\Omega_1} W_{x_1}^{p+1} - \int_{\Omega_1} (p+1) W_{x_1}^p \sum_{j=2}^{k} W_{x_j} \\
&= \frac{1}{2} p(p+1) \int_{\Omega_1} \Big(W_{x_1} + \theta \sum_{j=2}^{k} W_{x_j} \Big)^{p-1} \Big(\sum_{j=2}^{k} W_{x_j} \Big)^2 \\
&\leq C \int_{\Omega_1} \Big(W_{x_1}^{p-1} \Big(\sum_{j=2}^{k} W_{x_j} \Big)^2 + \Big(\sum_{j=2}^{k} W_{x_j} \Big)^{p+1} \Big).
\end{aligned}
$$

By Lemma A.1,

$$
\int_{\Omega_1} W_{x_1}^{p-1} \Big(\sum_{j=2}^{k} W_{x_j} \Big)^2 \leq \frac{C}{(k^{-1}r)^{2N+4s}} \int_{\Omega_1} W_{x_1}^{p-1} \leq \frac{C}{(k^{-1}r)^{2N+4s}}
$$

and

$$
\int_{\Omega_1} \Big(\sum_{j=2}^{k} W_{x_j} \Big)^{p+1} \leq \frac{C}{(k^{-1}r)^{(N+2s)p}} \sum_{j=2}^{k} \int_{\Omega_1} W_{x_j} \leq \frac{C}{(k^{-1}r)^{(N+2s)p}} \frac{C'}{(k^{-1}r)^{2s}}.
$$

Therefore,

$$
\int_{\Omega_1} U_r^{p+1} = \int_{\Omega_1} W_{x_1}^{p+1} + (p+1) W_{x_1}^p \sum_{j=2}^{k} W_{x_j} + O\Big(\frac{1}{(k^{-1}r)^{N+4s}} \Big). \qquad \text{(A.17)}
$$

Thus, by (A.13)–(A.17), we conclude that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} U_r \sum_{j=2}^{k} W_{x_j}^p - \frac{1}{p+1}\int_{\mathbb{R}^N} U_r^{p+1} \\
&= k\bigg(\frac{1}{2}\int_{\Omega_1} U_r \sum_{j=2}^{k} W_{x_j}^p - \frac{1}{p+1}\int_{\Omega_1} U_r^{p+1}\bigg) \\
&= k\bigg(\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\Omega_1} W_{x_1}^{p+1} - \frac{1}{2}\int_{\Omega_1} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} + O\Big(\frac{1}{(k^{-1}r)^{(N+2s)p}}\Big) + O\Big(\frac{1}{(k^{-1}r)^{N+4s}}\Big)\bigg) \\
&= k\bigg(\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N} W^{p+1} - \frac{1}{2}\int_{\Omega_1} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} + O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma_3}}\Big)\bigg) \qquad \text{(A.18)}
\end{aligned}$$

where $\sigma_3 = \min\{2s, (p-1)(N+2s)\}$.

Next, we claim that there exists a constant $B > 0$, such that

$$\frac{1}{2}\int_{\Omega_1} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} = \frac{B}{(k^{-1}r)^{N+2s}}.$$

It is easy to verify that

$$\frac{1}{2}\int_{\Omega_1} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} \leq \frac{B}{(k^{-1}r)^{N+2s}}.$$

Set $G_k = \{z \in \mathbb{R}^N : |z - x_1| < \frac{1}{4}|x_j - x_1|\}$. Then for a fixed $R > 0$, it follows that $B_R(x_1) \subset G_k \subset \Omega_1$. Then

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_1} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} &\geq \frac{1}{2}\int_{G_k} W_{x_1}^p \sum_{j=2}^{k} W_{x_j} \geq \frac{1}{2}\int_{G_k} W_{x_1}^p \sum_{j=2}^{k} \frac{C}{|z-x_j|^{N+2s}} \\
&\geq C'\int_{G_k} W_{x_1}^p \sum_{j=2}^{k} \frac{1}{\big|\frac{3}{2}r\sin\frac{(j-1)\pi}{k}\big|^{N+2s}} = C'\sum_{j=2}^{k} \frac{1}{\big|\frac{3}{2}r\sin\frac{(j-1)\pi}{k}\big|^{N+2s}}\int_{G_k} W_{x_1}^p \\
&\geq C'\sum_{j=2}^{k} \frac{1}{\big|\frac{3}{2}r\sin\frac{(j-1)\pi}{k}\big|^{N+2s}}\int_{B_R(0)} W^p \geq \frac{B}{(k^{-1}r)^{N+2s}}.
\end{aligned}$$

Combining this claim with Lemma A.2, Remark A.3, and (A.18), we obtain

$$\begin{aligned}
I(U_r) = k\bigg(&\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N} W^{p+1} + \frac{1}{2}\frac{a_1}{r^\alpha}\int_{\mathbb{R}^N} W^2 - \frac{1}{p+1}\frac{a_2}{r^\beta}\int_{\mathbb{R}^N} W^{p+1} \\
&- \frac{B}{(k^{-1}r)^{N+2s}} + O\Big(\frac{1}{r^{\alpha+\tau 1}}\Big) + O\Big(\frac{1}{r^{\beta+\tau 2}}\Big) + O\Big(\frac{1}{(k^{-1}r)^{N+2s+\sigma}}\Big)\bigg),
\end{aligned}$$

where $\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}$.

Denoting

$$D = \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^N} W^{p+1}, \qquad A_1 = \frac{1}{2}\int_{\mathbb{R}^N} W^2, \qquad A_2 = \frac{1}{p+1}\int_{\mathbb{R}^N} W^{p+1},$$

and using the fact that $r \in S$, we have

$$I(U_r) = k\bigg(D + \frac{a_1}{r^\alpha}A_1 - \frac{a_2}{r^\beta}A_2 - \frac{B}{(k^{-1}r)^{N+2s}} + O\Big(\frac{1}{k^{\frac{(N+2s)\tau}{(N+2s-\tau)}+\delta}}\Big)\bigg).$$

The proof is now complete. □

Acknowledgements

This work is partially supported by the National Natural Science Foundation of China (Nos. 11671364, 12071438). Vicențiu D. Rădulescu was supported by a grant of the Romanian Ministry of Education and Research, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2020-0068, within PNCDI III. Vicențiu D. Rădulescu was also supported by the Slovenian Research Agency program P1-0292.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **34**, 1–13; https://doi.org/10.14232/ejqtde.2021.1.34 www.math.u-szeged.hu/ejqtde/

Existence of a homoclinic orbit in a generalized Liénard type system

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Received 24 August 2020, appeared 14 April 2021
Communicated by Gabriele Villari

Abstract. The object of this paper is to study the existence and nonexistence of an important orbit in a generalized Liénard type system. This trajectory is doubly asymptotic to an equilibrium solution, i.e., an orbit which lies in the intersection of the stable and unstable manifolds of a critical point. Such an orbit is called a homoclinic orbit.

Keywords: Liénard system, homoclinic orbit, planar system, dynamical systems

2020 Mathematics Subject Classification: 34C37, 34A12, 34C10.

1 Introduction

Consider the planar system

$$\begin{aligned} \dot{x} &= P(Q(y) - F(x)) \\ \dot{y} &= -g(x), \end{aligned} \tag{1.1}$$

which is a generalized Liénard type system, where P, Q, F and g are continuous functions satisfying suitable assumptions in order to ensure the existence of a unique solution to the initial value problems. Moreover, suppose that the following assumptions hold under which the origin is the unique critical point of system (1.1).

$$\begin{gathered} P(u) \text{ and } Q(y) \text{ are strictly increasing and } F(0) = P(0) = Q(0) = 0, \\ uP(u) > 0 \text{ for } u \neq 0,\ yQ(y) > 0 \text{ for } y \neq 0 \text{ and } xg(x) > 0 \text{ for } x \neq 0. \end{gathered}$$

System (1.1) includes the classical Liénard system as a special case, which is of great importance in various applications (see [1] to [23] and the references cited therein).

Definition 1.1. In system (1.1), a trajectory is said to be a homoclinic orbit if its α- and ω-limit sets are the origin (see Fig. 1.1).

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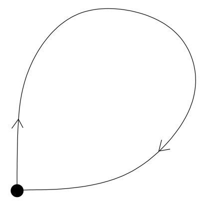

Figure 1.1: Homoclinic orbit

The main purpose of this paper is to give an implicit necessary and sufficient condition and some explicit sufficient conditions on $F(x)$, $g(x)$, $P(u)$ and $Q(y)$ under which system (1.1) has homoclinic orbits. These results extend and improve the results presented for special cases of system (1.1) in [3, 11, 19].

The existence of homoclinic orbit is an important problem in nonlinear dynamical systems and the theory of ordinary differential equations. The results about the existence of homoclinic orbits for the other systems, such as the Lorenz system, Schrödinger systems, predator–prey systems and Hamiltonian systems can be found in [13, 18, 22, 23], respectively. Moreover, various systems and equations such as generalized Euler equation [4] and predator–prey systems [22] can be transformed to the Liénard type systems.

The existence of homoclinic orbits in the Liénard-type systems is closely connected with the stability of the zero solution and the center problem (see [6, 11, 19, 21]). If system (1.1) has a homoclinic orbit, then the zero solution is no longer stable. A homoclinic orbit and a center cannot exist together in system (1.1). Our subject also has a near relation with the global attractivity of the origin and oscillation of solutions and so on (see [9, 12, 20]).

The curve $\Gamma = \{(x,y)|y = Q^{-1}(F(x))\}$ is called the characteristic curve of (1.1). Let

$$\Gamma_1 = \{(x,y) \mid y = Q^{-1}(F(x)) \text{ and } x > 0\},$$

and

$$\Gamma_2 = \{(x,y) \mid y = Q^{-1}(F(x)) \text{ and } x < 0\}.$$

Then, $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup (0,0)$. Positive and negative orbits of (1.1) passing through $p \in \mathbb{R}^2$ are shown by $O^+(p)$ and $O^-(p)$, respectively.

The following definitions are presented to state our main results.

Definition 1.2. System (1.1) has property (Z_1^+) (resp., (Z_3^+)) if there exists a point $p(x_0, y_0) \in \Gamma_1$ (resp., $p(x_0, y_0) \in \Gamma_2$), such that the $O^+(p)$ of (1.1) starting at p approaches the origin through only the first (resp., third) quadrant (see Fig. 1.2).

Definition 1.3. System (1.1) has property (Z_2^-) (resp., (Z_4^-)) if there exists a point $p(x_0, y_0) \in \Gamma_2$ (resp., $p(x_0, y_0) \in \Gamma_1$), such that the $O^-(p)$ of (1.1) starting at p approaches the origin through only the second (resp., fourth) quadrant.

If system (1.1) has both properties (Z_1^+) and (Z_2^-), then a homoclinic orbit exists in the upper half-plane. Similarly, if system (1.1) has both properties (Z_3^+) and (Z_4^-), then a homoclinic orbit exists in the lower half-plane. In this paper we will find conditions for deciding whether system (1.1) has homoclinic orbit.

Hara and Yoneyama in [9] considered system (1.1) with $Q(y) = y$ and $P(u) = u$ and presented some sufficient conditions under which the system has or fails to have property (Z_1^+). Also, Sugie presented an implicit necessary and sufficient condition for system (1.1) with $P(u) = u$ to have property (Z_1^+) [19]. Next, Aghajani and Moradifam in [3] considered

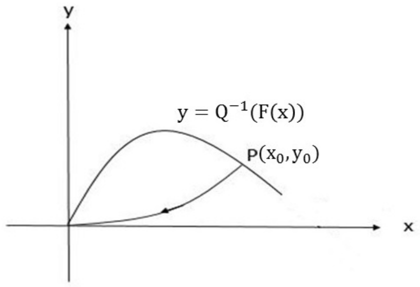

Figure 1.2: Property (Z_1^+)

system (1.1) with $P(u) = u$ and gave an implicit necessary and sufficient condition for the system to have property (Z_1^+) which improved some results in [19].

In the next section an implicit necessary and sufficient condition and some explicit sufficient conditions are provided for system (1.1) to have property (Z_1^+). Since some nonlinear functions are added to the classical Liénard system in this article, our results are proper extensions of the known ones in [3], [9], [11] and [19].

2 Necessary and sufficient conditions for property of (Z_1^+)

In this section we will give necessary and sufficient conditions for system (1.1) to have properties (Z_1^+) and (Z_2^-). First, consider the following lemma about asymptotic behavior of solutions of (1.1).

Lemma 2.1. *For each point $H(c, Q^{-1}(F(c)))$ with $c > 0$ or $c < 0$, the positive or negative semi-orbit of* (1.1) *starting at H crosses the negative y-axis if the following condition holds.*

(A$_1$) *There exists a $\delta > 0$ such that $F(x) < 0$ for $-\delta < x < \delta$ or $F(x)$ has an infinite number of positive zeroes clustering at $x = 0$.*

Remark 2.2. Lemma 2.1 implies that system (1.1) fails to have properties (Z_1^+) and (Z_2^-) if **(A$_1$)** holds. Hence, hereafter we assume that there exists a $\delta > 0$ such that $F(x) > 0$ for $-\delta < x < \delta$.

Theorem 2.3. *System* (1.1) *has property (Z_1^+) if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \le \phi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{-g(\eta)}{P(\phi(\eta) - F(\eta))} d\eta \le Q^{-1}(\phi(x)) \tag{2.1}$$

for $0 < x < \delta$.

Proof. First, note that the positive semi-orbit of (1.1) starting at $H(x_0, Q^{-1}(F(x_0)))$ is considered as a solution $y(x)$ of

$$\frac{dy}{dx} = \frac{-g(x)}{P(Q(y) - F(x))}, \tag{2.2}$$

with $y(x_0) = Q^{-1}(F(x_0))$.

Sufficiency: Suppose that system (1.1) fails to have property (Z_1^+). Thus, there exist a point $H(x_0, Q^{-1}(F(x_0)))$ and $x_0 > 0$ such that the positive semi-orbit of (1.1) starting at H does not approach the origin through the first quadrant. Taking the vector field of (1.1) into account, it is obvious that the positive semi-orbit rotates in clockwise direction about the origin. For this reason, it crosses the curve $y = Q^{-1}(\phi(x))$ and meets the y-axis at a point $(0, y_1)$ with $y_1 < 0$. Let

$$x_1 = \inf\{x : 0 < x < \delta \text{ and } y(x) > Q^{-1}(\phi(x))\}.$$

Then, $(x_1, y(x_1))$ is the intersection point of $O^+(H)$ and the curve $y = Q^{-1}(\phi(x))$ nearest to the origin, that is $y(x_1) = Q^{-1}(\phi(x_1))$ and $y < Q^{-1}(\phi(x))$ for $0 < x < x_1$. Hence, from (2.1), it can be concluded that

$$\begin{aligned} Q^{-1}(\phi(x_1)) < y(x_1) - y_1 &= \int_0^x \frac{-g(\eta)}{P(Q(y(\eta)) - F(\eta))} d\eta \\ &< \int_0^{x_1} \frac{-g(\eta)}{P(\phi(\eta) - F(\eta))} d\eta \leq Q^{-1}(\phi(x_1)), \end{aligned}$$

which is a contradiction.

Necessity: Suppose that $O^+(H)$ approaches the origin through the first quadrant. Then, its corresponding solution $y(x)$ satisfies

$$y(x) \to 0^+ \quad \text{as } x \to 0. \tag{2.3}$$

Let $\delta = x_0$ and $\phi(x) = Q(y(x))$ for $0 < x < \delta$. It is obvious that $\phi(x) \geq 0$. Thus,

$$Q^{-1}(\phi(x)) = y(x) < Q^{-1}(F(x)),$$

and therefore, $\phi(x) < F(x)$ for $0 < x < \delta$. Also, from (2.3) it can be easily seen that

$$\begin{aligned} \int_0^x \frac{-g(\eta)}{P(\phi(\eta) - F(\eta))} d\eta &= \int_0^x \frac{-g(\eta)}{P(Q(y(\eta)) - F(\eta))} d\eta = y(x) - \lim_{\epsilon \to 0} y(\epsilon) \\ &= Q^{-1}(\phi(x)). \end{aligned}$$

Thus, (2.1) holds and the proof is complete. □

Remark 2.4. For $P(u) = u$, Theorem 2.3 gives the corresponding result of Sugie in [19].

Corollary 2.5. *Suppose that there exists $k \in (0, 1)$ and $\delta > 0$ such that*

$$\frac{1}{Q^{-1}(kF(x))} \int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))} d\eta \leq 1 \quad \text{for } 0 < x < \delta. \tag{2.4}$$

Then, system (1.1) has property (Z_1^+).

Proof. Let $\phi(x) = kF(x)$. The following inequality is obtained from (2.4).

$$\int_0^x \frac{-g(\eta)}{P(\phi(\eta) - F(\eta))} d\eta = \int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))} d\eta \leq Q^{-1}(kF(x)),$$

for $0 < x < \delta$. Thus, by Theorem 2.3 system (1.1) has property (Z_1^+). □

Corollary 2.6. *Suppose that $P(au) \le aP(u)$ for $a \in (-1, 0)$ and $u > 0$. If there exist $k \in (0, 1)$ and $\delta > 0$ such that*

$$\frac{1}{(1-k)Q^{-1}(kF(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta \le 1 \quad \text{for } 0 < x < \delta,$$

then system (1.1) has property (Z_1^+).

Remark 2.7. For $P(u) = u$ and $Q(y) = y$ and taking $k = \frac{1}{2}$, Corollary 2.6 gives the result of Hara and Yoneyama in [9].

Corollary 2.8. *If for every $k \in [0, 1]$ there exists a constant $\gamma_k > 0$ such that*

$$\liminf_{x\to 0^+}\left(\frac{1}{Q^{-1}((k+\gamma_k)F(x))}\int_0^x \frac{-g(\eta)}{P((k-\gamma_k-1)F(\eta))}d\eta\right) > 1, \tag{2.5}$$

then system (1.1) fails to have property (Z_1^+).

Proof. Suppose that there exist a constant $\delta > 0$ and a continuous function ϕ such that condition (2.1) holds. Define $k' = \liminf_{x\to 0^+}\frac{\phi(x)}{F(x)}$. Then $0 \le k' \le 1$, and from the definition of k' it follows that for every $\epsilon > 0$, there exist a b and a sequence $\{x_n\}$ with $0 < b < \delta$, $0 < x_n \le b$, and $x_n \to 0$ as $n \to +\infty$ such that

$$\frac{\phi(x)}{F(x)} > k' - \epsilon \quad \text{for } 0 < x \le b \quad \text{and} \quad \frac{\phi(x_n)}{F(x_n)} < k' + \epsilon.$$

Hence,

$$\phi(x) > (k'-\epsilon)F(x) \quad \text{for } 0 < x \le b \quad \text{and} \quad \phi(x_n) < (k'+\epsilon)F(x_n).$$

Thus, from (2.1) it can be concluded that

$$\begin{aligned} 0 &\ge \int_0^{x_n}\frac{-g(\eta)}{P(\phi(\eta)-F(\eta))}d\eta - Q^{-1}(\phi(x_n)) \\ &> \int_0^{x_n}\frac{-g(\eta)}{P((k'-\epsilon)F(\eta)-F(\eta))}d\eta - Q^{-1}((k'+\epsilon)F(x_n)). \end{aligned}$$

Consequently, for $n \ge 1$ the following inequality holds.

$$\frac{1}{Q^{-1}((k'+\epsilon)F(x_n))}\int_0^{x_n}\frac{-g(\eta)}{P((k'-\epsilon-1)F(\eta))}d\eta < 1. \tag{2.6}$$

Thus, (2.6) contradicts (2.5) and the proof is complete. □

Corollary 2.9. *Suppose that $P(au) \ge aP(u)$ for $a \in [-2, -1)$ and $u > 0$. If there exists $\beta \in (1, 2]$ such that*

$$\liminf_{x\to 0^+}\left(\frac{1}{2Q^{-1}((\beta+1)F(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta\right) > 1, \tag{2.7}$$

then system (1.1) fails to have property (Z_1^+).

Proof. Suppose that (2.7) holds. Then, in (2.5) for every $k \in [0, 1]$ let $\gamma_k = (\beta - 1)k + 1$. By this argument, we have $k - 1 - \gamma_k = 2k - \beta k - 2$ and $k + \gamma_k = \beta k + 1$. Since $1 < \beta \le 2$ and $0 \le k \le 1$, then

$$-2 \le 2k - \beta k - 2 < -1, \qquad \frac{1}{2} \le \frac{1}{2+(\beta-2)k} < 1 \quad \text{and} \quad \beta k + 1 \le \beta + 1.$$

Now, put the last relations in the left-hand side of (2.5) and get

$$\begin{aligned}
&\liminf_{x\to 0^+}\left(\frac{1}{Q^{-1}((k+\gamma_k)F(x))}\int_0^x \frac{-g(\eta)}{P((k-\gamma_k-1)F(\eta))}d\eta\right)\\
&\quad=\liminf_{x\to 0^+}\left(\frac{1}{Q^{-1}((\beta k+1)F(x))}\int_0^x \frac{-g(\eta)}{P((2k-\beta k-2)F(\eta))}d\eta\right)\\
&\quad\ge\liminf_{x\to 0^+}\left(\frac{1}{(2+(\beta-2)k)Q^{-1}((\beta+1)F(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta\right)\\
&\quad\ge\liminf_{x\to 0^+}\left(\frac{1}{2Q^{-1}((\beta+1)F(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta\right)>1.
\end{aligned}$$

This completes the proof. □

By choosing $k=0$ in the proof of Corollary 2.9, the following corollary can be presented with weaker conditions.

Corollary 2.10. *Suppose that $P(au)\ge aP(u)$ for $a\in[-2,-1)$ and $u>0$. If*

$$\liminf_{x\to 0^+}\left(\frac{1}{2Q^{-1}(F(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta\right)>1, \tag{2.8}$$

then system (1.1) fails to have property (Z_1^+).

The following corollaries can be obtained as results of Theorem 2.3 which are very useful in applications.

Corollary 2.11. *Suppose that system (1.1) with $P(u)=P_1(u)$ has (resp., fails to have) property (Z_1^+). If $P_2(u)\le P_1(u)$ (resp., $P_2(u)\ge P_1(u)$) for $u<0$, then system (1.1) with $P(u)=P_2(u)$ has (resp., fails to have) property (Z_1^+).*

Corollary 2.12. *Suppose that system (1.1) with $Q(y)=Q_1(y)$ has (resp., fails to have) property (Z_1^+). If $Q_2(y)\le Q_1(y)$ (resp., $Q_2(y)\ge Q_1(y)$) for $y>0$ sufficiently small, then system (1.1) with $Q(y)=Q_2(y)$ has (resp., fails to have) property (Z_1^+).*

By the same way, we can prove the following theorem about property (Z_2^-).

Theorem 2.13. *System (1.1) has property (Z_2^-) if and only if there exist a constant $\delta>0$ and a continuous function $\phi(x)$ such that*

$$0\le\phi(x)<F(x)\quad\text{and}\quad\int_0^x\frac{-g(\eta)}{P(\phi(\eta)-F(\eta))}d\eta\le Q^{-1}(\phi(x))$$

for $-\delta<x<0$.

Similarly, other obtained results (Corollaries 2.5–2.10) can be formulated for property (Z_2^-).

3 Some explicit results

Condition (2.1) is implicit necessary and sufficient for system (1.1) to possess property (Z_1^+). However, in some cases, it is very difficult to find a suitable function $\phi(x)$ with a constant δ satisfying (2.1). Therefore, in the following, some explicit sufficient conditions are provided for system (1.1) to have property (Z_1^+). The results can also be formulated for the property (Z_3^+), (Z_2^-) or (Z_4^-). We leave the details to the reader. To state the results, define

$$H(y) = \int_0^y Q(\eta)d\eta \quad \text{and} \quad G(x) = \int_0^x g(\eta)d\eta.$$

Also, the inverse function of $\omega(y) = H(y)\mathrm{sgn}(y)$ is denoted by $H^{-1}(\omega)$.

Theorem 3.1. *Suppose that $P(au) \leq aP(u)$ for $a \in (-1, 0)$ and $u > 0$ and there exist $\alpha > 0$ and $k \in [0, 1)$ such that*

$$Q\left(\frac{x}{\alpha(1-k)}\right) \leq kP^{-1}(\alpha Q(x)) \tag{3.1}$$

for $x > 0$ sufficiently small. Then, system (1.1) has property (Z_1^+) if

$$F(x) \geq P^{-1}(\alpha Q(H^{-1}(G(x)))), \tag{3.2}$$

for $x > 0$ sufficiently small.

Proof. From (3.1) it is obvious that

$$\frac{u}{\alpha(1-k)Q^{-1}(kP^{-1}(\alpha Q(u)))} \leq 1, \tag{3.3}$$

for $u > 0$ sufficiently small. Since the function $u(x) = H^{-1}(G(x))$ is increasing and continuous on $[0, \infty)$ and $u(0) = 0$, by (3.2) we obtain

$$\frac{H^{-1}(G(x))}{\alpha(1-k)Q^{-1}(kP^{-1}(\alpha Q(H^{-1}(G(x)))))} \leq 1, \tag{3.4}$$

for $x > 0$ sufficiently small. Since

$$\frac{d}{dx}H^{-1}(G(x)) = \frac{g(x)}{Q(H^{-1}(G(x)))},$$

from (3.4) we conclude that

$$\begin{aligned}
&\frac{1}{(1-k)Q^{-1}(kF(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta \\
&\quad\leq \frac{1}{(1-k)Q^{-1}(kP^{-1}(\alpha Q(H^{-1}(G(x)))))}\int_0^x \frac{g(\eta)}{\alpha Q(H^{-1}(G(\eta)))}d\eta \\
&\quad= \frac{H^{-1}(G(x))}{\alpha(1-k)Q^{-1}(kP^{-1}(\alpha Q(H^{-1}(G(x)))))} \leq 1,
\end{aligned}$$

for $x > 0$ sufficiently small. Hence, by Corollary 2.6 system (1.1) has property (Z_1^+). □

By choosing $\alpha = 2$, $k = \frac{1}{2}$ and $P(u) = u$, condition (3.1) holds for any function Q. In this case, the following corollary is obtained about property (Z_1^+) which is the corresponding result of Sugie in [19].

Corollary 3.2. *Suppose that*

$$F(x) \geq 2Q(H^{-1}(G(x))), \tag{3.5}$$

for $x > 0$ sufficiently small. Then, system (1.1) with $P(u) = u$ has property (Z_1^+).

Theorem 3.3. *Suppose that $\alpha > 0$ and $P(au) \geq aP(u)$ for $a \in [-2,-1)$ and $u > 0$. Also, assume that there exists $\beta \in (1,2]$ such that*

$$Q\left(\frac{x}{2\alpha}\right) \geq (\beta+1)P^{-1}(\alpha Q(x)) \tag{3.6}$$

for $x > 0$ sufficiently small. Then, system (1.1) fails to have property (Z_1^+) if

$$F(x) \leq P^{-1}(\lambda \alpha Q(H^{-1}(G(x)))), \tag{3.7}$$

for some $\lambda < 1$.

Proof. By (3.6) it is obvious that

$$\frac{u}{\alpha Q^{-1}((\beta+1)P^{-1}(\alpha Q(u)))} \geq 2.$$

By the similar argument to the proof of Theorem 3.1, it can be concluded that if (3.6) and (3.7) hold, then

$$\liminf_{x\to 0^+}\left(\frac{1}{2Q^{-1}((\beta+1)F(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta\right) > 1.$$

Hence, by Corollary 2.9 system (1.1) fails to have property (Z_1^+). □

4 Homoclinic orbit

In this section some results will be presented about the existence of homoclinic orbit in the upper half-plane for system (1.1). The following theorem is obtained by combining Theorem 2.3 and 2.13.

Theorem 4.1. *System (1.1) has homoclinic orbit in the upper half-plane if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \leq \phi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{-g(\eta)}{P(\phi(\eta)-F(\eta))}d\eta \leq Q^{-1}(\phi(x)) \tag{4.1}$$

for $0 < |x| < \delta$.

The following two corollaries are obtained from Theorem 4.1, which provide explicit conditions for system (1.1) to have homoclinic orbit in upper half-plane. Note that, in Remark 2.2, it is assumed that there exists a $\delta > 0$ such that $F(x) > 0$ for $-\delta < x < \delta$.

Corollary 4.2. *Suppose that there exist $k \in (0,1)$ and $\delta > 0$ such that*

$$\frac{1}{Q^{-1}(kF(x))}\int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))}d\eta \leq 1 \quad \text{for } 0 < |x| < \delta. \tag{4.2}$$

Then, system (1.1) has homoclinic orbit in the upper half-plane.

Corollary 4.3. *Suppose that $P(au) \leq aP(u)$ for $a \in (-1,0)$ and $u > 0$. If there exist $k \in (0,1)$ and $\delta > 0$ such that*

$$\frac{1}{(1-k)Q^{-1}(kF(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta \leq 1 \quad \text{for } 0 < |x| < \delta, \tag{4.3}$$

then system (1.1) has homoclinic orbit in the upper half-plane.

Remark 4.4. Suppose that F is an even and g is an odd function. It is easy to see that system (1.1) has property (Z_1^+) if and only if it has property (Z_2^-). Therefore, if system (1.1) has property (Z_1^+), then it has a homoclinic orbit in the upper half-plane.

Similarly, Theorem 3.1 and Corollary 3.2 and some other results can be formulated about property (Z_2^-) and the existence of homoclinic orbits in the upper half-plane. Turning our attention to the lower half-plane, all presented results can be formulated about properties (Z_3^+) and (Z_4^-) and finally about the existence of homoclinic orbit in the lower half-plane.

In the following, two examples will be presented to illustrate our results and show the applications of the results.

Example 4.5. Consider the following Gause-type Predator-Prey system

$$\begin{aligned} \dot{u} &= ur(u) - vsf(u) \\ \dot{v} &= v(q(u) - D), \end{aligned} \tag{4.4}$$

with $f(u) = u$, $r(u) = \beta - \gamma|u - \alpha|$, $q(u) = u^2$, $D = \alpha^2$ and $\beta > \alpha\gamma$. In system (4.4), $u(t)$ and $v(t)$ represent prey and predator densities, the function $f(u)$ is functional response, $q(u)$ is the growth rate of the predator, $r(u)$ is the growth rate of the prey in the absence of any predators, and $D > 0$ is the natural death rate of the predator in the absence of any prey. The constants α, β and γ are positive ecological parameters. System (4.4) has the positive equilibrium $E^* = (\alpha, \beta)$. By the change of variables

$$x = u - \alpha, \qquad y = \ln\beta - \ln v \quad \text{and} \quad dt = uds,$$

system (4.4) will be transformed into system (1.1) with

$$P(u) = u, \qquad Q(y) = \beta(1 - e^{-y}), \qquad F(x) = \gamma|x| \quad \text{and} \quad g(x) = x + \alpha - \frac{\alpha^2}{x + \alpha}. \tag{4.5}$$

Functions $F(x)$ and $g(x)$ are defined on $(-\alpha, +\infty)$ and satisfy $F(0) = 0$ and $xg(x) > 0$ for $x \neq 0$. Also, $Q(y)$ is defined on $\mathbb{R}$ satisfying $Q(0) = 0$ and $yQ(y) > 0$ for $y \neq 0$. The inverse function of $Q(y)$ is $Q^{-1}(y) = \ln\left(\frac{\beta}{\beta - y}\right)$ where defined on $(-\infty, \beta)$. For $0 < x < \frac{\beta}{k\gamma}$, by using Corollary 4.3, it can be written that

$$\begin{aligned} \frac{1}{(1-k)Q^{-1}(kF(x))} \int_0^x \frac{g(\eta)}{P(F(\eta))} d\eta &= \frac{1}{\gamma(1-k)\ln\left(\dfrac{\beta}{\beta - k\gamma x}\right)} \left(x + \alpha\ln\left(1 + \frac{x}{\alpha}\right)\right) \\ &< \frac{2\beta}{\gamma^2(1-k)k}. \end{aligned}$$

By choosing $k = \frac{1}{2}$, it can be concluded that

$$\frac{1}{(1-k)Q^{-1}(kF(x))} \int_0^x \frac{g(\eta)}{P(F(\eta))} d\eta < \frac{8\beta}{\gamma^2}.$$

If $0 < 8\beta \leq \gamma^2$, then

$$\frac{1}{(1-k)Q^{-1}(kF(x))} \int_0^x \frac{g(\eta)}{P(F(\eta))} d\eta < 1.$$

By a similar argument, it can be shown that for $-\alpha < x < 0$

$$\frac{1}{(1-k)Q^{-1}(kF(x))} \int_0^x \frac{g(\eta)}{P(F(\eta))} d\eta < 1.$$

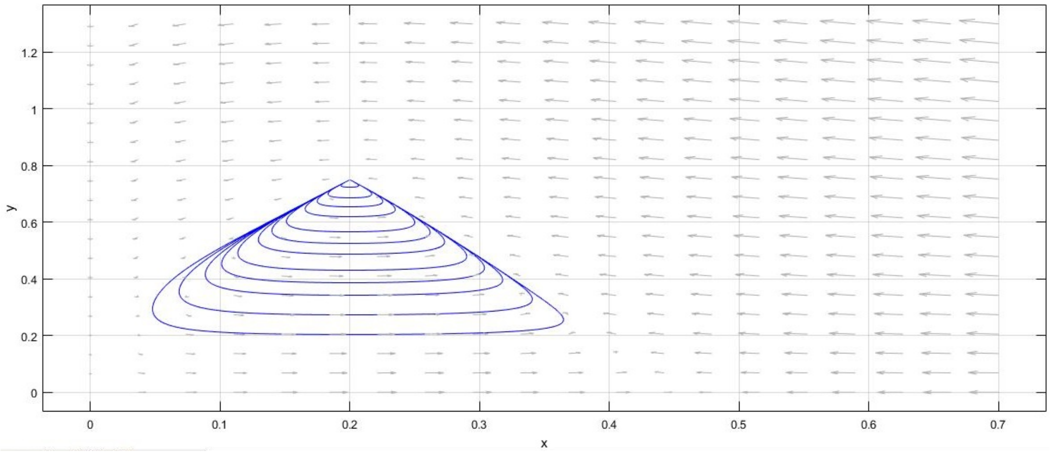

Figure 4.1: Phase portrait for system (4.4) with $\alpha = 0.2, \beta = 0.75$ and $\gamma = 3$.

Therefore, by Corollary 4.3 this system has a homoclinic orbit in the upper half-plane (see Fig. 4.1).

Remark 4.6. Sugie and Kimoto in [22], under the assumption $Q(y) \leq my$ for $y > 0$, showed that system (1.1) with functions in (4.5) has homoclinic orbits in the upper half-plane if $0 < 8\beta \leq \gamma^2$. In this work, the existence of homoclinic orbits has been presented without the assumption $Q(y) \leq my$ for $y > 0$.

Example 4.7. Consider system (1.1) with functions

$$P(u) = u^3, \qquad Q(y) = \operatorname{sgn}(y)\sqrt{|y|}, \qquad F(x) = \sqrt[4]{|x|} \quad \text{and} \quad g(x) = x. \tag{4.6}$$

By Corollary 2.5, it can be written that

$$\frac{1}{Q^{-1}(kF(x))} \int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))} d\eta = \frac{4\sqrt[4]{x^3}}{5k^2(1-k)^3} \leq 1$$

for $0 < x < \left(\frac{3}{5}\right)^4 \frac{1}{5\sqrt[3]{5}}$. Therefore, this system has property (Z_1^+). Since F is even and g is odd, Remark 4.4 implies that this system has a homoclinic orbit in the upper half-plane (see Fig. 4.2).

The next example shows a new application which comes from articles treating the Liénard equation with the differential operator related to the relativistic acceleration, that is

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-(\dot{x})^2}}\right) + f(x)\dot{x} + g(x) = 0, \tag{4.7}$$

which, nowadays, is a quite interesting topic in works concerning the case of generalized Liénard equations. The existence of a stable limit cycle and periodic solutions of relativistic Liénard equation (4.7) has been investigated by Mawhin and Villari in [15]. Now, we apply our results to a special case of this equation.

Equation (4.7) can easily be transformed to system (1.1) with

$$P(u) = \frac{u}{\sqrt{1+u^2}}, \qquad Q(y) = y \quad \text{and} \quad F(x) = \int_0^x f(\eta) d\eta.$$

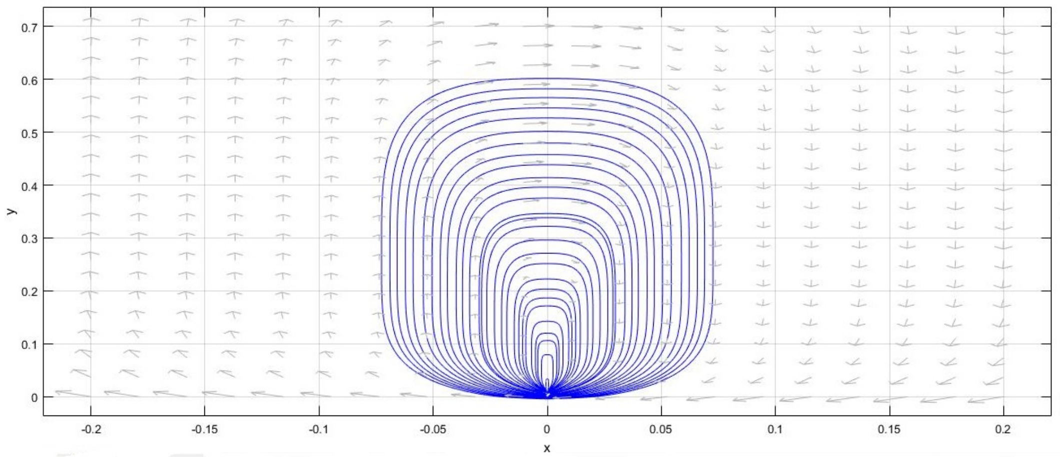

Figure 4.2: Phase portrait for system (4.6).

Example 4.8. Consider system (1.1) with

$$P(u) = \frac{u}{\sqrt{1+u^2}}, \qquad Q(y) = y, \qquad F(x) = x^2 \quad \text{and} \quad g(x) = \frac{x^3}{2\sqrt{1+x^4}}. \tag{4.8}$$

Since $P(au) \le aP(u)$ for $-1 < a < 0$ and $u > 0$, from Corollary 2.6, by choosing $k = \frac{1}{2}$, we have

$$\frac{1}{(1-k)Q^{-1}(kF(x))} \int_0^x \frac{g(\eta)}{P(F(\eta))} d\eta = \frac{2}{x^2} \int_0^x \eta d\eta = 1.$$

Therefore, this system has property (Z_1^+). Since F is even and g is odd, Remark 4.4 implies that this system has a homoclinic orbit in the upper half-plane (see Fig. 4.3).

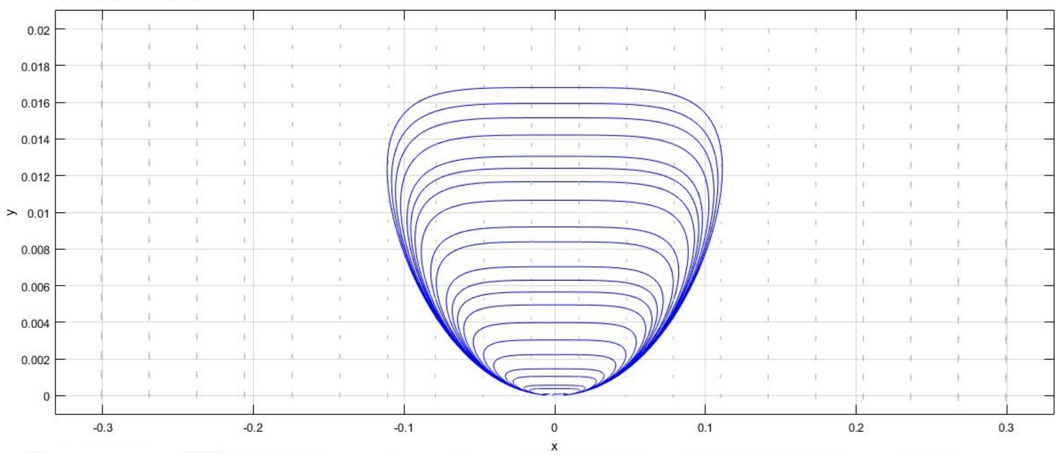

Figure 4.3: Phase portrait for system (4.8).

Example 4.9. Consider system (1.1) with

$$F(x) = x^m \quad (m > 0 \text{ and even number}), \qquad Q(y) = y^3$$
$$P(u) = u^3 \quad \text{and} \quad g(x) = |x^q| \operatorname{sgn}(x) \quad \text{with } q = \frac{10}{3}m + 1.$$

By choosing $k = \frac{1}{2}$, $\delta = \sqrt{\frac{q-3m+1}{8\sqrt[3]{2}}}$ and using Corollary 4.2 we have:

$$\frac{1}{Q^{-1}(kF(x))}\int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))}d\eta = 8\sqrt[3]{2}\left(\frac{\int_0^x \eta^{q-3m}d\eta}{x^{\frac{m}{3}}}\right) = \frac{8\sqrt[3]{2}}{q-3m+1}x^{q-\frac{10}{3}m+1} < 1$$

for $0 < |x| < \sqrt{\frac{q-3m+1}{8\sqrt[3]{2}}}$.

Thus, system (1.1) has homoclinic orbit in the upper half-plane.

Acknowledgment

The authors would like to thank anonymous referees for their carefully reading the manuscript and such valuable comments, which has improved the manuscript significantly.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **35**, 1–89; https://doi.org/10.14232/ejqtde.2021.1.35 www.math.u-szeged.hu/ejqtde/

Structurally unstable quadratic vector fields of codimension two: families possessing a finite saddle-node and an infinite saddle-node

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Received 9 December 2020, appeared 15 April 2021
Communicated by Gabriele Villari

Abstract. In 1998, Artés, Kooij and Llibre proved that there exist 44 structurally stable topologically distinct phase portraits modulo limit cycles, and in 2018 Artés, Llibre and Rezende showed the existence of at least 204 (at most 211) structurally unstable topologically distinct codimension-one phase portraits, modulo limit cycles. Artés, Oliveira and Rezende (2020) started the study of the codimension-two systems by the set (AA), of all quadratic systems possessing either a triple saddle, or a triple node, or a cusp point, or two saddle-nodes. They got 34 topologically distinct phase portraits modulo limit cycles. Here we consider the sets (AB) and (AC). The set (AB) contains all quadratic systems possessing a finite saddle-node and an infinite saddle-node obtained by the coalescence of an infinite saddle with an infinite node. The set (AC) describes all quadratic systems possessing a finite saddle-node and an infinite saddle-node, obtained by the coalescence of a finite saddle (respectively, finite node) with an infinite node (respectively, infinite saddle). We obtain all the potential topological phase portraits of these sets and we prove their realization. From the set (AB) we got 71 topologically distinct phase portraits modulo limit cycles and from the set (AC) we got 40 ones.

Keywords: quadratic differential system, structural stability, codimension two, phase portrait, saddle-node.

2020 Mathematics Subject Classification: Primary: 34A34, 34C23, 34C40. Secondary: 58-02.

1 Introduction and statement of the main results

Mathematicians are fascinated in closing problems. Having a question solved or even sign with a “q.e.d” a question asked in the past is a pleasure which is directly proportional to the time elapsed between the formulation of the question and the moment of the answer.

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The advent of the differential calculus opened the possibility of solving many questions that medieval mathematicians asked, but at the same time it opened the possibility of formulating many new other questions. The search for primitive functions that could not be expressed algebraically or with a finite number of analytic terms complicated the future research lines, and even new areas of Mathematics were created to give answers to these questions. And beside the problem of finding a primitive to a differential equation in a single dimension, if we add the possibility of more dimensions, the problem becomes much more difficult.

Therefore, it took almost 200 years between the appearance of the first system of linear differential equations and its complete resolution by Laplace in 1812. After the resolution of linear differential systems, for any dimension, it seemed natural to address the classification of quadratic differential systems. However, it was found that the problem would not have an easy and fast solution. Unlike the linear systems that can be solved analytically, quadratic systems (or higher degree systems) do not generically admit a solution of that kind, calculable in a finite number of terms.

Therefore, for the resolution of non-linear differential systems, another strategy was chosen and it allowed the creation of a new area of knowledge in Mathematics: the Qualitative Theory of Ordinary Differential Equations [27]. Since we are not able to give a concrete mathematical expression to the solution of a system of differential equations, this theory intends to express by means of a complete and precise drawing the behavior of any particle located in a vector field governed by such a differential equation, i.e. its phase portrait.

Even with all the reductions made to the problem until now, there are still difficulties. The most expressive difficulty is that the phase portraits of differential systems may have invariant sets as limit cycles and graphics. A linear system cannot generate limit cycles; at most they can present a completely circular phase portrait where all the orbits are periodic. But a differential system in the plane, polynomial or not, and starting with the quadratic ones, may present several limit cycles. It is natural to find an infinite number of these cycles in non-polynomial problems, but the intuition seems to indicate that a polynomial system should not have an infinite number of limit cycles in a similar way as it cannot have an infinite number of isolated singular points. And because the number of singular points is linked to the degree of the polynomial system, it also seems logical to think that the number of limit cycles could also have a similar link, either directly as the number of singular points, or even in an indirect way from the number the parameters of such systems.

In 1900, David Hilbert [21] proposed a set of 23 problems to be solved in the 20th century, and among them, the second part of his well-known 16th problem asks for the maximum number of limit cycles that a polynomial differential system in the plane with degree n may have. More than one hundred years after, we do not have an uniform upper bound for this generic problem, only for specific families of such a system.

During discussions, in 1966 Coppel [16] expressed the belief that we could obtain the classification of phase portraits of quadratic systems by purely algebraic means. That is, by means of algebraic equalities and inequalities, it should be possible to determine the phase portrait of a quadratic system. This claim was not easy to refute at that time, since the isolated finite singular points of a quadratic system can be found by means of the resultant that is of fourth degree, and its solutions can be calculated algebraically, like those of infinity. Moreover, at that time it was known how to generate limit cycles by a Hopf bifurcation, whose conditions are also determined algebraically.

On the other hand, in 1991, Dumortier and Fiddelears [17] showed that, starting with the quadratic systems (and following all the higher-degree systems), there exist geometric and topological phenomena in phase portraits of such a system whose determination cannot be fixed by means of algebraic expressions. More specifically, most part of the connections among separatrices and the occurrence of double or semi-stable limit cycles cannot be algebraically determined.

Therefore, the complete classification of quadratic systems is a very difficult task at the moment and it depends on the solution of the second part of Hilbert's 16th problem, even at least partially for the quadratic case.

Even so, a lot of problems have been appearing related to quadratic systems to which it has been possible to give an answer. In fact, there are more than one thousand articles published that are directly related to quadratic systems. John Reyn, from Delft University (Netherlands), prepared a bibliography that was published several times until his retirement (see [28,30–33]). It is worth mentioning that in the last two decades many other articles related to quadratic systems have appeared, so that the number of one thousand published papers on the subject may have been widely exceeded.

Many of the questions proposed and the problems solved have dealt with subclassifications of quadratic systems, that is, classifications of systems that shared some characteristic in common. For instance, we have systems with a center [26,35,36,38], with a weak focus of third order [3,24], with a nilpotent singularity [22], without real singular points [20], with two invariant lines [28] and so on, up to a thousand articles. In some of them complete answers could be given, including the problem of limit cycles (the existence and the number of limit cycles), but in other cases, the classification was done modulo limit cycles, that is, all the possible phase portraits without taking into account the presence and number of cycles. Since in quadratic systems a limit cycle can only surround a single finite singular point, which must necessarily be a focus [16], then it is enough to identify the outermost limit cycle of a nesting of cycles with a point, and interpret the stability of that point as the outer stability of this cycle, and study everything that can happen to the phase portrait in the rest of the space.

Within the families of quadratic systems that were studied in the 20th century, we would highlight the study of the structurally stable quadratic systems, modulo limit cycles. That is, the goal was to determine how many and which phase portraits of a quadratic system cannot be modified by small perturbations in their coefficients. To obtain a structurally stable system modulo limits cycles we need a few conditions: we do not allow the existence of multiple singular points and the existence of connections of separatrices. Centers, weak foci, semi-stable cycles, and all other unstable elements belong to the quotient modulo limit cycles. This systematic analysis [2] showed that the structurally stable quadratic systems have a total of 44 topologically distinct phase portraits.

From this scenario we observe that if we intent to work with classification of phase portraits of quadratic systems before the solution of the second part of Hilbert's 16th problem, this will have to be done modulo limit cycles.

Additionally, the entire family of quadratic systems by definition depends on twelve parameters, but due to the action of the group of the real affine transformations and time rescaling, this family ultimately depends on five parameters, but this is still a large number.

There are two ways to carry out a systematic study of all the phase portraits of the quadratic systems. One of them is the one initiated by Reyn in which he began by studying the phase portraits of all the quadratic systems in which all the finite singular points have coalesced with infinite singular points [29]. Later, he studied those in which exactly three finite singular points have coalesced with points of infinity, so there remains one real finite singularity. And then he completed the study of the cases in which two finite singular points have coalesced with points of infinity, originating two real points, or one double point, or two complex points. His work on finite multiplicity three was incomplete and the one on finite multiplicity four was inaccessible.

In another approach, instead of working from the highest degrees of degeneracy to the lower ones, is going to reverse direction. We already know that the structurally stable quadratic systems produce 44 topologically distinct phase portrait, as already mentioned before. In [6] the authors classified the structurally unstable quadratic systems of codimension one modulo limit cycles, which are systems having one and only one of the simplest structurally unstable objects: a saddle-node of multiplicity two (finite or infinite), a separatrix from one saddle point to another, or a separatrix forming a loop for a saddle point with its divergence nonzero. All the phase portraits of codimension one are split into four sets according to the possession of a structurally unstable element: (A) possessing a finite semi-elemental saddle-node, (B) possessing an infinite semi-elemental saddle-node $\overline{\binom{0}{2}}SN$, (C) possessing an infinite semi-elemental saddle-node $\overline{\binom{1}{1}}SN$, and (D) possessing a separatrix connection. This last set is split into five subsets according to the type of the connection: (a) finite-finite (heteroclinic orbit), (b) loop (homoclinic orbit), (c) finite-infinite, (d) infinite-infinite between symmetric points, and (e) infinite-infinite between adjacent points. The study of the codimension-one systems was done in approximately 20 years and finally it was obtained at least 204 (and at most 211) topologically distinct phase portraits of codimension one modulo limit cycles.

The next step is to study the structurally unstable quadratic systems of codimension two (see [12]), modulo limit cycles. Up to now, we have mentioned many times the word "codimension" and this is a clear concept in Geometry. However, in this classification we want to obtain topologically distinct phase portraits, and we want to group them according to their level of degeneracy. So, what was clear for structurally stable phase portraits and for codimension-one phase portraits (modulo limit cycles) may become a little weird if we continue in this same way, so we must give a definition of codimension adapted to this specific set that we want to classify.

Definition 1.1. We say that a phase portrait of a quadratic vector field is structurally stable if any sufficiently small perturbation in the parameter space leaves the phase portrait topologically equivalent to the previous one.

Definition 1.2. We say that a phase portrait of a quadratic vector field is structurally unstable of codimension $k \in \mathbb{N}$ if any sufficiently small perturbation in the parameter space either leaves the phase portrait topologically equivalent the previous one or it moves it to a lower codimension one, and there is at least one perturbation that moves it to the codimension $k-1$.

Remark 1.3.

1. When applying these definitions, modulo limit cycles, to phase portraits with centers, it would say that some phase portraits with centers would be of codimension as low as two, while geometrically they occupy a much smaller region in $\mathbb{R}^{12}$. So, the best way to avoid inconsistencies in the definitions is to tear apart the phase portraits with centers, that we know they are in number 31 (see [36]), and just work with systems without centers.

2. Starting in cubic systems, the definition of topologically equivalence, modulo limit cycles, becomes more complicated since we can have limit cycles having only one singu-

larity in its interior or more than one. So we cannot collapse the limit cycle because its interior is also relevant for the phase portrait.

3. Moreover, our definition of codimension needs also more precision starting with cubic systems due to new phenomena that may happen there.

Let $P_n(\mathbb{R}^2)$ be the set of all vector fields in $\mathbb{R}^2$ of the form $X(x,y) = (P(x,y), Q(x,y))$, with P and Q polynomials in the variables x and y of degree at most n (with $n \in \mathbb{N}$). In this set we consider the *coefficient topology* by identifying each vector field $X \in P_n(\mathbb{R}^2)$ with a point of $\mathbb{R}^{(n+1)(n+2)}$ (see more details in [6]). According to the previous definition concerning codimension two, and also according to the previously known results of codimension one, we have the result.

Theorem 1.4. *A polynomial vector field in $P_2(\mathbb{R}^2)$ is structurally unstable of codimension two modulo limit cycles if and only if all its objects are stable except for the break of exactly two stable objects. In other words, we allow the presence of two unstable objects of codimension one or one of codimension two.*

In what follows, instead of talking about codimension one modulo limit cycles, we will simply say *codimension one**. Analogously we will simply say *codimension two** instead of talking about codimension two modulo limit cycles.

Combining the classes of *codimension one** quadratic vector fields one to each other, we obtain 10 new classes, where one of them is split into 15 subsets, according to Tables 1.1 and 1.2.

	(A)	(B)	(C)	(D)
(A)	(AA)	-	-	-
(B)	(AB)	(BB)	-	-
(C)	(AC)	(BC)	(CC)	-
(D)	(AD) (5 cases)	(BD) (5 cases)	(CD) (5 cases)	see Table 1.2

Table 1.1: Sets of structurally unstable quadratic vector fields of codimension two considered from combinations of the classes of *codimension one**: (A), (B), (C), and (D) (which in turn is split into (a), (b), (c), (d), and (e)).

	(a)	(b)	(c)	(d)	(e)
(a)	(aa)				
(b)	(ab)	(bb)			
(c)	(ac)	(bc)	(cc)		
(d)	(ad)	(bd)	(cd)	(dd)	
(e)	(ae)	(be)	(ce)	(de)	(ee)

Table 1.2: Sets of structurally unstable quadratic vector fields of *codimension two** in the class (DD) (see Table 1.1).

Geometrically, the *codimension two** classes can be described as follows. Let X be a *codimension one** quadratic vector field. We have the following classes:

(AA) When X already has a finite saddle-node and either a finite saddle (respectively a finite node) of X coalesces with the finite saddle-node, giving birth to a semi-elemental triple saddle: $\overline{s}_{(3)}$ (respectively a triple node: $\overline{n}_{(3)}$), or when both separatrices of the saddle-node limiting its parabolic sector coalesce, giving birth to a cusp of multiplicity two: $\widehat{cp}_{(2)}$, or when another finite saddle-node is formed, having then two finite saddle-nodes: $\overline{sn}_{(2)}+\overline{sn}_{(2)}$. Since the phase portraits with $\overline{s}_{(3)}$ and with $\overline{n}_{(3)}$ would be topologically equivalent to structurally stable phase portraits and we are mainly interested in new phase portraits, we will skip them in this classification. Anyway, we may find them in the papers [11] and [13].

(AB) When X already has a finite saddle-node and an infinite saddle, and an infinite node of X coalesce with a finite saddle-node: $\overline{sn}_{(2)}+\overline{\binom{0}{2}}SN$.

(AC) When X already has a finite saddle-node and a finite saddle (respectively node), and an infinite node (respectively saddle) of X coalesce: $\overline{sn}_{(2)}+\overline{\binom{1}{1}}SN$.

(AD) When X has already a finite saddle-node and a separatrix connection is formed, considering all five types of class (D).

(BB) When an infinite saddle (respectively an infinite node) of X coalesces with an existing infinite saddle-node $\overline{\binom{0}{2}}SN$ of X, leading to a triple saddle: $\overline{\binom{0}{3}}S$ (respectively a triple node: $\overline{\binom{0}{3}}N$). This case is irrelevant to the production of new phase portraits since all the possible phase portraits that may produce are topologically equivalent to an structurally stable one.

(BC) When a finite antisaddle (respectively finite saddle) of X coalesces with an existing infinite saddle-node $\overline{\binom{0}{2}}SN$ of X, leading to a nilpotent elliptic saddle $\widehat{\binom{1}{2}}E-H$ (respectively nilpotent saddle $\widehat{\binom{1}{2}}HHH-H$). Or it may also happen that a finite saddle (respectively node) coalesces with an elemental node (respectively saddle) in a phase portrait having already an $\overline{\binom{0}{2}}SN$, having then in total $\overline{\binom{1}{1}}SN+\overline{\binom{0}{2}}SN$.

(BD) When we have an infinite saddle-node $\overline{\binom{0}{2}}SN$ plus a separatrix connection, considering all five types of class (D).

(CC) This case has two possibilities:

i) a finite saddle (respectively finite node) of X coalesces with an existing infinite saddle-node $\overline{\binom{1}{1}}SN$, leading to an semi-elemental triple saddle $\overline{\binom{2}{1}}S$ (respectively an semi-elemental triple node $\overline{\binom{2}{1}}N$),

ii) a finite saddle (respectively node) and an infinite node (respectively saddle) of X coalesce plus an another existing infinite saddle-node $\overline{\binom{1}{1}}SN$, leading to two infinite saddle-nodes $\overline{\binom{1}{1}}SN+\overline{\binom{1}{1}}SN$.

The first case is irrelevant to the production of new phase portraits since all the possible phase portraits that may produce are topologically equivalent to an structurally stable one.

(CD) When we have an infinite saddle-node $\overline{\binom{1}{1}}SN$ plus a saddle to saddle connection, considering all five types of class (D).

(DD) When we have two saddle to saddle connections, which are grouped as follows:

(aa) two finite-finite heteroclinic connections;

(ab) a finite-finite heteroclinic connection and a loop;

(ac) a finite-finite heteroclinic connection and a finite-infinite connection;

(ad) a finite-finite heteroclinic connection and an infinite-infinite connection between symmetric points;

(ae) a finite-finite heteroclinic connection and an infinite-infinite connection between adjacent points;

(bb) two loops;

(bc) a loop and a finite-infinite connection;

(bd) a loop and an infinite-infinite connection between symmetric points;

(be) a loop and an infinite-infinite connection between adjacent points;

(cc) two finite-infinite connections;

(cd) a finite-infinite connection and an infinite-infinite connection between symmetric points;

(ce) a finite-infinite connection and an infinite-infinite connection between adjacent points;

(dd) two infinite-infinite connections between symmetric points;

(de) an infinite-infinite connection between symmetric points and an infinite-infinite connection between adjacent points;

(ee) two infinite-infinite connections between adjacent points.

Some of these cases have also been proved to be empty in an on course paper [8].

In [12] the authors begin the study of codimension-two quadratic systems. The approach is the same used in the previous two works [2,6]. One must start by looking for all the potential topological phase portraits (i.e. phase portraits that can be drawn on paper) of codimension two modulo limit cycles, and then try to realize all of them (i.e. to find examples of quadratic differential systems whose phase portraits are exactly those phase portraits obtained previously) or to show that some of them are non-realizable or impossible (i.e. in case of absence of examples for the realization of a phase portrait, say Ψ, it is necessary to prove that there is no quadratic differential system whose phase portrait is topologically equivalent to Ψ).

In [12] the authors have considered the set (AA) obtained by the coalescence of two finite singular points, yielding either a triple saddle, or a triple node, or a cusp point, or two saddle-nodes. They obtained all the potential topological phase portraits modulo limit cycles of the set (AA) and proved their realization. In their study they got 34 new topologically distinct phase portraits (of codimension two) in the Poincaré disc modulo limit cycles. Moreover, they also proved the impossibility of one phase portrait among the 204 phase portraits from [6]. Therefore, in [6] they actually have at least 203 (and at most 210) topologically distinct phase portraits of codimension one modulo limit cycles. Additionally, more recent studies (in a preprint level) have shown the impossibility of another phase portrait among the 203 cited above. In that study it was also verified that, in fact, there exist at least 202 (and at most 209) topologically distinct phase portraits of codimension one modulo limit cycles.

In this paper we intend to contribute to the classification of the phase portraits of planar quadratic differential systems of codimension two, modulo limit cycles. According to what was explained before, since there are more than 10 cases of codimension two to be analyzed, it is impracticable to write a single paper with all the results. So, in [12] the authors have decided to split this study in several papers and this present article is the second one of this series. We indicate [2,6,12] for more details of the context of this study as well for all related definitions.

Here we present all the global phase portraits of the vector fields $X \in P_2(\mathbb{R}^2)$ belonging to sets (AB) and (AC) and we study their realization. The set (AB) contains all quadratic systems possessing a finite saddle-node $\overline{sn}_{(2)}$ and an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ obtained by the coalescence of an infinite saddle with an infinite node. The set (AC) describes all quadratic systems possessing a finite saddle-node $\overline{sn}_{(2)}$ and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$, obtained by the coalescence of a finite saddle (respectively, a finite node) with an infinite node (respectively, an infinite saddle). Notice that the finite singularity that coalesces with an infinite singularity cannot be the finite saddle-node since then what we would obtain at infinity would not be a saddle-node of type $\overline{\binom{1}{1}}SN$ but a multiplicity three singularity. Even this is also a *codimension two** case and somehow can be considered inside the set (AC), we have preferred to put it into the set (CC), which will be studied in a future paper.

We point out that in each picture representing a phase portrait we only draw the *skeleton of separatrices*, according to the next definition.

Definition 1.5. Let $p(X) \in P_n(\mathbb{S}^2)$ (respectively $X \in P_n(\mathbb{R}^2)$). A *separatrix* of $p(X)$ (respectively of X) is an orbit which is either a singular point (respectively a finite singular point), or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point (respectively a finite singular point). In [25] the author proved that the set formed by all separatrices of $p(X)$, denoted by $S(p(X))$, is closed. The open connected components of $\mathbb{S}^2 \setminus S(p(X))$ are called *canonical regions* of $p(X)$. We define a *separatrix configuration* as the union of $S(p(X))$ plus one representative solution chosen from each canonical region. Two separatrix configurations S_1 and S_2 of vector fields of $P_n(\mathbb{S}^2)$ (respectively $P_n(\mathbb{R}^2)$) are said to be *topologically equivalent* if there exists an orientation-preserving homeomorphism of $\mathbb{S}^2$ (respectively $\mathbb{R}^2$) which maps the trajectories of S_1 onto the trajectories of S_2. The *skeleton of separatrices* is defined as the union of $S(p(X))$ without the representative solution of each canonical region. Thus, a skeleton of separatrices can still produce different separatrix configurations.

Let $\sum_0^2$ denote the set of all planar structurally stable vector fields and $\sum_i^2(S)$ denote the set of all structurally unstable vector fields $X \in P_2(\mathbb{R}^2)$ of codimension i, modulo limit cycles belonging to the set S, where S is a set of vector fields with the same type of instability modulo orientation. For instance, in this paper we consider the sets $\sum_2^2(AB)$ and $\sum_2^2(AC)$, which denote, respectively, the set of all structurally unstable vector fields $X \in P_2(\mathbb{R}^2)$ of *codimension two** belonging to the sets (AB) and (AC).

The main goal of this paper is to prove the following two theorems.

Theorem 1.6. *If $X \in \sum_2^2(AB)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 71 *phase portraits of Figures 1.1 to 1.3.*

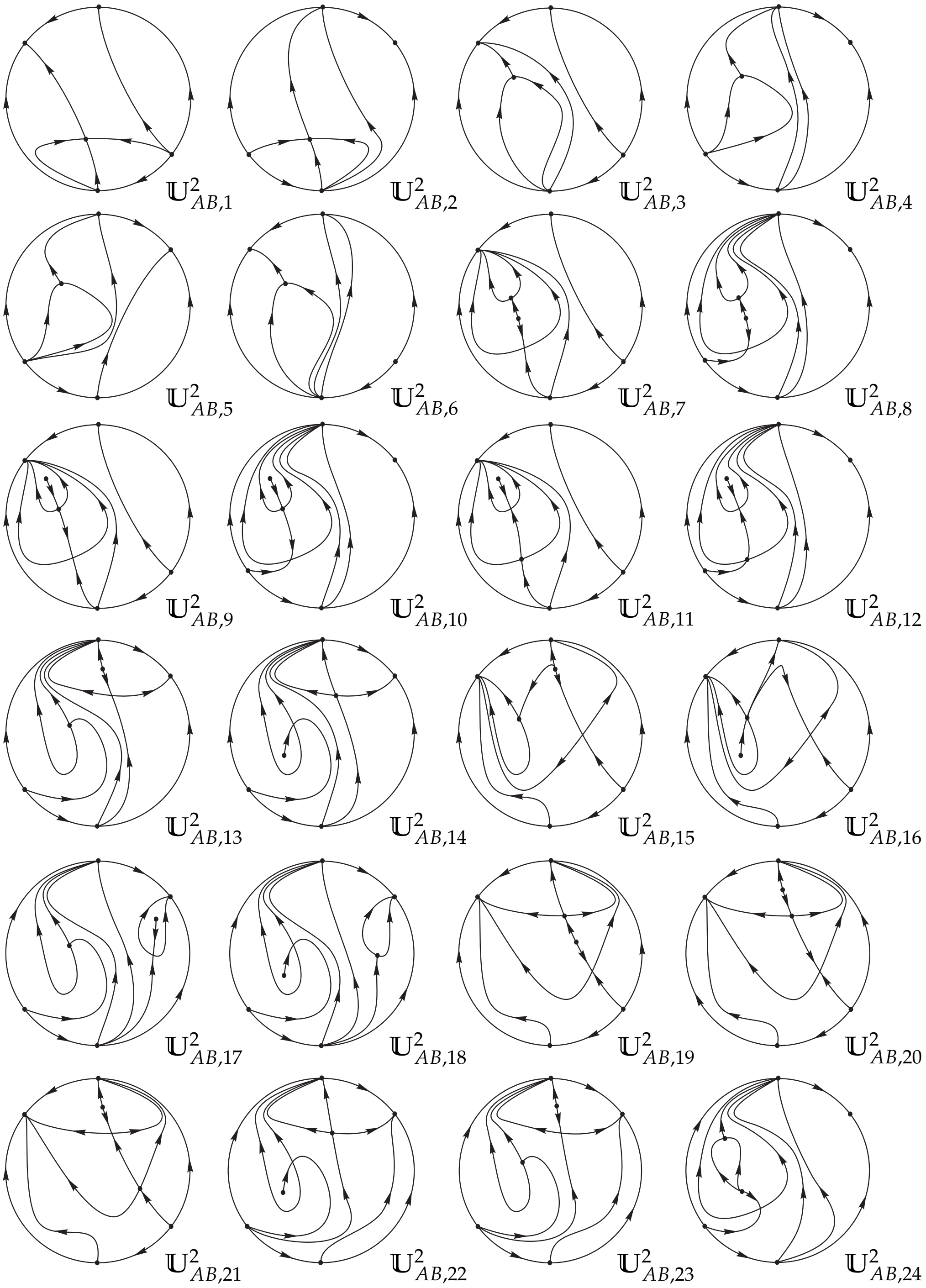


Figure 1.1: Structurally unstable quadratic phase portraits of *codimension two** of the set (AB).

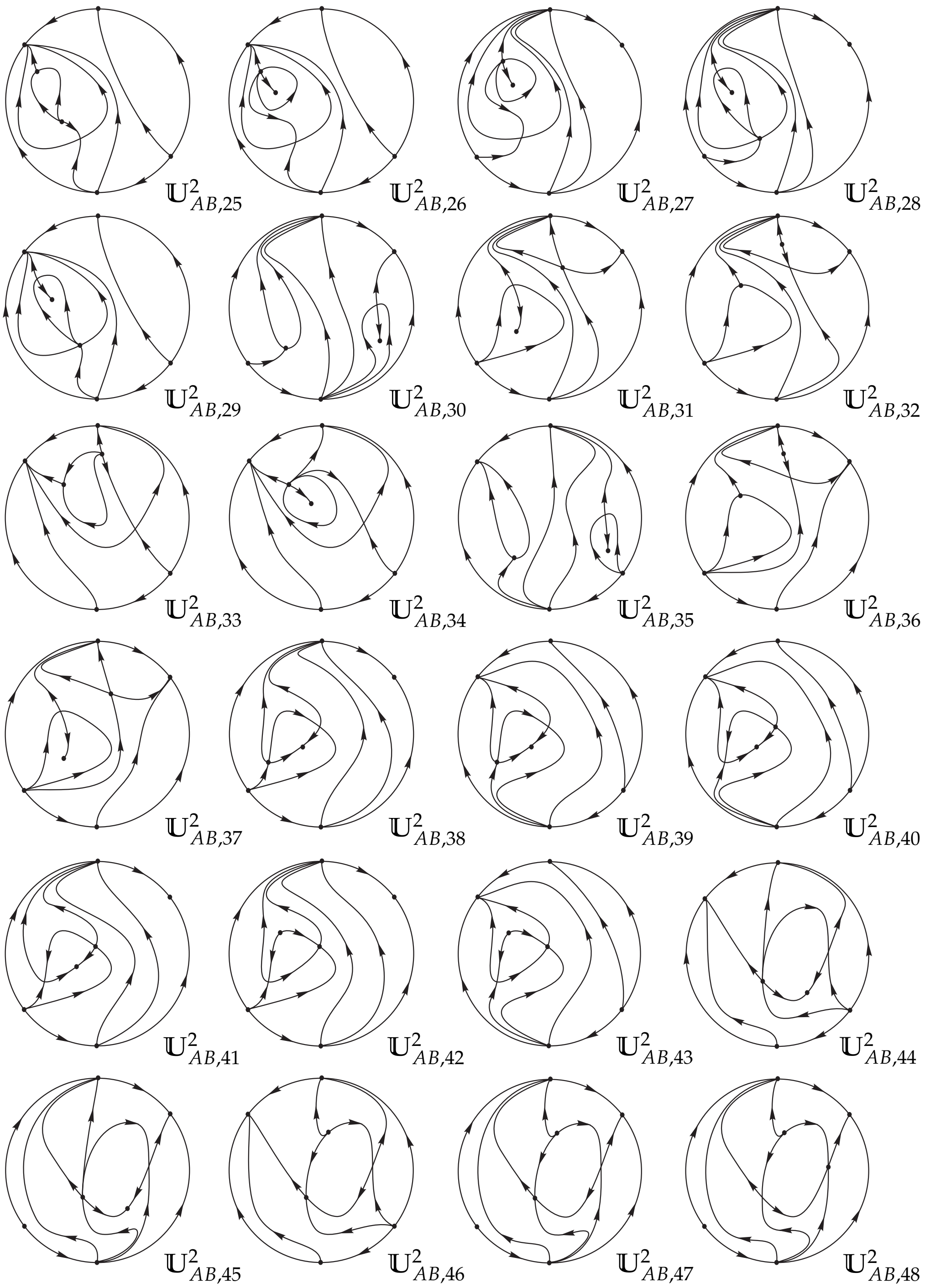


Figure 1.2: *(Cont.)* Structurally unstable quadratic phase portraits of *codimension two** of the set (AB).

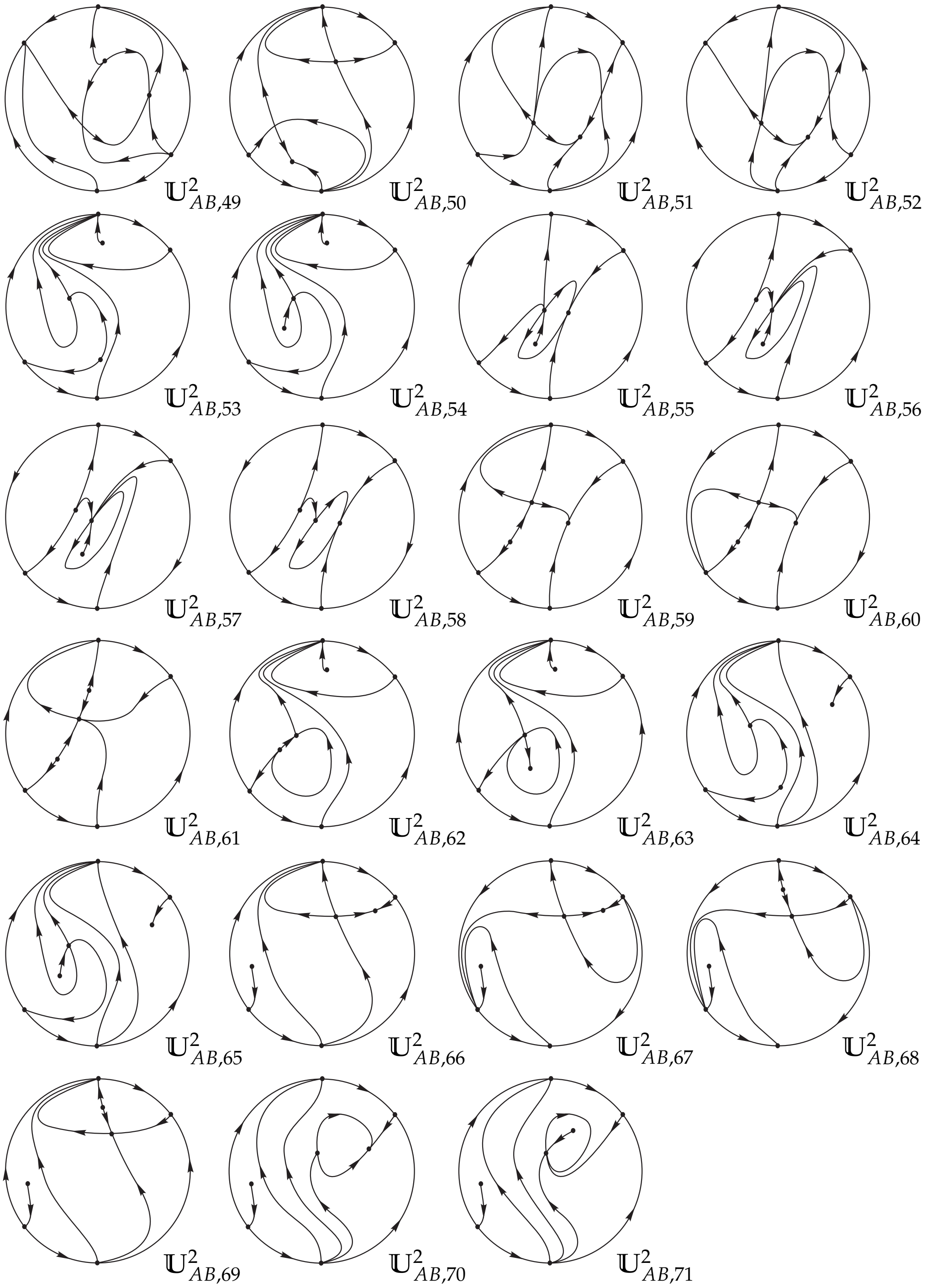


Figure 1.3: *(Cont.)* Structurally unstable quadratic phase portraits of *codimension two** of the set (AB).

Theorem 1.7. *If $X \in \sum_2^2(AC)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 40 *phase portraits of Figures 1.4 and 1.5.*

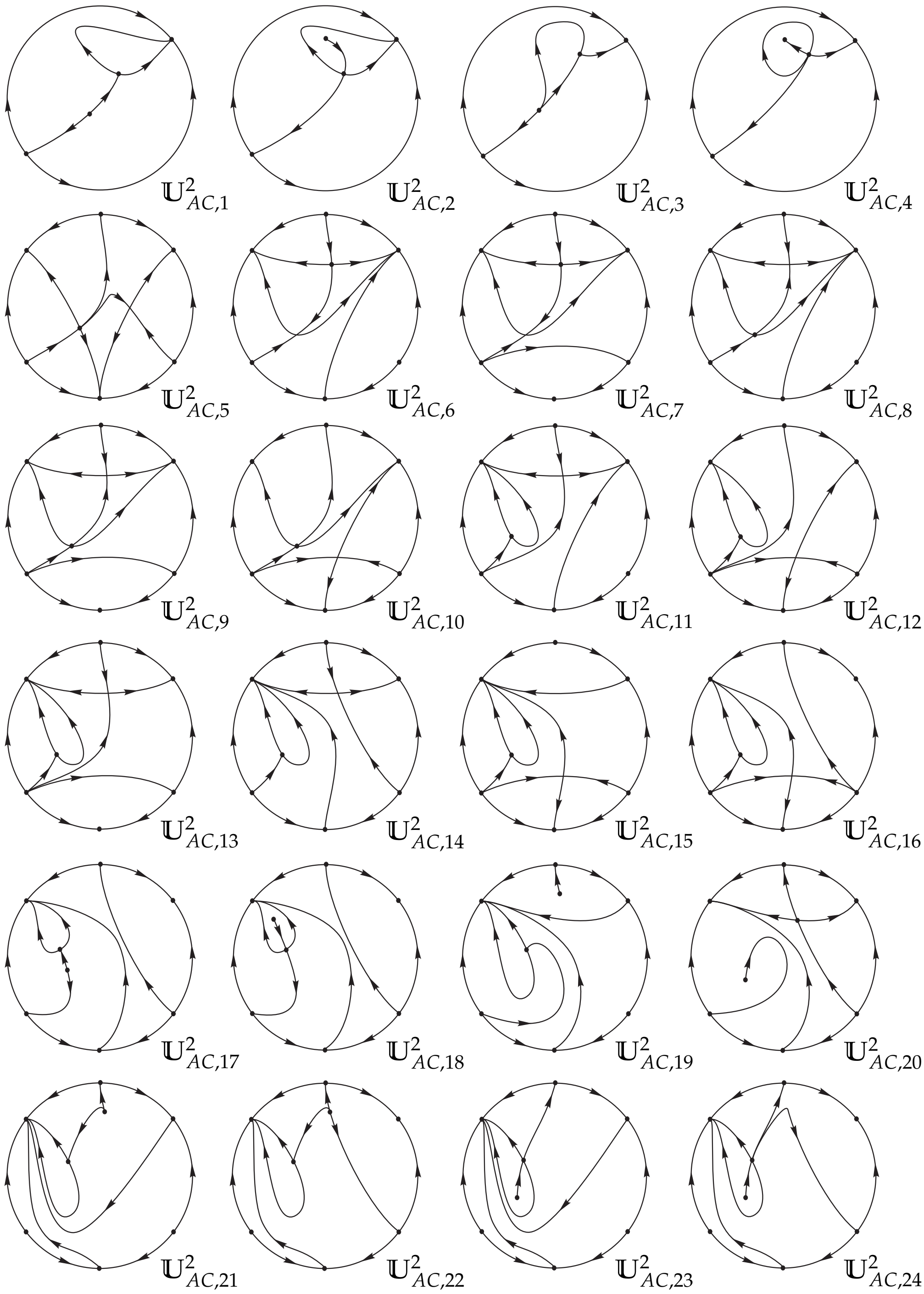


Figure 1.4: Structurally unstable quadratic phase portraits of *codimension two** of the set (AC).

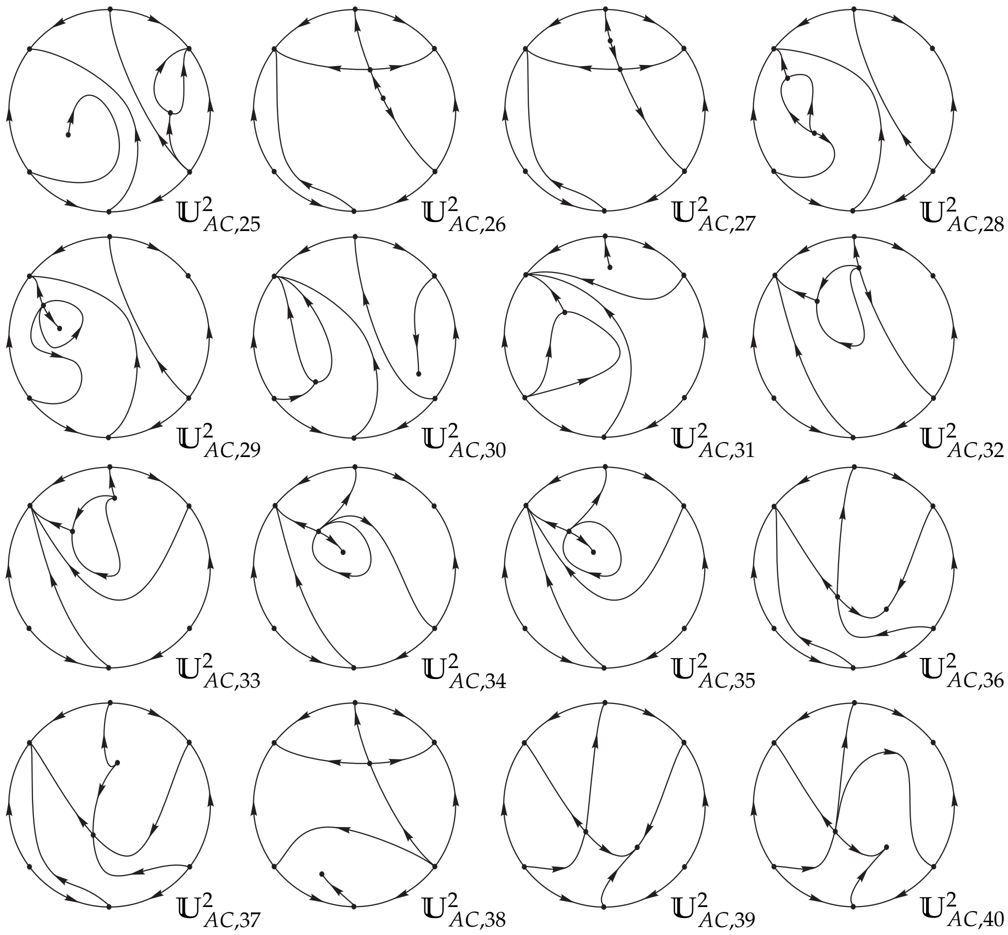


Figure 1.5: *(Cont.)* Structurally unstable quadratic phase portraits of *codimension two** of the set (AC).

This paper is organized as follows. In Section 2 we make a brief description of phase portraits of codimensions zero and one that are needed in this paper.

In Section 3 we prove Theorem 1.6 and in Section 4 we prove Theorem 1.7. We point out that in order to verify the realization of the corresponding phase portraits we compute each one of them with the numerical program P4 [1, 18].

Once again, remember that by modulo limit cycles we mean all eyes with limit cycles are assimilated with the unique singular point (a focus) within such an eye, i.e. we may say that the phase portraits are *blind* to limit cycles. Additionally, the phase portraits are also blind with respect to distinguishing if a singular point is a focus or a node, because these are not topological properties. But as the phase portraits are not blind to detecting other important features like various types of graphics, in Section 5 we discuss about the existence of graphics and also limit cycles in this study.

2 Quadratic vector fields of codimension zero and one

In this section we summarize all the needed results from the book of Artés, Llibre and Rezende [6]. The following three results are the restriction of Theorem 1.1 from book [6] to the sets (A), (B), and (C), respectively (see page 4). We denote by $\sum_1^2(A)$ (respectively, $\sum_1^2(B)$, and $\sum_1^2(C)$) the set of all structurally unstable vector fields $X \in P_2(\mathbb{R}^2)$ of *codimension one** belonging to the set (A) (respectively, (B), and (C)).

Theorem 2.1. *If $X \in \sum_1^2(A)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 69 *phase portraits of Figures 2.1 to 2.3, and all of them are realizable.*

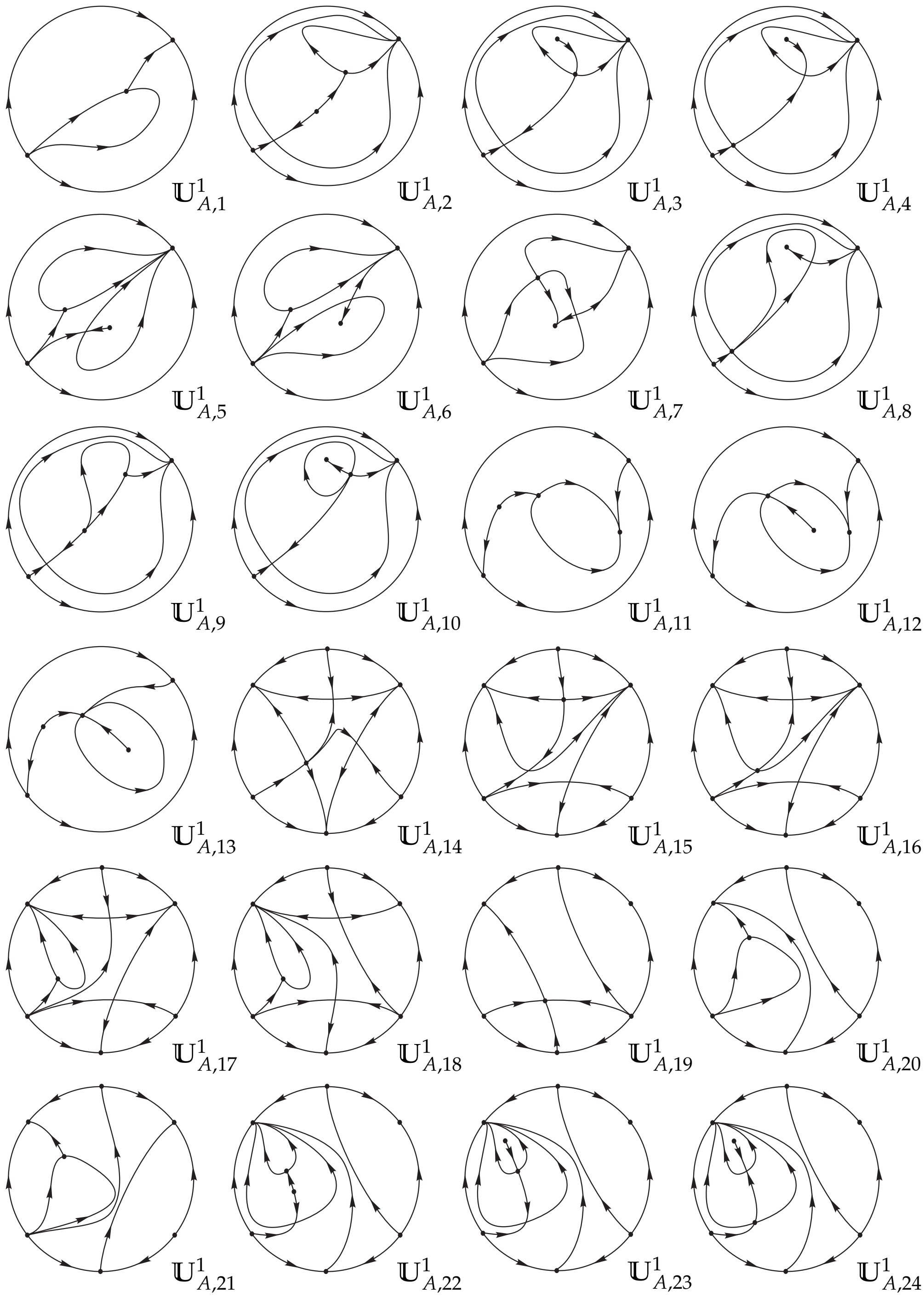


Figure 2.1: Unstable quadratic systems of *codimension one** of the set (A) (cases with a finite saddle-node $\overline{sn}_{(2)}$).

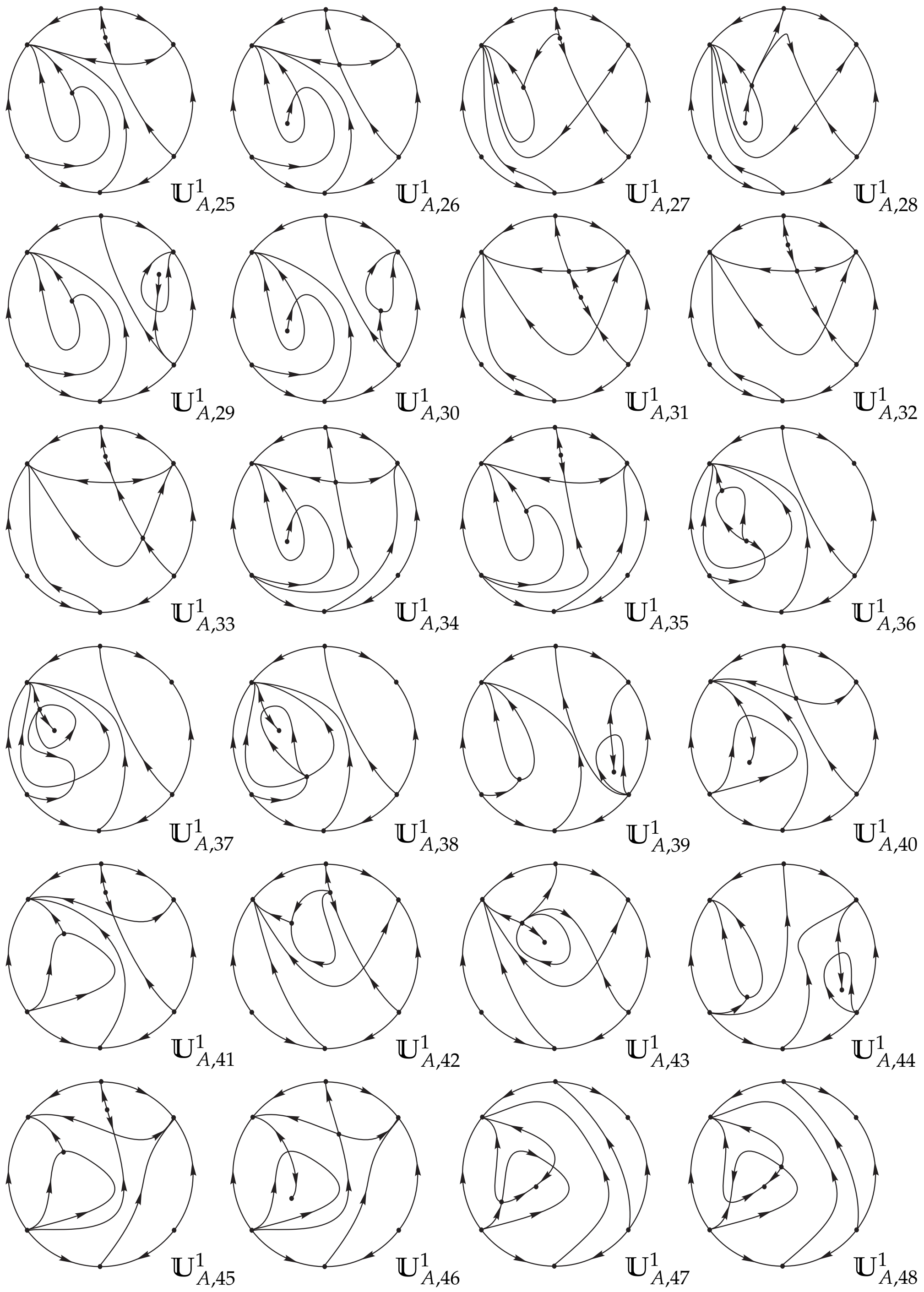


Figure 2.2: *(Cont.)* Unstable quadratic systems of *codimension one** of the set (A) (cases with a finite saddle-node $\overline{sn}_{(2)}$).

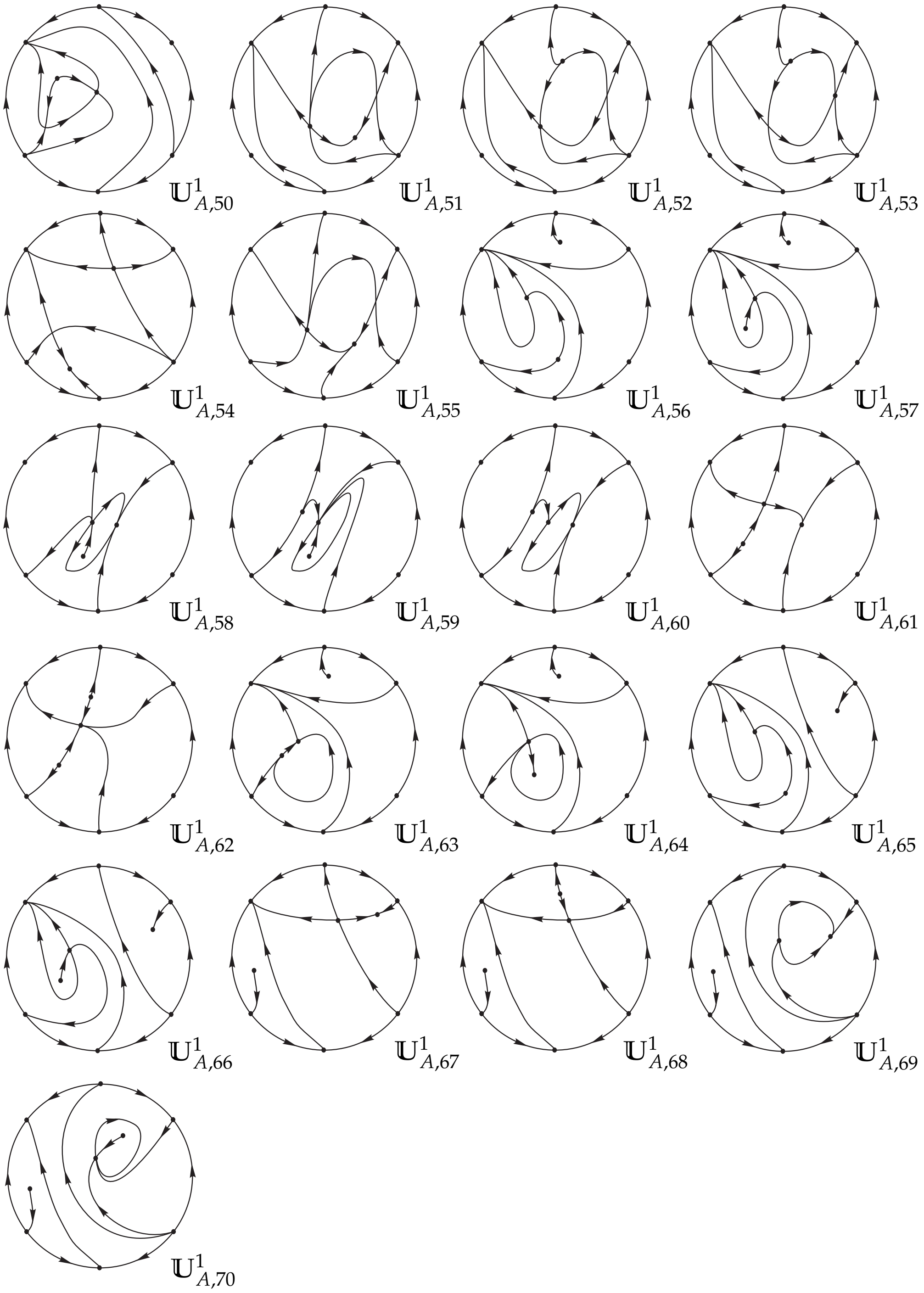


Figure 2.3: *(Cont.)* Unstable quadratic systems of *codimension one** of the set (A) (cases with a finite saddle-node $\overline{sn}_{(2)}$).

Remark 2.2. In [12] the authors proved that the phase portrait $\mathbb{U}^1_{A,49}$ from Figure 1.4 of [6] is actually impossible. Therefore, in our Figures 2.1 to 2.3 we have simply "skipped" this phase portrait, since all of the remaining ones are proved to be realizable in [6]. We present this impossible phase portrait in Figure 2.8 and there we denote it by $\mathbb{U}^{1,I}_{A,49}$.

Theorem 2.3. *If $X \in \sum_1^2(B)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 40 *phase portraits of Figures 2.4 and 2.5, and all of them are realizable.*

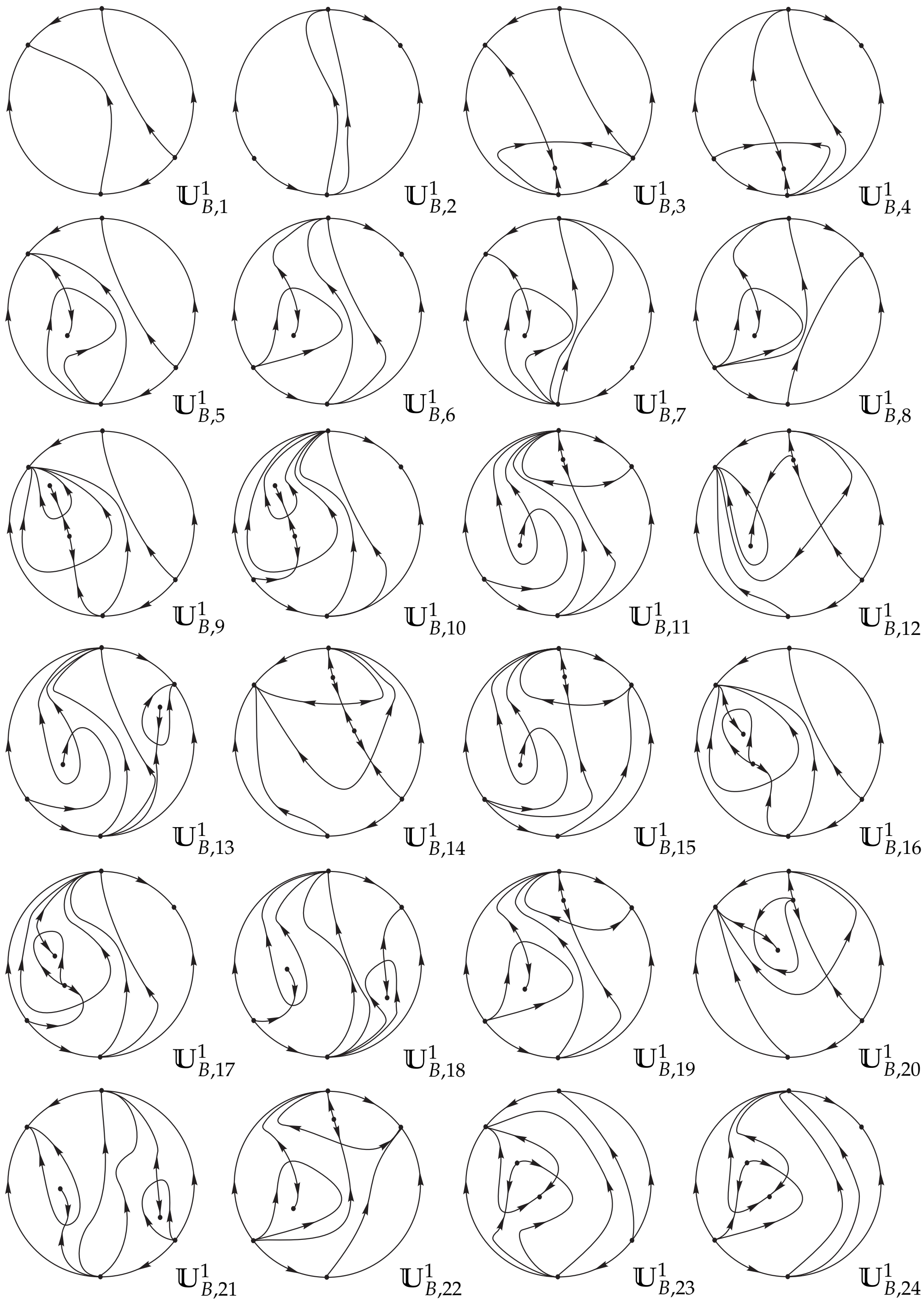


Figure 2.4: Unstable quadratic systems of *codimension one** of the set (B) (cases with an infinite saddle-node of type $\overline{\binom{0}{2}}SN$).

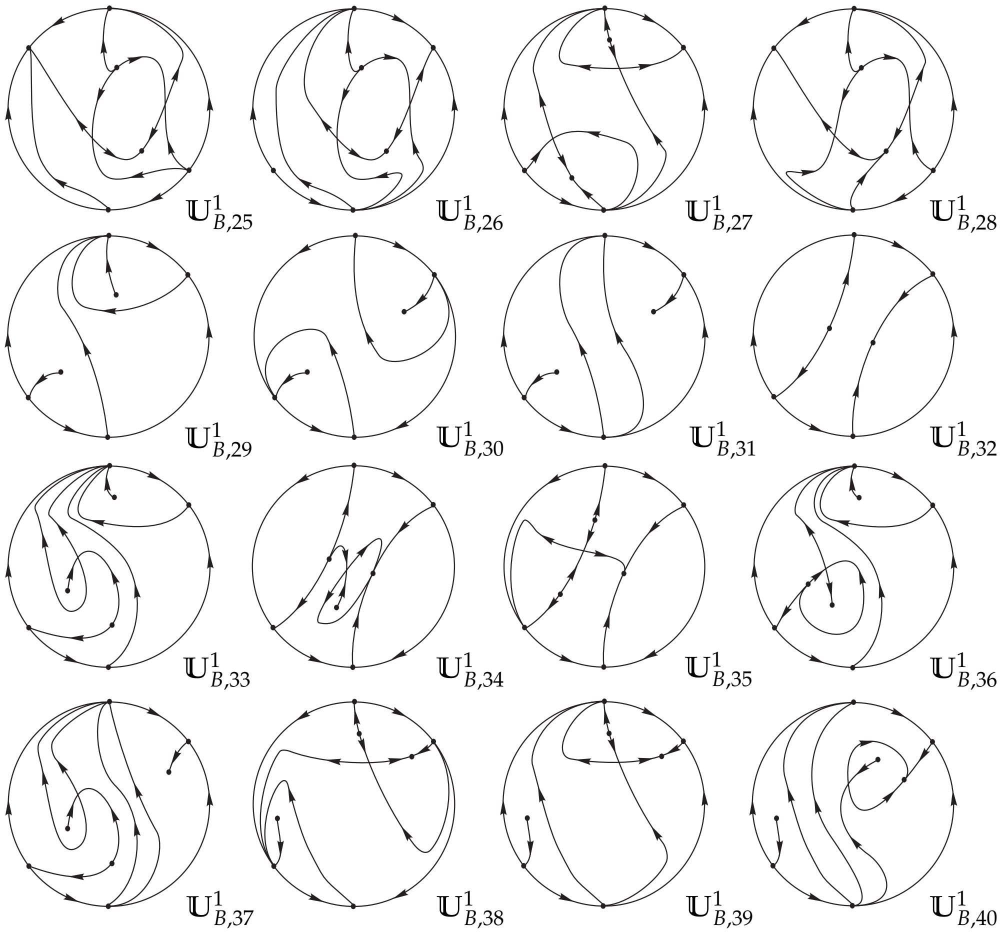


Figure 2.5: *(Cont.)* Unstable quadratic systems of *codimension one** of the set (B) (cases with an infinite saddle-node of type $\overline{\binom{0}{2}}SN$).

Theorem 2.4. *If $X \in \sum_1^2(C)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 32 *phase portraits of Figures 2.6 and 2.7, and all of them are realizable.*

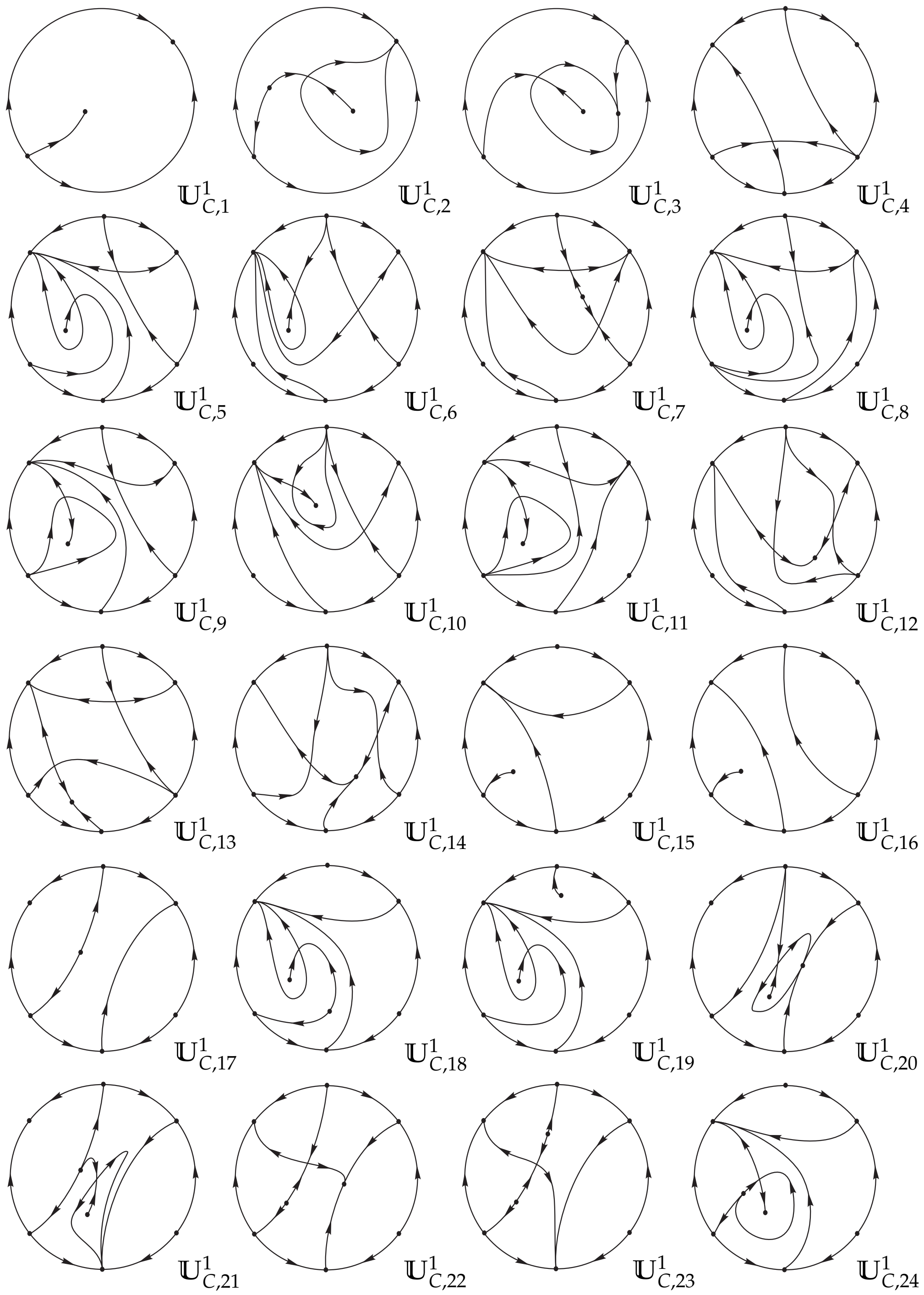


Figure 2.6: Unstable quadratic systems of *codimension one** of the set (C) (cases with an infinite saddle-node of type $\overline{\binom{1}{1}}SN$).

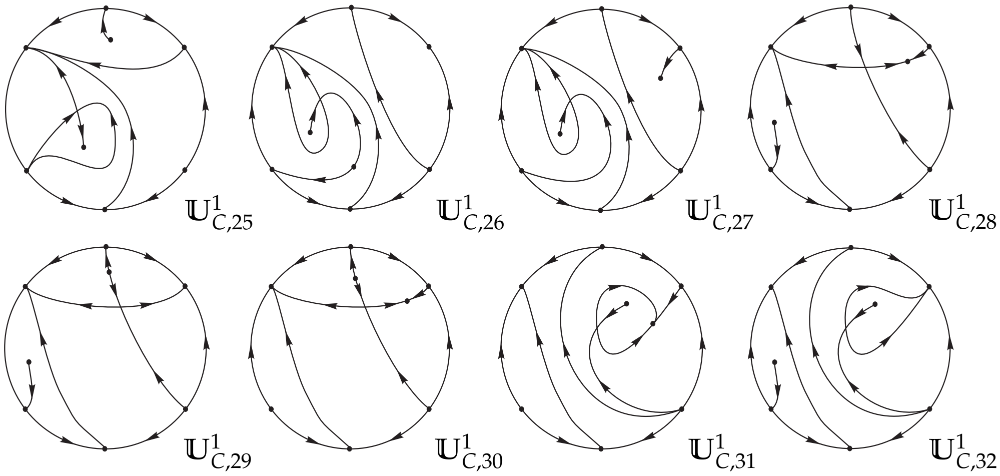


Figure 2.7: *(Cont.)* Unstable quadratic systems of *codimension one** of the set (C) (cases with an infinite saddle-node of type $\overline{\binom{1}{1}}SN$).

Before we state our next theorem, consider the following remark.

Remark 2.5. Consider all the impossible phase portraits from the book [6]. In that book these phase portraits are described with a specific notation. However, in this paper we changed a little bit their notation in order to associate each impossible phase portrait with the set in which such a phase portrait is proved to be impossible, but we keep the respective indexes. For instance, in that book we have the presence of the impossible phase portrait $\mathbb{U}^1_{I,105}$, which is a non-realizable case from the set (A). Such a phase portrait is denoted in this paper by $\mathbb{U}^{1,I}_{A,105}$. We also use this new notation for phase portraits which are proved to be impossible in the sets (B) and (C).

The next result describes which phase portraits were discarded in the set (A) in [6] because they were not realizable, but their role now is important in the process of discarding impossible phase portraits of *codimension two**.

Theorem 2.6. *In order to obtain a phase portrait of a structurally unstable quadratic vector field of codimension one** *from the set* (A) *it is necessary and sufficient to coalesce a finite saddle and a finite node from a structurally stable quadratic vector field, which leads to a finite saddle-node, and after some small perturbation it disappears. For the vector fields in the set* (A), *the following statements hold.*

(a) In Table 2.1 we see in the first and fifth columns the structurally stable quadratic vector fields (following the notation present in [2, 6]) which, after the coalescence of singularities cited above, lead to at least one phase portrait of codimension one* *from the set* (A).

(b) Inside this set (A), *we have a total of* 77 *topologically distinct phase portraits according to the different* α*-limit or* ω*-limit of the separatrices of their saddles,* 7 *of which are proved non-realizable in [6] and another one is proved non-realizable in [12] (all of these eight non-realizable phase portraits are given in Table 2.2). These numbers are given in the second and sixth columns of Table 2.1.*

(c) From these potential phase portraits, most of them are realizable. That is, even though there is the topological possibility of their existence, some of them break some analytical property which makes them not realizable inside quadratic vector fields. We have a total of 69 *realizable phase portraits. In the third and seventh columns of Table 2.1 we present the number of realizable*

cases coming from the bifurcation of each structurally stable phase portrait, and in the fourth and eighth columns we present the bifurcated phase portraits of codimension one associated to each one.*

(d) *There are then 8 non-realizable cases from the set (A) which we now collect in a single picture (see Figure 2.8) and denote by $\mathbb{U}^{1,I}_{A,k}$, where $\mathbb{U}^{1,I}_{A}$ stands for Impossible of codimension one* from the set (A) and $k \in \{1, 2, 3, 49, 103, 104, 105, 106\}$, see Remark 2.5. These phase portraits are all drawn in [6]. Anyway, we provide Table 2.2 in order to relate easily (giving also the page where they appear first and the page they are proved to be impossible).*

SSQVF [2]	$\#_\mathbf{p}$	$\#_\mathbf{r}$	**SU1 [6]**	**SSQVF [2]**	$\#_\mathbf{p}$	$\#_\mathbf{r}$	**SU1 [6]**
$\mathbb{S}^2_{2,1}$	1	1	$\mathbb{U}^1_{A,1}$	$\mathbb{S}^2_{10,6}$	2	2	$\mathbb{U}^1_{A,34}, \mathbb{U}^1_{A,35}$
$\mathbb{S}^2_{3,1}$	3	3	$\mathbb{U}^1_{A,2}, \mathbb{U}^1_{A,3}, \mathbb{U}^1_{A,4}$	$\mathbb{S}^2_{10,7}$	4	3	$\mathbb{U}^1_{A,36}, \mathbb{U}^1_{A,37}, \mathbb{U}^1_{A,38}$
$\mathbb{S}^2_{3,2}$	1	1	$\mathbb{U}^1_{A,5}$	$\mathbb{S}^2_{10,8}$	1	1	$\mathbb{U}^1_{A,39}$
$\mathbb{S}^2_{3,3}$	1	1	$\mathbb{U}^1_{A,6}$	$\mathbb{S}^2_{10,9}$	2	2	$\mathbb{U}^1_{A,40}, \mathbb{U}^1_{A,41}$
$\mathbb{S}^2_{3,4}$	1	1	$\mathbb{U}^1_{A,7}$	$\mathbb{S}^2_{10,10}$	4	2	$\mathbb{U}^1_{A,42}, \mathbb{U}^1_{A,43}$
$\mathbb{S}^2_{3,5}$	3	3	$\mathbb{U}^1_{A,8}, \mathbb{U}^1_{A,9}, \mathbb{U}^1_{A,10}$	$\mathbb{S}^2_{10,11}$	1	1	$\mathbb{U}^1_{A,44}$
$\mathbb{S}^2_{5,1}$	3	3	$\mathbb{U}^1_{A,11}, \mathbb{U}^1_{A,12}, \mathbb{U}^1_{A,13}$	$\mathbb{S}^2_{10,12}$	2	2	$\mathbb{U}^1_{A,45}, \mathbb{U}^1_{A,46}$
$\mathbb{S}^2_{7,1}$	1	1	$\mathbb{U}^1_{A,14}$	$\mathbb{S}^2_{10,13}$	4	4	$\mathbb{U}^1_{A,47}, \mathbb{U}^1_{A,48}, \mathbb{U}^1_{A,50}$
$\mathbb{S}^2_{7,2}$	2	2	$\mathbb{U}^1_{A,15}, \mathbb{U}^1_{A,16}$	$\mathbb{S}^2_{10,14}$	4	3	$\mathbb{U}^1_{A,51}, \mathbb{U}^1_{A,52}, \mathbb{U}^1_{A,53}$
$\mathbb{S}^2_{7,3}$	1	1	$\mathbb{U}^1_{A,17}$	$\mathbb{S}^2_{10,15}$	1	1	$\mathbb{U}^1_{A,54}$
$\mathbb{S}^2_{7,4}$	1	1	$\mathbb{U}^1_{A,18}$	$\mathbb{S}^2_{10,16}$	1	1	$\mathbb{U}^1_{A,55}$
$\mathbb{S}^2_{9,1}$	1	1	$\mathbb{U}^1_{A,19}$	$\mathbb{S}^2_{12,1}$	2	2	$\mathbb{U}^1_{A,56}, \mathbb{U}^1_{A,57}$
$\mathbb{S}^2_{9,2}$	1	1	$\mathbb{U}^1_{A,20}$	$\mathbb{S}^2_{12,2}$	3	3	$\mathbb{U}^1_{A,58}, \mathbb{U}^1_{A,59}, \mathbb{U}^1_{A,60}$
$\mathbb{S}^2_{9,3}$	1	1	$\mathbb{U}^1_{A,21}$	$\mathbb{S}^2_{12,3}$	2	2	$\mathbb{U}^1_{A,61}, \mathbb{U}^1_{A,62}$
$\mathbb{S}^2_{10,1}$	3	3	$\mathbb{U}^1_{A,22}, \mathbb{U}^1_{A,23}, \mathbb{U}^1_{A,24}$	$\mathbb{S}^2_{12,4}$	3	2	$\mathbb{U}^1_{A,63}, \mathbb{U}^1_{A,64}$
$\mathbb{S}^2_{10,2}$	2	2	$\mathbb{U}^1_{A,25}, \mathbb{U}^1_{A,26}$	$\mathbb{S}^2_{12,5}$	2	2	$\mathbb{U}^1_{A,65}, \mathbb{U}^1_{A,66}$
$\mathbb{S}^2_{10,3}$	3	2	$\mathbb{U}^1_{A,27}, \mathbb{U}^1_{A,28}$	$\mathbb{S}^2_{12,6}$	2	2	$\mathbb{U}^1_{A,67}, \mathbb{U}^1_{A,68}$
$\mathbb{S}^2_{10,4}$	2	2	$\mathbb{U}^1_{A,29}, \mathbb{U}^1_{A,30}$	$\mathbb{S}^2_{12,7}$	3	2	$\mathbb{U}^1_{A,69}, \mathbb{U}^1_{A,70}$
$\mathbb{S}^2_{10,5}$	3	3	$\mathbb{U}^1_{A,31}, \mathbb{U}^1_{A,32}, \mathbb{U}^1_{A,33}$				

Table 2.1: Potential and realizable bifurcated phase portraits for a given structurally stable quadratic vector field. In this table, **SSQVF** stands for structurally stable quadratic vector fields, $\#_\mathbf{p}$ (respectively $\#_\mathbf{r}$) for the number of topologically potential (respectively realizable) phase portraits of *codimension one** bifurcated from the respective **SSQVF**, and **SU1** for the respective phase portraits of *codimension one**.

SSQVF [2]	**Page** [6]	**Impossible** [6]	**SSQVF** [2]	**Page** [6]	**Impossible** [6]
$\mathbb{S}^2_{10,3}$	70	$\mathbb{U}^{1,I}_{A,1}$	$\mathbb{S}^2_{10,14}$	77	$\mathbb{U}^{1,I}_{A,3}$
$\mathbb{S}^2_{10,7}$	(73) 190	$\mathbb{U}^{1,I}_{A,103}$	$\mathbb{S}^2_{12,4}$	(80) 191	$\mathbb{U}^{1,I}_{A,105}$
$\mathbb{S}^2_{10,10}$	75; 191	$\mathbb{U}^{1,I}_{A,2}$; $\mathbb{U}^{1,I}_{A,104}$	$\mathbb{S}^2_{12,7}$	(82) 188	$\mathbb{U}^{1,I}_{A,106}$
$\mathbb{S}^2_{10,13}$	76	$\mathbb{U}^{1,I}_{A,49}$ (see [12])			

Table 2.2: Non-realizable phase portraits from the set (A) which could bifurcate (if existed) from structurally stable quadratic vector fields. The first and fourth columns indicate the structurally stable quadratic vector field (**SSQVF**) which suffers a bifurcation, the second and fifth columns indicate the pages where they appear in [6] and the third and sixth columns present the corresponding impossible phase portraits (remember that phase portrait $\mathbb{U}^1_{A,49}$ from Figure 1.4 of [6] is proved to be impossible in [12]).

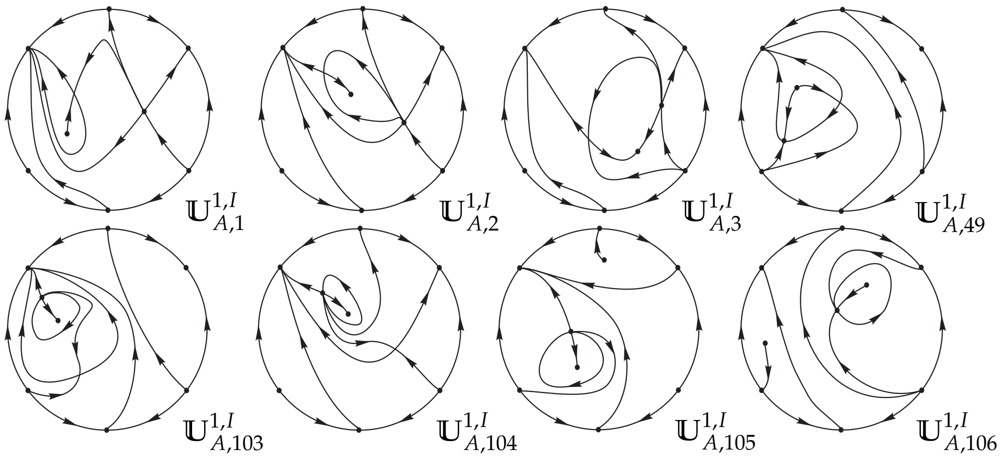


Figure 2.8: Phase portraits of the non-realizable structurally unstable quadratic vector fields of *codimension one** from the set (A).

In what follows we present an analogous theorem regarding discarded phase portraits from the set (B) in [6].

Theorem 2.7. *In order to obtain a phase portrait of a structurally unstable quadratic vector field of codimension one** *from the set (B) it is necessary and sufficient to coalesce an infinite saddle with an infinite node from a structurally stable quadratic vector field, which leads to an infinite saddle-node of type* $\overline{\binom{0}{2}}SN$*, and after some small perturbation it disappears. For the vector fields in set* (B)*, the following statements hold.*

(a) In Table 2.3 we see in the first and fifth columns the structurally stable quadratic vector fields (following the notation present in [2, 6]) which, after the coalescence of singularities cited above, lead to at least one phase portrait of codimension one* *from the set (B).*

(b) Inside this set (B), we have a total of 55 *topologically distinct phase portraits according to the different* α*-limit or* ω*-limit of the separatrices of their saddles,* 15 *of which are non-realizable (they are given in Table 2.4). These numbers are given in the second and sixth columns of Table 2.3.*

(c) From these potential phase portraits, most of them are realizable. That is, even though there is the topological possibility of their existence, some of them break some analytical property which

makes them not realizable inside quadratic vector fields. We have a total of 40 *realizable phase portraits. In the third and seventh columns of Table 2.3 we present the number of realizable cases coming from the bifurcation of each structurally stable phase portrait, and in the fourth and eighth columns we present the bifurcated phase portraits of codimension one** *associated to each one.*

(d) *There are then* 15 *non-realizable cases from the set (B) which we now collect in a single picture (see Figure 2.9) and denote by* $\mathbb{U}^{1,I}_{B,k}$, *where* $\mathbb{U}^{1,I}_{B}$ *stands for Impossible of codimension one** *from the set (B) and* $k \in \{4, 5, 6, 7, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117\}$, *see Remark 2.5. These phase portraits are all drawn in [6]. Anyway, we provide Table 2.4 in order to relate easily (giving also the page where they appear first and the page they are proved to be impossible).*

SSQVF [2]	$\#_p$	$\#_r$	**SU1 [6]**	**SSQVF [2]**	$\#_p$	$\#_r$	**SU1 [6]**
$\mathbb{S}^2_{8,1}$	2	2	$\mathbb{U}^1_{B,1}, \mathbb{U}^1_{B,2}$	$\mathbb{S}^2_{10,12}$	2	1	$\mathbb{U}^1_{B,22}$
$\mathbb{S}^2_{9,1}$	2	2	$\mathbb{U}^1_{B,3}, \mathbb{U}^1_{B,4}$	$\mathbb{S}^2_{10,13}$	2	2	$\mathbb{U}^1_{B,23}, \mathbb{U}^1_{B,24}$
$\mathbb{S}^2_{9,2}$	2	2	$\mathbb{U}^1_{B,5}, \mathbb{U}^1_{B,6}$	$\mathbb{S}^2_{10,14}$	2	2	$\mathbb{U}^1_{B,25}, \mathbb{U}^1_{B,26}$
$\mathbb{S}^2_{9,3}$	2	2	$\mathbb{U}^1_{B,7}, \mathbb{U}^1_{B,8}$	$\mathbb{S}^2_{10,15}$	2	1	$\mathbb{U}^1_{B,27}$
$\mathbb{S}^2_{10,1}$	2	2	$\mathbb{U}^1_{B,9}, \mathbb{U}^1_{B,10}$	$\mathbb{S}^2_{10,16}$	1	1	$\mathbb{U}^1_{B,28}$
$\mathbb{S}^2_{10,2}$	2	1	$\mathbb{U}^1_{B,11}$	$\mathbb{S}^2_{11,1}$	1	1	$\mathbb{U}^1_{B,29}$
$\mathbb{S}^2_{10,3}$	2	1	$\mathbb{U}^1_{B,12}$	$\mathbb{S}^2_{11,2}$	2	2	$\mathbb{U}^1_{B,30}, \mathbb{U}^1_{B,31}$
$\mathbb{S}^2_{10,4}$	2	1	$\mathbb{U}^1_{B,13}$	$\mathbb{S}^2_{11,3}$	1	1	$\mathbb{U}^1_{B,32}$
$\mathbb{S}^2_{10,5}$	2	1	$\mathbb{U}^1_{B,14}$	$\mathbb{S}^2_{12,1}$	2	1	$\mathbb{U}^1_{B,33}$
$\mathbb{S}^2_{10,6}$	2	1	$\mathbb{U}^1_{B,15}$	$\mathbb{S}^2_{12,2}$	1	1	$\mathbb{U}^1_{B,34}$
$\mathbb{S}^2_{10,7}$	2	2	$\mathbb{U}^1_{B,16}, \mathbb{U}^1_{B,17}$	$\mathbb{S}^2_{12,3}$	1	1	$\mathbb{U}^1_{B,35}$
$\mathbb{S}^2_{10,8}$	2	1	$\mathbb{U}^1_{B,18}$	$\mathbb{S}^2_{12,4}$	2	1	$\mathbb{U}^1_{B,36}$
$\mathbb{S}^2_{10,9}$	2	1	$\mathbb{U}^1_{B,19}$	$\mathbb{S}^2_{12,5}$	2	1	$\mathbb{U}^1_{B,37}$
$\mathbb{S}^2_{10,10}$	2	1	$\mathbb{U}^1_{B,20}$	$\mathbb{S}^2_{12,6}$	2	2	$\mathbb{U}^1_{B,38}, \mathbb{U}^1_{B,39}$
$\mathbb{S}^2_{10,11}$	2	1	$\mathbb{U}^1_{B,21}$	$\mathbb{S}^2_{12,7}$	2	1	$\mathbb{U}^1_{B,40}$

Table 2.3: Potential and realizable bifurcated phase portraits for a given structurally stable quadratic vector field. In this table, **SSQVF** stands for structurally stable quadratic vector fields, $\#_p$ (respectively $\#_r$) for the number of topologically potential (respectively realizable) phase portraits of *codimension one** bifurcated from the respective **SSQVF**, and **SU1** for the respective phase portraits of *codimension one**.

SSQVF [2]	**Page** [6]	**Impossible** [6]	**SSQVF** [2]	**Page** [6]	**Impossible** [6]
$\mathbb{S}^2_{10,2}$	86; 200	$\mathbb{U}^{1,I}_{B,107}$	$\mathbb{S}^2_{10,11}$	90; 200	$\mathbb{U}^{1,I}_{B,115}$
$\mathbb{S}^2_{10,3}$	86; 203	$\mathbb{U}^{1,I}_{B,108}$	$\mathbb{S}^2_{10,12}$	91; 200	$\mathbb{U}^{1,I}_{B,116}$
$\mathbb{S}^2_{10,4}$	87; 200	$\mathbb{U}^{1,I}_{B,109}$	$\mathbb{S}^2_{10,15}$	92; 200	$\mathbb{U}^{1,I}_{B,117}$
$\mathbb{S}^2_{10,5}$	87; 207	$\mathbb{U}^{1,I}_{B,110}$	$\mathbb{S}^2_{12,1}$	94	$\mathbb{U}^{1,I}_{B,4}$
$\mathbb{S}^2_{10,6}$	88; 200	$\mathbb{U}^{1,I}_{B,111}$	$\mathbb{S}^2_{12,4}$	96	$\mathbb{U}^{1,I}_{B,5}$
$\mathbb{S}^2_{10,8}$	89; 200	$\mathbb{U}^{1,I}_{B,112}$	$\mathbb{S}^2_{12,5}$	96	$\mathbb{U}^{1,I}_{B,6}$
$\mathbb{S}^2_{10,9}$	89; 200	$\mathbb{U}^{1,I}_{B,113}$	$\mathbb{S}^2_{12,7}$	97	$\mathbb{U}^{1,I}_{B,7}$
$\mathbb{S}^2_{10,10}$	90; 203	$\mathbb{U}^{1,I}_{B,114}$			

Table 2.4: Non-realizable phase portraits from the set (B) which could bifurcate (if existed) from structurally stable quadratic vector fields. The first and fourth columns indicate the structurally stable quadratic vector field (**SSQVF**) which suffers a bifurcation, the second and fifth columns indicate the pages where they appear in [6] and the third and sixth columns present the corresponding impossible phase portraits.

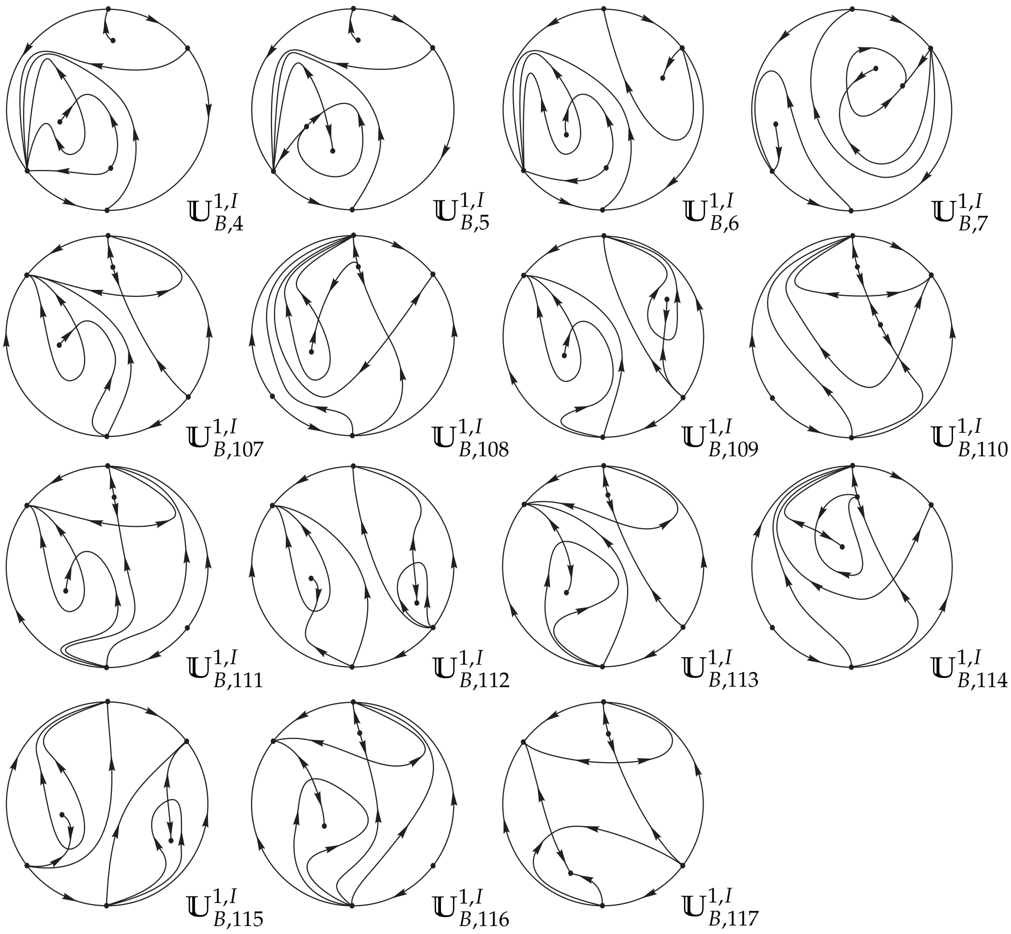


Figure 2.9: Phase portraits of the non-realizable structurally unstable quadratic vector fields of *codimension one** from the set (B).

Remark 2.8. Regarding the phase portraits of the non-realizable structurally unstable quadratic vector fields of *codimension one** from the set (B), we point out that in page 91 from [6], phase portrait $\mathbb{U}^{1,I}_{B,116}$ (which corresponds to B_{31} from such a book) is wrongly drawn. In fact, it possesses an extra infinite node and such a phase portrait should be drawn exactly as we present in our Figure 2.9.

Finally we present an analogous theorem regarding discarded phase portraits from the set (C) in [6].

Theorem 2.9. *In order to obtain a phase portrait of a structurally unstable quadratic vector field of codimension one* from the set (C) it is necessary and sufficient to coalesce a finite node (respectively, a finite saddle) with an infinite saddle (respectively, an infinite node) from a structurally stable quadratic vector field, which leads to an infinite saddle-node of type $\overline{\binom{1}{1}}SN$, and after some small perturbation, this saddle-node is split into a finite saddle (respectively, a finite node) and an infinite node (respectively, an infinite saddle). For the vector fields in the set (C), the following statements hold.*

(a) In Table 2.5 we see in the first and fifth columns the structurally stable quadratic vector fields (following the notation present in [2, 6]) which, after the coalescence of singularities cited above, lead to at least one phase portrait of codimension one from the set (C).*

(b) Inside this set (C), we have a total of 34 topologically distinct phase portraits according to the different α-limit or ω-limit of the separatrices of their saddles, two of which are non-realizable (they are given in Table 2.6). These numbers are given in the second and sixth columns of Table 2.5.

(c) From these potential phase portraits, only two of them are not realizable. That is, even though there is the topological possibility of their existence, two of them break some analytical property which makes them not realizable inside quadratic vector fields. We have a total of 32 realizable phase portraits. In the third and seventh columns of Table 2.5 we present the number of realizable cases coming from the bifurcation of each structurally stable phase portrait, and in the fourth and eighth columns we present the bifurcated phase portraits of codimension one associated to each one.*

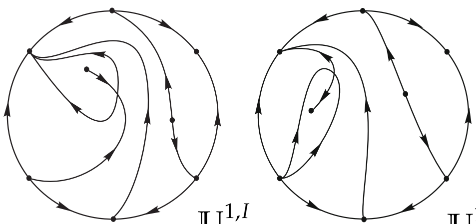
(d) There are then two non-realizable cases from the set (C) which we present in Figure 2.10 and denote by $\mathbb{U}^{1,I}_{C,k}$, where $\mathbb{U}^{1,I}_{C}$ stands for Impossible of codimension one from the set (C) and $k \in \{8, 9\}$, see Remark 2.5. These phase portraits are drawn in [6]. Anyway, we provide Table 2.6 in order to relate easily (giving also the page where they appear first and the page they are proved to be impossible).*

SSQVF [2]	$\#_\mathbf{p}$	$\#_\mathbf{r}$	**SU1 [6]**	**SSQVF [2]**	$\#_\mathbf{p}$	$\#_\mathbf{r}$	**SU1 [6]**
$\mathbb{S}^2_{4,1}$	1	1	$\mathbb{U}^1_{C,1}$	$\mathbb{S}^2_{10,16}$	1	1	$\mathbb{U}^1_{C,14}$
$\mathbb{S}^2_{5,1}$	2	2	$\mathbb{U}^1_{C,2}, \mathbb{U}^1_{C,3}$	$\mathbb{S}^2_{11,1}$	1	1	$\mathbb{U}^1_{C,15}$
$\mathbb{S}^2_{9,1}$	1	1	$\mathbb{U}^1_{C,4}$	$\mathbb{S}^2_{11,2}$	1	1	$\mathbb{U}^1_{C,16}$
$\mathbb{S}^2_{10,2}$	2	1	$\mathbb{U}^1_{C,5}$	$\mathbb{S}^2_{11,3}$	1	1	$\mathbb{U}^1_{C,17}$
$\mathbb{S}^2_{10,3}$	1	1	$\mathbb{U}^1_{C,6}$	$\mathbb{S}^2_{12,1}$	2	2	$\mathbb{U}^1_{C,18}, \mathbb{U}^1_{C,19}$
$\mathbb{S}^2_{10,5}$	1	1	$\mathbb{U}^1_{C,7}$	$\mathbb{S}^2_{12,2}$	2	2	$\mathbb{U}^1_{C,20}, \mathbb{U}^1_{C,21}$
$\mathbb{S}^2_{10,6}$	1	1	$\mathbb{U}^1_{C,8}$	$\mathbb{S}^2_{12,3}$	2	2	$\mathbb{U}^1_{C,22}, \mathbb{U}^1_{C,23}$
$\mathbb{S}^2_{10,9}$	2	1	$\mathbb{U}^1_{C,9}$	$\mathbb{S}^2_{12,4}$	2	2	$\mathbb{U}^1_{C,24}, \mathbb{U}^1_{C,25}$
$\mathbb{S}^2_{10,10}$	1	1	$\mathbb{U}^1_{C,10}$	$\mathbb{S}^2_{12,5}$	2	2	$\mathbb{U}^1_{C,26}, \mathbb{U}^1_{C,27}$
$\mathbb{S}^2_{10,12}$	1	1	$\mathbb{U}^1_{C,11}$	$\mathbb{S}^2_{12,6}$	3	3	$\mathbb{U}^1_{C,28}, \mathbb{U}^1_{C,29}, \mathbb{U}^1_{30}$
$\mathbb{S}^2_{10,14}$	1	1	$\mathbb{U}^1_{C,12}$	$\mathbb{S}^2_{12,7}$	2	2	$\mathbb{U}^1_{C,31}, \mathbb{U}^1_{C,32}$
$\mathbb{S}^2_{10,15}$	1	1	$\mathbb{U}^1_{C,13}$				

Table 2.5: Potential and realizable bifurcated phase portraits for a given structurally stable quadratic vector field. In this table, **SSQVF** stands for structurally stable quadratic vector fields, $\#_\mathbf{p}$ (respectively $\#_\mathbf{r}$) for the number of topologically potential (respectively realizable) phase portraits of *codimension one** bifurcated from the respective **SSQVF**, and **SU1** for the respective phase portraits of *codimension one**.

SSQVF [2]	**Page [6]**	**Impossible [6]**
$\mathbb{S}^2_{10,2}$	101	$\mathbb{U}^{1,I}_{C,8}$
$\mathbb{S}^2_{10,9}$	103	$\mathbb{U}^{1,I}_{C,9}$

Table 2.6: Non-realizable phase portraits from the set (C) which could bifurcate (if existed) from structurally stable quadratic vector fields. The first column indicates the structurally stable quadratic vector field (**SSQVF**) which suffers a bifurcation, the second column indicates the pages where they appear in [6] and the third column present the corresponding impossible phase portrait.



$\mathbb{U}^{1,I}_{C,8}$ $\mathbb{U}^{1,I}_{C,9}$

Figure 2.10: Phase portraits of the non-realizable structurally unstable quadratic vector fields of *codimension one** from the set (C).

An important result to study the impossibility of some phase portraits is Corollary 3.29 of [6].

Corollary 2.10. *If one of the structurally stable vector fields that bifurcates from a potential structurally unstable vector field of codimension one** *is not realizable, then this unstable system is also not realizable.*

Our aim is to prove the following result, which is the analogous of the previous corollary for the sets (AB) and (AC).

Theorem 2.11. *If one of the phase portraits of codimension one* that bifurcates from a potential codimension two* phase portrait from the sets (AB) and (AC) is not realizable, then this latter phase portrait is also not realizable.*

Proof. In what follows we prove the equivalent statement: If a potential *codimension two** phase portrait X from the sets (AB) and (AC) is realizable, then the phase portraits of *codimension one** that bifurcates from X are also realizable.

We start from the set (AB). We already know that a realizable phase portrait belongs to the set (AB) if and only if it has a finite saddle-node $\overline{sn}_{(2)}$ and an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ obtained by the coalescence of an infinite saddle with an infinite node. In [14] the authors classified the set of all real quadratic polynomial differential systems with a finite semi-elemental saddle-node $\overline{sn}_{(2)}$ located at the origin of the plane and an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ located in the bisector of first and third quadrants. Such a classification was done with respect to the normal form

$$\begin{aligned} \dot{x} &= gx^2 + 2hxy + (n-g-2h)y^2, \\ \dot{y} &= y + lx^2 + (2g+2h-2l-n)xy + (l-2g-2h+2n)y^2, \end{aligned} \tag{2.1}$$

where g, h, l, and n are real parameters. The parameter space of this normal form is a four-dimensional space, which can be projectivized, as it was done in [14] and the authors proved that all generic phenomena occur for $g = 1$. In the paper under discussion the authors used the *Invariant Theory* (developed in Sibirsky School – Moldova, see a very nice summary of this theory in Sec. 7 of [7]) in order to construct and study their bifurcation diagram. In Lemma 5.5 from the book [9] the authors proved that a necessary and sufficient condition for a generic quadratic system to possess an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ and another simple infinite singularity is that the comitants η and $\widetilde{M}$ verify the conditions

$$\eta = 0, \quad \widetilde{M} \neq 0,$$

for all the possible values of the parameters of the system. Additionally, in Table 5.1 from that book the authors present the invariant polynomials which are responsible for the number, kinds (real or/and complex), and multiplicities of finite singularities of a generic quadratic system. In particular, they show that if the invariant polynomial $\mathbb{D}$ verifies the condition

$$\mathbb{D} = 0,$$

then we have a finite singularity of multiplicity at least two. In fact, for systems (2.1) calculations show that these systems verify such conditions, since for that normal form (with $g = 1$) we obtain

$$\eta = 0, \quad \widetilde{M} = -8(1+2h+l-n)^2(x-y)^2 \neq 0, \quad \mathbb{D} = 0.$$

Now, for $g = 1$, consider the perturbation of systems (2.1)

$$\begin{aligned} \dot{x} &= (1-\varepsilon)x^2 + 2hxy + (n-1-2h)y^2, \\ \dot{y} &= y + l(1-\varepsilon)x^2 + ((2+2h-n)(1-\varepsilon)-2l)xy + (l-2-2h+2n)y^2, \end{aligned} \tag{2.2}$$

where $|\varepsilon|$ is small enough. For these systems, calculations show that

$$\eta = 4\varepsilon((1+2h+l-n)^2 - (-1-2h+n)^2\varepsilon)^2 \neq 0, \quad \mathbb{D} = 0.$$

So, according to Lemma 5.5 from the mentioned book, we have three distinct infinite singularities (all of them are real if $\varepsilon > 0$ and, if $\varepsilon < 0$, we have one real infinite singularity and two complex ones). Additionally, as $\mathbb{D} = 0$, perturbation (2.2) leaves unperturbed the finite saddle-node.

On the other hand, for $g = 1$ consider the perturbation of systems (2.1)

$$\begin{aligned} \dot{x} &= -\varepsilon + x^2 + 2hxy + (n - 1 - 2h)y^2, \\ \dot{y} &= -\varepsilon l + y + lx^2 + (2 + 2h - 2l - n)xy + (l - 2 - 2h + 2n)y^2, \end{aligned} \tag{2.3}$$

where $|\varepsilon|$ is small enough. For systems (2.3) we have

$$\eta = 0, \quad \widetilde{M} = -8(1 + 2h + l - n)^2(x - y)^2 \neq 0,$$

and

$$\mathbb{D} = -768\varepsilon(-1 + (2(1 + h)(-1 + l) + n)^2\varepsilon)^2(1 + 2h + h^2 - n + n^2((-1 + l)(1 + 2h + l) + n)\varepsilon).$$

According to Lemma 5.5 mentioned before, the perturbation (2.3) has not affected the infinite singular points and, according to Table 5.1 from the mentioned book, we no longer have finite multiple singularities, i.e. the perturbation splits the origin into two points (which are real or complex, depending on the sign of ε).

Therefore the result holds for the set (AB).

Now, consider the set (AC). A realizable phase portrait belongs to the set (AC) if and only if it has a finite saddle-node $\overline{sn}_{(2)}$ and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$, obtained by the coalescence of a finite saddle (respectively, finite node) with an infinite node (respectively, infinite saddle). Remember that, as we discussed in page 8, the case in which the finite saddle-node is the finite singularity that coalesces with an infinite singularity will be considered in the future during the study of the set (CC). With the *Invariant Theory* as the main tool, in [10] we classified the set of all real quadratic polynomial differential systems with a finite semi-elemental saddle-node $\overline{sn}_{(2)}$ located at the origin of the plane and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$. Such a classification was done with respect to the normal form

$$\begin{aligned} \dot{x} &= cx + cy - cx^2 + 2hxy, \\ \dot{y} &= ex + ey - ex^2 + 2mxy, \end{aligned} \tag{2.4}$$

where c, h, e, and m are real parameters, with the (nondegeneracy) condition $eh \neq cm$. The parameter space of this normal form is a four-dimensional space, which can be projectivized, as it was done in that paper where we proved that all generic phenomena occur for $h = 1$. In Lemma 5.2 from the book [9] the authors proved that a necessary and sufficient condition for a generic quadratic system to possess an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ is that the comitants μ_0 and μ_1 verify the conditions

$$\mu_0 = 0, \quad \mu_1 \neq 0,$$

for all the possible values of the parameters of the system. Additionally, as in the previous case, from Table 5.1 it is possible to conclude that if the invariant polynomial $\mathbb{D}$ verifies the condition

$$\mathbb{D} = 0,$$

then we have a finite singularity of multiplicity at least two. Indeed, for systems (2.4) with $h = 1$ calculations show that such conditions are fulfilled, since

$$\mu_0 = 0, \quad \mu_1 = -8(e - cm)^2 x \neq 0, \quad \mathbb{D} = 0.$$

Now, for $h = 1$, consider the perturbation of systems (2.4)

$$\begin{aligned} \dot{x} &= cx + cy - cx^2 + 2xy + \varepsilon y^2, \\ \dot{y} &= ex + ey - ex^2 + 2mxy + \varepsilon y^2, \end{aligned} \tag{2.5}$$

where $|\varepsilon|$ is small enough, calculations show that for systems (2.5) the comitant μ_0 is given by

$$\mu_0 = \varepsilon(-4(1-m)(e-cm) + (c-e)^2\varepsilon).$$

So the perturbation under consideration splits the infinite saddle-node $\overline{\binom{1}{1}}SN$. Additionally, we conclude that the perturbation maintains the finite saddle-node, since for systems (2.5) calculations show that the invariant polynomial $\mathbb{D}$ vanishes.
Finally, for $h = 1$ (as we did for the set (AB)), consider the perturbation of systems (2.4)

$$\begin{aligned} \dot{x} &= -\varepsilon + cx + cy - cx^2 + 2xy, \\ \dot{y} &= -\varepsilon e + ex + ey - ex^2 + 2mxy, \end{aligned} \tag{2.6}$$

where $|\varepsilon|$ is small enough. For systems (2.6) we have

$$\mu_0 = 0, \quad \mu_1 = -4(e - cm)^2 x \neq 0,$$

and

$$\begin{aligned} \mathbb{D} =& 768\varepsilon(e-cm)^3\left(16\varepsilon^2(e-m)^3 - 8(c-1)e(e-cm)^2\right) \\ &+ 768\varepsilon^2(e-cm)^4\left((9c(3c-2)-13)e^2 + 4(11-9c)em - 4m^2\right). \end{aligned}$$

According to the results (from the book [9]) presented before, we conclude that systems (2.6) have the infinite saddle-node $\overline{\binom{1}{1}}SN$ and do not have the finite saddle-node $\overline{sn}_{(2)}$, i.e. the perturbation (2.6) of systems (2.4) keeps the infinite saddle-node and splits the finite saddle-node.
Then the theorem also holds for the set (AC), as we wanted to prove. ☐

As at the moment we are not interested in giving a proof for a general case of the previous theorem, in what follows we present a conjecture.

Conjecture 2.12. *If one of the phase portraits of codimension k that bifurcates from a potential codimension $k+1$ phase portrait is not realizable, then this latter phase portrait is also not realizable.*

Remark 2.13. In Qualitative Theory of Ordinary Differential Equations is quite common to use the term "perturbation" to denote an infinitesimal modification of the parameters of a system such that a different phase portrait bifurcates from it. In this paper we use the term "evolution" in order to say that we "move a *codimension one** phase portrait to its border and detect which phase portraits are in the other side of this border", so with an evolution of a *codimension one** phase portrait we produce a *codimension two** phase portrait. In this sense we mean that we modify (in a continuous way) the first system inside the region of parameters in which it is defined up to the other side of the border of this region where we obtain a system having one codimension more. In a certain way, with this modification we are provoking an "evolution" of the first system. Note that we *contrast* "perturbation" with "evolution".

3 Proof of Theorem 1.6

In this section we present the proof of Theorem 1.6. More precisely, in Subsection 3.1 we obtain all the topologically potential phase portraits belonging to the set (AB) (we have 110 topologically distinct phase portraits) and we prove that 39 of them are impossible. In Subsection 3.2 we show the realization of each one of the remaining 71 phase portraits.

3.1 The topologically potential phase portraits

The main goal of this subsection is to obtain all the topologically potential phase portraits from the set (AB).

We already know that in the set (AB), the unstable objects of *codimension two** belong to the set of saddle-nodes $\{\overline{sn}_{(2)}+\overline{\binom{0}{2}}SN\}$. Considering all the different ways of obtaining phase portraits belonging to the set (AB) of *codimension two**, we have to consider all the possible ways of coalescing specific singular points in both sets (A) and (B). However, as the sets (AB) and (BA) are the same (i.e. their elements are obtained independently of the order of the evolution in the elements of the sets (A) or (B)), it is necessary to consider only all the possible ways of obtaining an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ in each element from the set (A) (phase portraits possessing a finite saddle-node $\overline{sn}_{(2)}$). Anyway, in order to make things clear, in page 54 we discuss briefly how should we perform if we start by considering the set (B).

In order to obtain phase portraits from the set (AB) by starting our study from the set (A), we have to consider Theorem 2.7 and also Lemma 3.25 from [6] (regarding phase portraits from the set (B)) which we state as follows.

Lemma 3.1. *Suppose that a polynomial vector field X of codimension one* has an infinite saddle-node p of multiplicity two with $\rho_0 = (\partial P/\partial x + \partial Q/\partial y)_p \neq 0$ and first eigenvalue equal to zero.*

(a) Any perturbation of X in a sufficiently small neighborhood of this point will produce a structurally stable system (with one infinite saddle and one infinite node, or with no singular points in the neighborhood) or a system topologically equivalent to X.

(b) Both possibilities of structurally stable system (with one saddle and one node at infinity, or with no singular points in the neighborhood) are realizable.

Here we consider all the 69 realizable structurally unstable quadratic vector fields of *codimension one** from the set (A). In order to obtain a phase portrait of *codimension two** belonging to the set (AB) starting from a phase portrait of *codimension one** of the set (A), we keep the existing finite saddle-node and using Lemma 3.1 we build an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ by the coalescence of an infinite saddle with an infinite node. On the other hand, from the phase portraits of *codimension two** from the set (AB), one can obtain phase portraits of *codimension one** belonging to the set (A) after perturbation of the infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node, or into complex singularities.

In what follows we denote by $\mathbb{U}^2_{AB,k}$, where $\mathbb{U}^2_{AB}$ stands for structurally unstable quadratic vector field of *codimension two** from the set (AB) and $k \in \{1,\ldots,71\}$ (note that the notation $\mathbb{U}^2_{AB}$ is simpler than $\mathbb{U}^2_{(AB)}$). The impossible phase portraits will be denoted by $\mathbb{U}^{2,I}_{AB,j}$, where $\mathbb{U}^{2,I}_{AB}$ stands for *Impossible of codimension two* from the set (AB)* and $j \in \mathbb{N}$. We need to enumerate also the impossible phase portraits, not for the completeness of this paper, but for the future papers in which someone will study *codimension three** families. Just in the same way as impossible *codimension one** phase portraits are a crucial tool for the study of our families.

Note that phase portraits $\mathbb{U}^1_{A,1}$ to $\mathbb{U}^1_{A,13}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ as an evolution, since each one of them has only one infinite singularity. Analogously, phase portraits $\mathbb{U}^1_{A,14}$ to $\mathbb{U}^1_{A,18}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ as an evolution, since each one of them has three infinite singularities (which are nodes).

Phase portrait $\mathbb{U}^1_{A,19}$ has phase portraits $\mathbb{U}^2_{AB,1}$ and $\mathbb{U}^2_{AB,2}$ as evolution (see Figure 3.1, where the arrows starting from the phase portrait $\mathbb{U}^1_{A,19}$ and pointing towards the phase portraits $\mathbb{U}^2_{AB,1}$ and $\mathbb{U}^2_{AB,2}$ indicate that these last two phase portraits are evolution of the phase portrait $\mathbb{U}^1_{A,19}$). After bifurcation we get phase portrait $\mathbb{U}^1_{A,1}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear (split into two complex singularities). In Figure 3.1 we present the corresponding unfoldings on the right-hand side of the *codimension two** phase portraits.

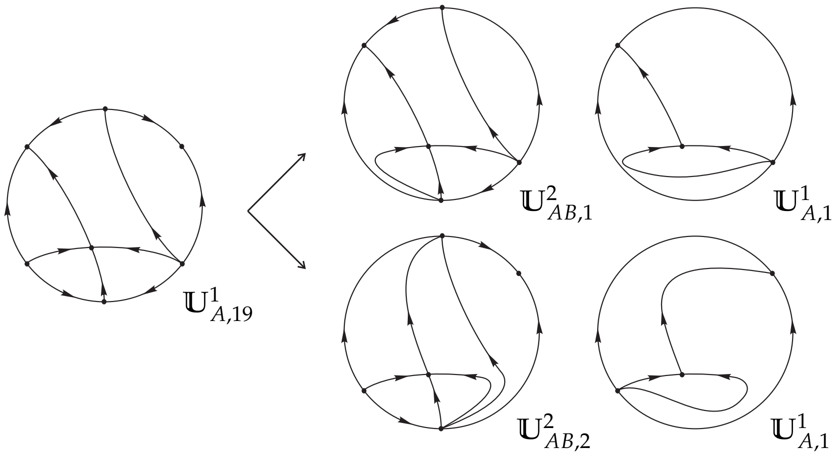


Figure 3.1: Unstable systems $\mathbb{U}^2_{AB,1}$ and $\mathbb{U}^2_{AB,2}$.

Note that $\mathbb{U}^1_{A,19}$ possesses two pairs of infinite nodes and only one pair of infinite saddles, so from $\mathbb{U}^1_{A,19}$ there are only two ways of obtaining a phase portrait possessing an infinite saddle-node of type $\overline{\binom{0}{2}}SN$, and these cases are represented exactly by the phase portraits $\mathbb{U}^2_{AB,1}$ and $\mathbb{U}^2_{AB,2}$ from Figure 3.1. From now on, we will always omit the proof of the nonexistence of other cases apart from those ones that we discuss by words or by presenting in figures, since the argument of nonexistence is in general quite simple.

Before we continue with the study of the remaining *codimension one** phase portraits, we highlight that it is very important to have the "structure" of all the figures very well understood, since the proofs of Theorems 1.6 and 1.7 require and are done based on several figures. So, in this paragraph we discuss about it. In the next cases, when from a *codimension one** phase portrait we have more than one *codimension two** phase portraits which are evolution of the *codimension one** phase portrait, we will present figures with the same "structure" of Figure 3.1. More precisely, all the arrows that appear starting from an unstable phase portrait of *codimension one** will have the same meaning as explained for Figure 3.1, i.e., they will point towards the phase portraits of *codimension two** which are evolution of the respective *codimension one** phase portrait. Moreover, we will present the corresponding unfoldings on the right-hand side of the *codimension two** phase portraits. On the other hand, when from a *codimension one** phase portrait we have only one *codimension two** phase portrait which is an evolution of the *codimension one** phase portrait, we will present figures like Figure 3.7, for instance, where on the left-hand side we have a *codimension one** phase portrait, on the center we have the corresponding *codimension two** phase portrait and on the right-hand side we have the respective unfolding of the *codimension two** phase portrait.

Phase portrait $\mathbb{U}^1_{A,20}$ has phase portraits $\mathbb{U}^2_{AB,3}$ and $\mathbb{U}^2_{AB,4}$ as evolution (see Figure 3.2). After bifurcation we get phase portrait $\mathbb{U}^1_{A,1}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

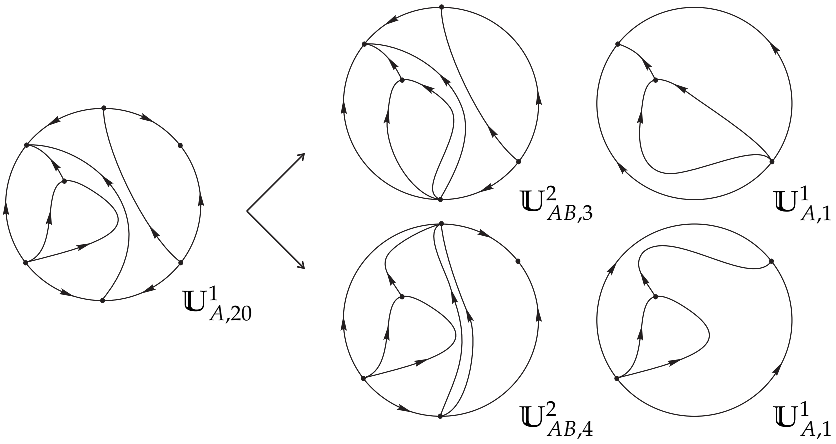


Figure 3.2: Unstable systems $\mathbb{U}^2_{AB,3}$ and $\mathbb{U}^2_{AB,4}$.

Phase portrait $\mathbb{U}^1_{A,21}$ has phase portraits $\mathbb{U}^2_{AB,5}$ and $\mathbb{U}^2_{AB,6}$ as evolution (see Figure 3.3). After bifurcation we get phase portrait $\mathbb{U}^1_{A,1}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

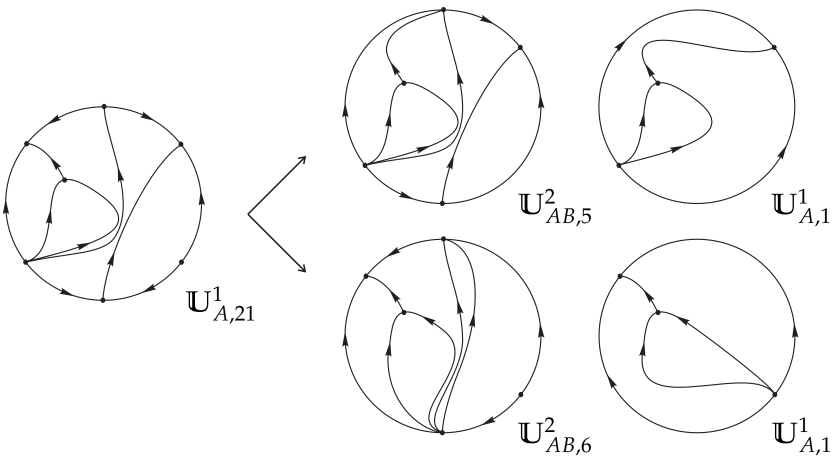


Figure 3.3: Unstable systems $\mathbb{U}^2_{AB,5}$ and $\mathbb{U}^2_{AB,6}$.

Phase portrait $\mathbb{U}^1_{A,22}$ has phase portraits $\mathbb{U}^2_{AB,7}$ and $\mathbb{U}^2_{AB,8}$ as evolution (see Figure 3.4). After bifurcation we get phase portrait $\mathbb{U}^1_{A,2}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,23}$ has phase portraits $\mathbb{U}^2_{AB,9}$ and $\mathbb{U}^2_{AB,10}$ as evolution (see Figure 3.5). After bifurcation we get phase portrait $\mathbb{U}^1_{A,3}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

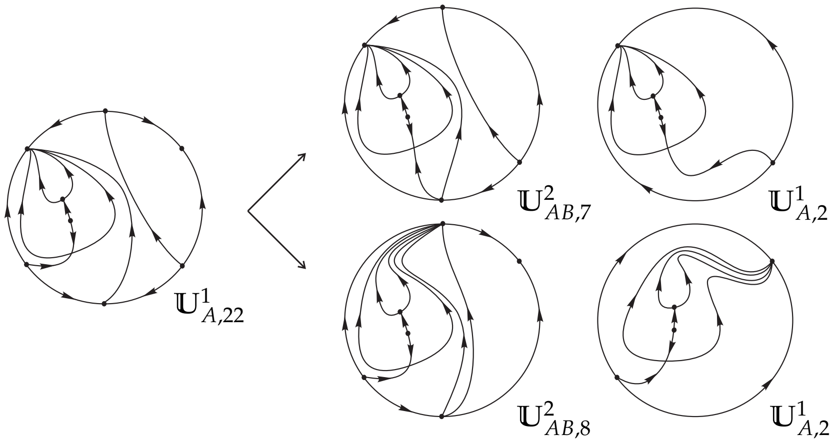


Figure 3.4: Unstable systems $\mathbb{U}^2_{AB,7}$ and $\mathbb{U}^2_{AB,8}$.

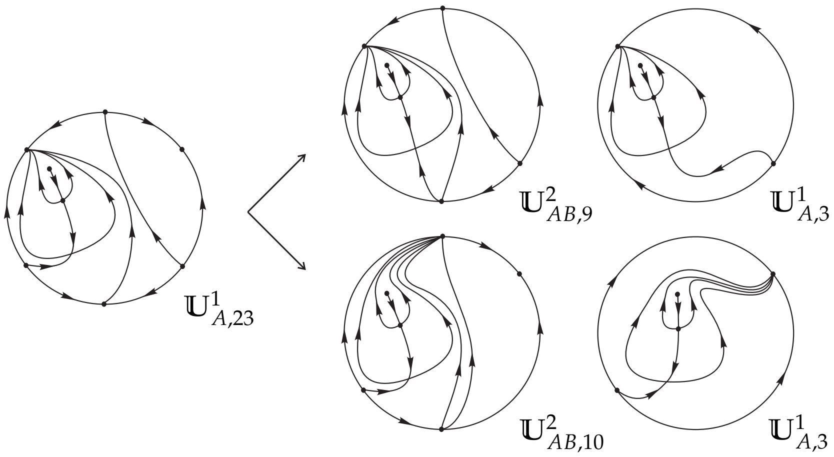


Figure 3.5: Unstable systems $\mathbb{U}^2_{AB,9}$ and $\mathbb{U}^2_{AB,10}$.

Phase portrait $\mathbb{U}^1_{A,24}$ has phase portraits $\mathbb{U}^2_{AB,11}$ and $\mathbb{U}^2_{AB,12}$ as evolution (see Figure 3.6). After bifurcation we get phase portrait $\mathbb{U}^1_{A,4}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

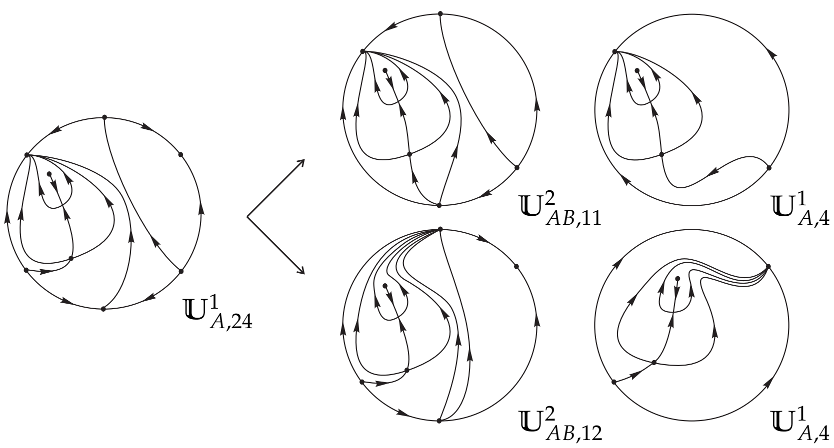


Figure 3.6: Unstable systems $\mathbb{U}^2_{AB,11}$ and $\mathbb{U}^2_{AB,12}$.

Phase portrait $\mathbb{U}^1_{A,25}$ has phase portrait $\mathbb{U}^2_{AB,13}$ as an evolution (see Figure 3.7). After

bifurcation we get phase portrait $\mathbb{U}^1_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,25}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,1}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,107}$ of *codimension one**, see Figure 3.8. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,1}$ unfolds in $\mathbb{U}^1_{A,5}$.

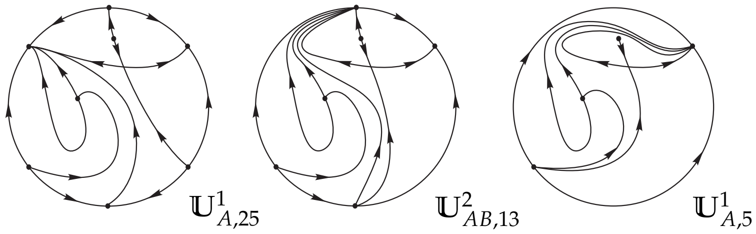


Figure 3.7: Unstable system $\mathbb{U}^2_{AB,13}$.

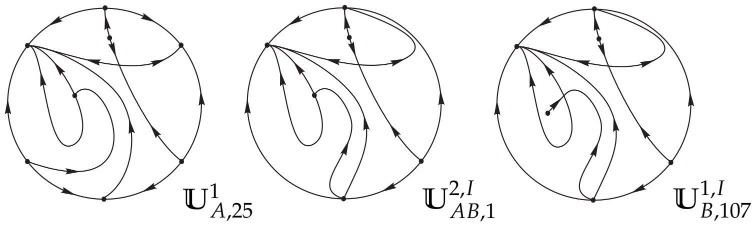


Figure 3.8: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,1}$.

Phase portrait $\mathbb{U}^1_{A,26}$ has phase portrait $\mathbb{U}^2_{AB,14}$ as an evolution (see Figure 3.9). After bifurcation we get phase portrait $\mathbb{U}^1_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,26}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,2}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,107}$ of *codimension one**, see Figure 3.10. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,2}$ unfolds in $\mathbb{U}^1_{A,5}$.

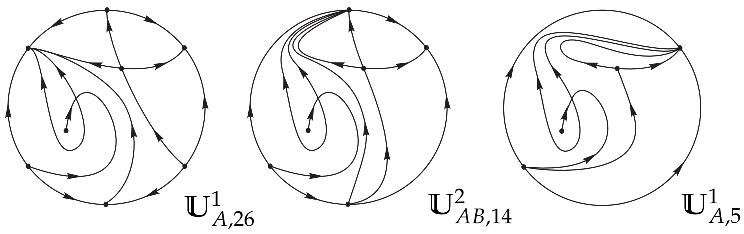


Figure 3.9: Unstable system $\mathbb{U}^2_{AB,14}$.

Phase portrait $\mathbb{U}^1_{A,27}$ has phase portrait $\mathbb{U}^2_{AB,15}$ as an evolution (see Figure 3.11). After bifurcation we get phase portrait $\mathbb{U}^1_{A,2}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,27}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,3}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,108}$ of *codimension one**, see Figure 3.12. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,3}$ unfolds in $\mathbb{U}^1_{A,2}$.

Phase portrait $\mathbb{U}^1_{A,28}$ has phase portrait $\mathbb{U}^2_{AB,16}$ as an evolution (see Figure 3.13). After

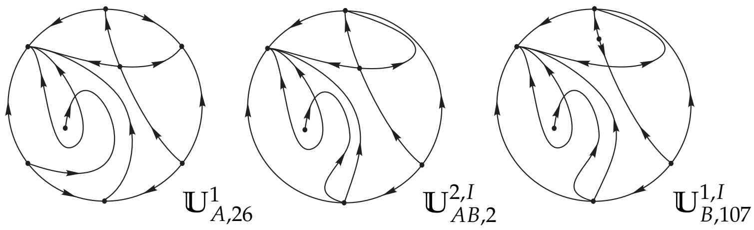


Figure 3.10: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,2}$.

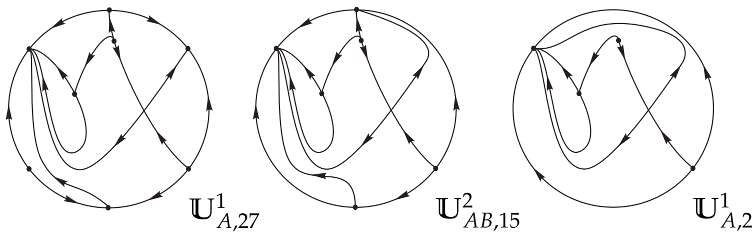


Figure 3.11: Unstable system $\mathbb{U}^{2}_{AB,15}$.

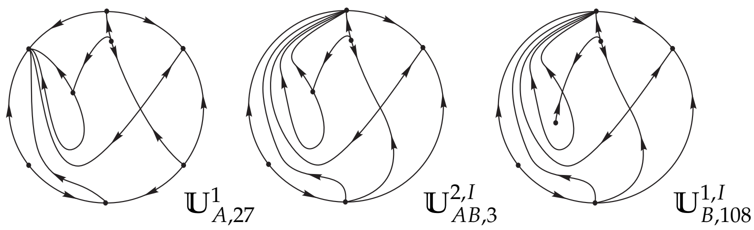


Figure 3.12: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,3}$.

bifurcation we get phase portrait $\mathbb{U}^{1}_{A,3}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^{1}_{A,28}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,4}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,108}$ of *codimension one**, see Figure 3.14. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,4}$ unfolds in $\mathbb{U}^{1}_{A,3}$.

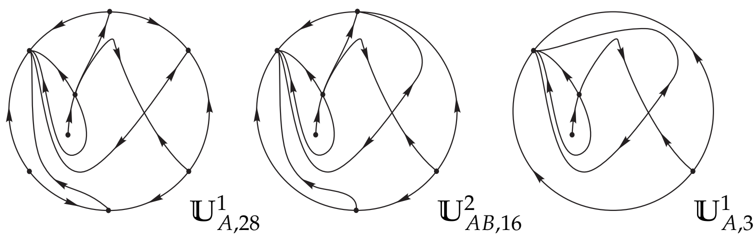


Figure 3.13: Unstable system $\mathbb{U}^{2}_{AB,16}$.

Phase portrait $\mathbb{U}^{1}_{A,29}$ has phase portrait $\mathbb{U}^{2}_{AB,17}$ as an evolution (see Figure 3.15). After bifurcation we get phase portrait $\mathbb{U}^{1}_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^{1}_{A,29}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,5}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,109}$ of *codimension one**, see Figure 3.16. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,5}$ unfolds in $\mathbb{U}^{1}_{A,5}$.

Phase portrait $\mathbb{U}^{1}_{A,30}$ has phase portrait $\mathbb{U}^{2}_{AB,18}$ as an evolution (see Figure 3.17). After

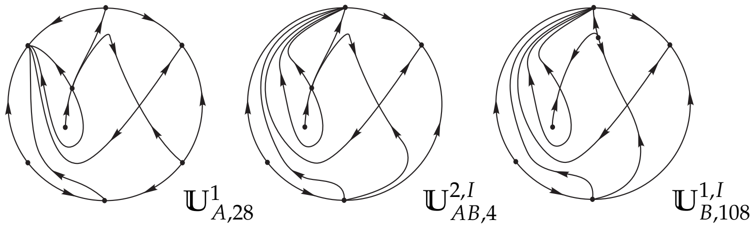


Figure 3.14: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,4}$.

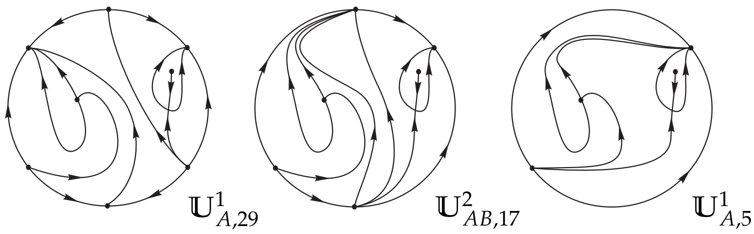


Figure 3.15: Unstable system $\mathbb{U}^{2}_{AB,17}$.

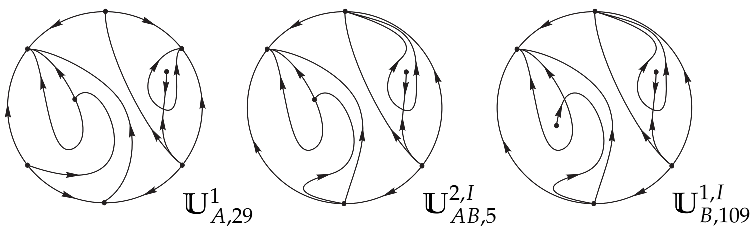


Figure 3.16: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,5}$.

bifurcation we get phase portrait $\mathbb{U}^{1}_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^{1}_{A,30}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,6}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,109}$ of *codimension one**, see Figure 3.18. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,6}$ unfolds in $\mathbb{U}^{1}_{A,5}$.

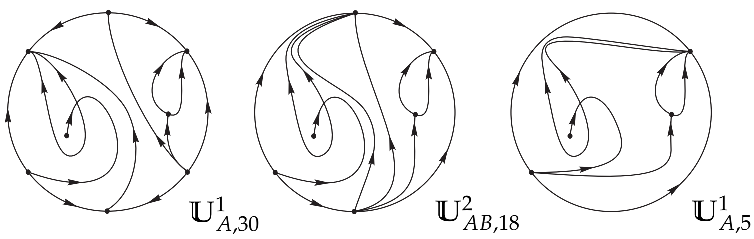


Figure 3.17: Unstable system $\mathbb{U}^{2}_{AB,18}$.

Phase portrait $\mathbb{U}^{1}_{A,31}$ has phase portrait $\mathbb{U}^{2}_{AB,19}$ (see Figure 3.19) as an evolution. After bifurcation we get phase portrait $\mathbb{U}^{1}_{A,2}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^{1}_{A,31}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,7}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,110}$ of *codimension one**, see Figure 3.20. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,7}$ unfolds in $\mathbb{U}^{1}_{A,2}$.

Phase portrait $\mathbb{U}^{1}_{A,32}$ has phase portrait $\mathbb{U}^{2}_{AB,20}$ as an evolution (see Figure 3.21). After

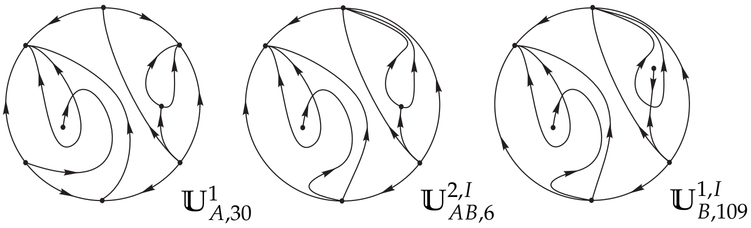


Figure 3.18: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,6}$.

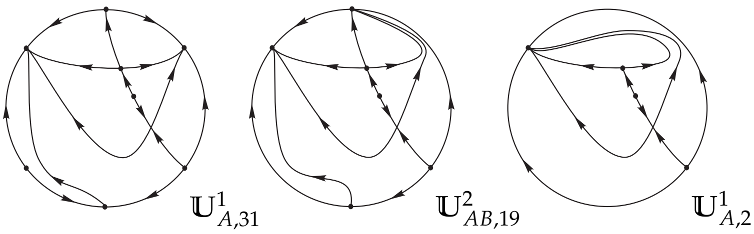


Figure 3.19: Unstable system $\mathbb{U}^2_{AB,19}$.

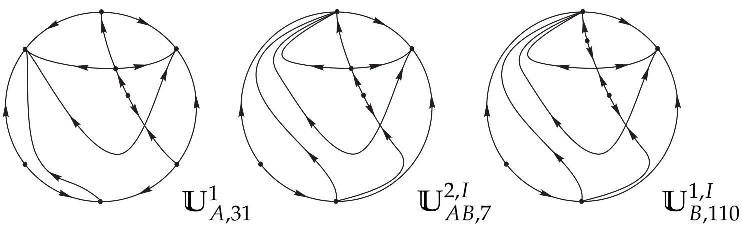


Figure 3.20: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,7}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,3}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,32}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,8}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,110}$ of *codimension one**, see Figure 3.22. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,8}$ unfolds in $\mathbb{U}^1_{A,3}$.

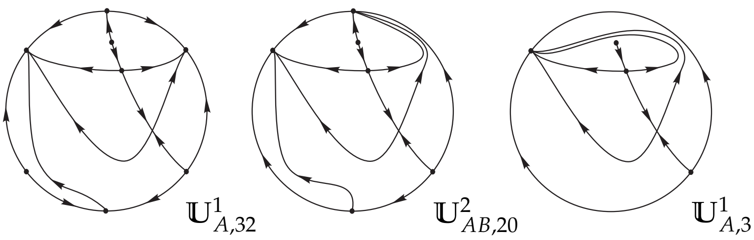


Figure 3.21: Unstable system $\mathbb{U}^2_{AB,20}$.

Phase portrait $\mathbb{U}^1_{A,33}$ has phase portrait $\mathbb{U}^2_{AB,21}$ as an evolution (see Figure 3.23). After bifurcation we get phase portrait $\mathbb{U}^1_{A,4}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,33}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,9}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,110}$ of *codimension one**, see Figure 3.24. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,9}$ unfolds in $\mathbb{U}^1_{A,4}$.

Phase portrait $\mathbb{U}^1_{A,34}$ has phase portrait $\mathbb{U}^2_{AB,22}$ as an evolution (see Figure 3.25). After

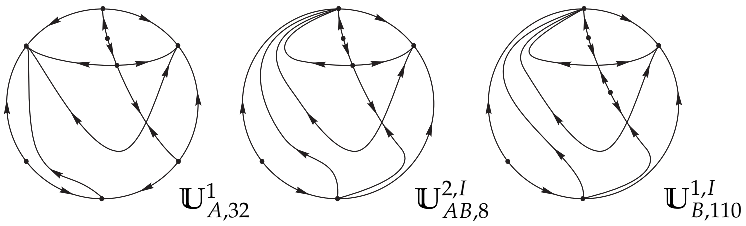


Figure 3.22: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,8}$.

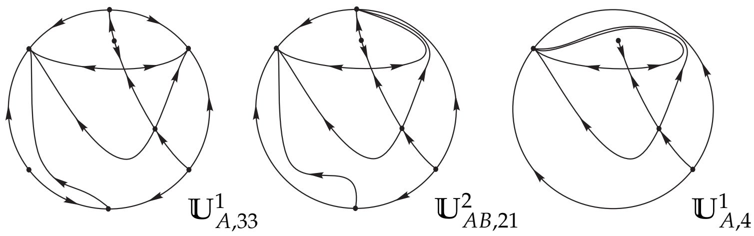


Figure 3.23: Unstable system $\mathbb{U}^2_{AB,21}$.

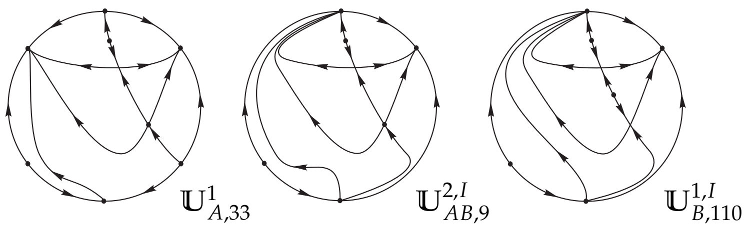


Figure 3.24: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,9}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,34}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,10}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,111}$ of *codimension one**, see Figure 3.26. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,10}$ unfolds in $\mathbb{U}^1_{A,5}$.

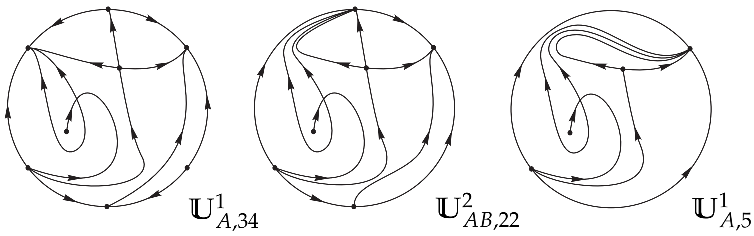


Figure 3.25: Unstable system $\mathbb{U}^2_{AB,22}$.

Phase portrait $\mathbb{U}^1_{A,35}$ has phase portrait $\mathbb{U}^2_{AB,23}$ as an evolution (see Figure 3.27). After bifurcation we get phase portrait $\mathbb{U}^1_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,35}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,11}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,111}$ of *codimension one**, see Figure 3.28. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,11}$ unfolds in $\mathbb{U}^1_{A,5}$.

Phase portrait $\mathbb{U}^1_{A,36}$ has phase portraits $\mathbb{U}^2_{AB,24}$ and $\mathbb{U}^2_{AB,25}$ as evolution (see Figure 3.29).

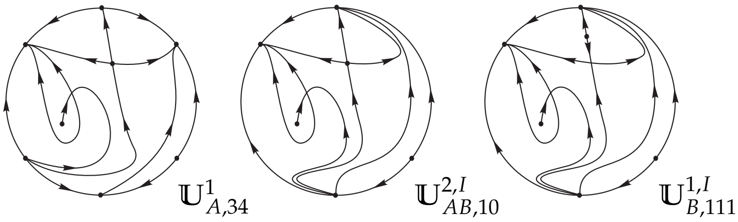


Figure 3.26: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,10}$.

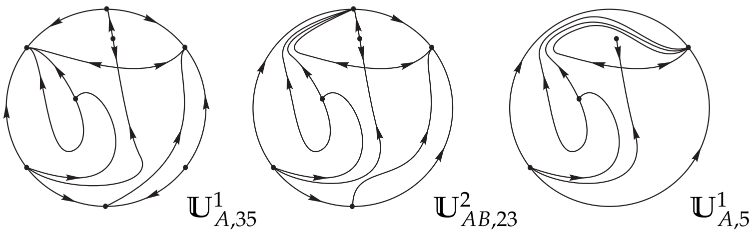


Figure 3.27: Unstable system $\mathbb{U}^2_{AB,23}$.

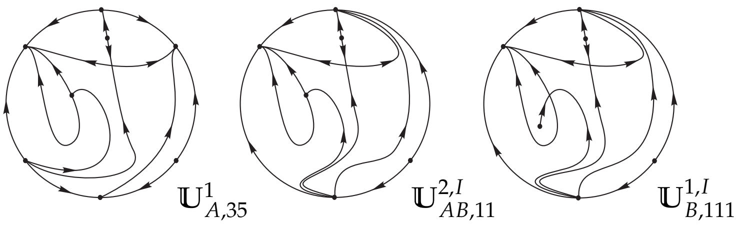


Figure 3.28: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,11}$.

After bifurcation we get phase portrait $\mathbb{U}^1_{A,9}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

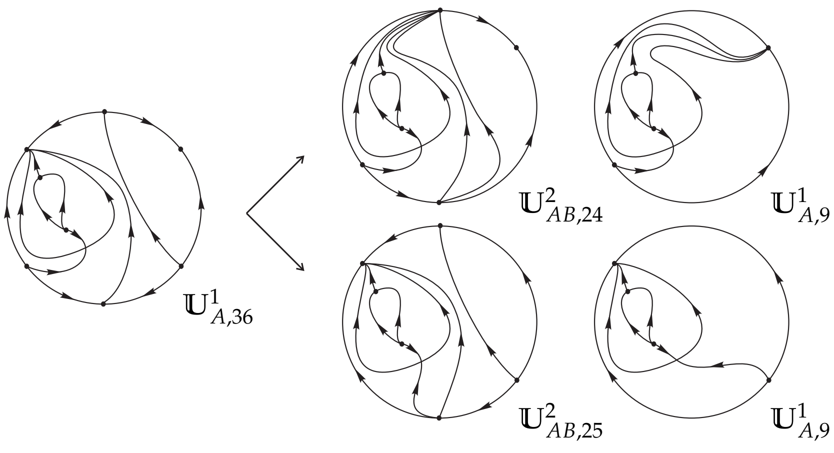


Figure 3.29: Unstable systems $\mathbb{U}^2_{AB,24}$ and $\mathbb{U}^2_{AB,25}$.

Phase portrait $\mathbb{U}^1_{A,37}$ has phase portraits $\mathbb{U}^2_{AB,26}$ and $\mathbb{U}^2_{AB,27}$ as evolution (see Figure 3.30). After bifurcation we get phase portrait $\mathbb{U}^1_{A,10}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,38}$ has phase portraits $\mathbb{U}^2_{AB,28}$ and $\mathbb{U}^2_{AB,29}$ as evolution (see Figure 3.31). After bifurcation we get phase portrait $\mathbb{U}^1_{A,8}$, in both cases, by making the infinite saddle-node

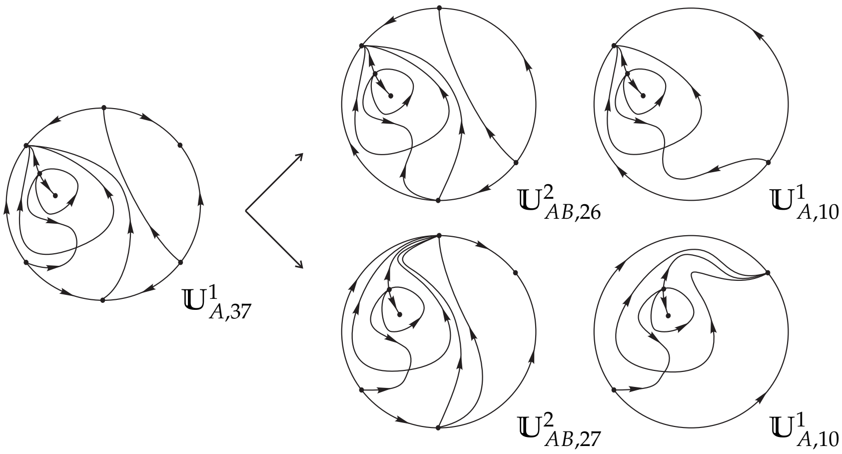


Figure 3.30: Unstable systems $\mathbb{U}^2_{AB,26}$ and $\mathbb{U}^2_{AB,27}$.

$\overline{\binom{0}{2}}SN$ disappear.

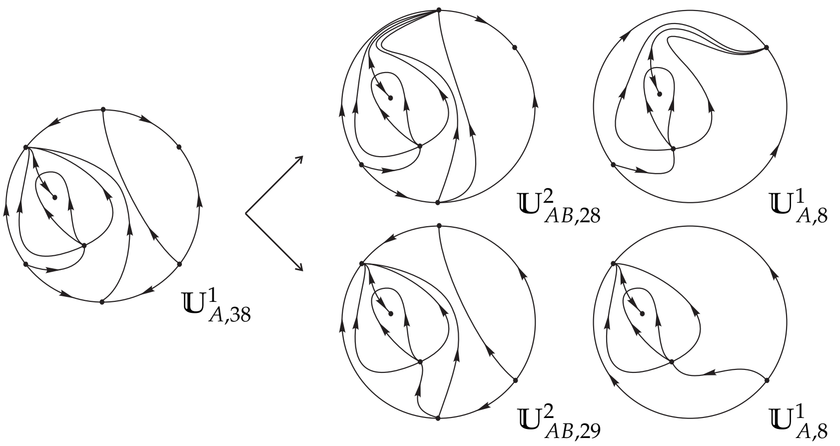


Figure 3.31: Unstable systems $\mathbb{U}^2_{AB,28}$ and $\mathbb{U}^2_{AB,29}$.

Phase portrait $\mathbb{U}^1_{A,39}$ has phase portrait $\mathbb{U}^2_{AB,30}$ as an evolution (see Figure 3.32). After bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,39}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,12}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,112}$ of *codimension one**, see Figure 3.33. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,12}$ unfolds in $\mathbb{U}^1_{A,6}$.

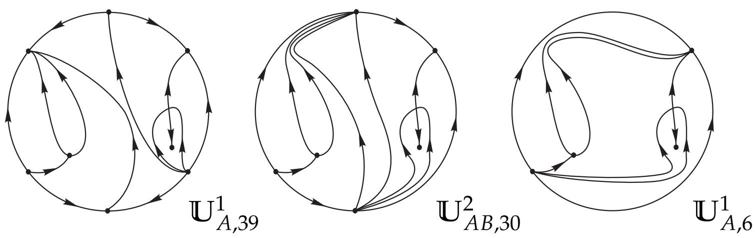


Figure 3.32: Unstable system $\mathbb{U}^2_{AB,30}$.

Phase portrait $\mathbb{U}^1_{A,40}$ has phase portrait $\mathbb{U}^2_{AB,31}$ as an evolution (see Figure 3.34). After

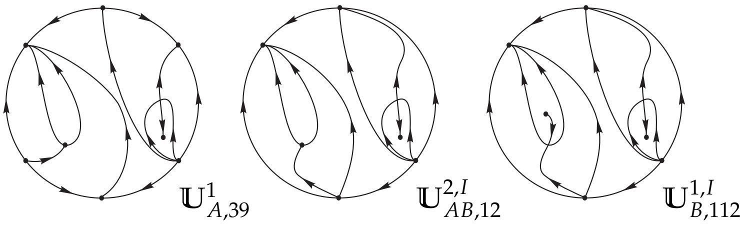


Figure 3.33: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,12}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,40}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,13}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,113}$ of *codimension one**, see Figure 3.35. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,13}$ unfolds in $\mathbb{U}^1_{A,6}$.

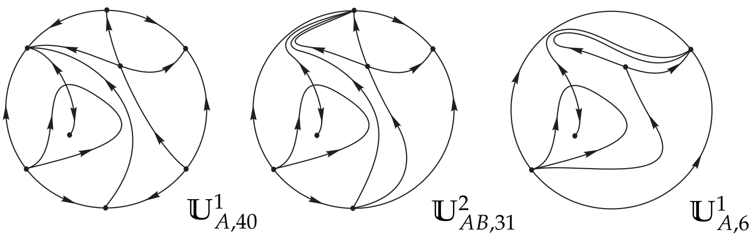


Figure 3.34: Unstable system $\mathbb{U}^2_{AB,31}$.

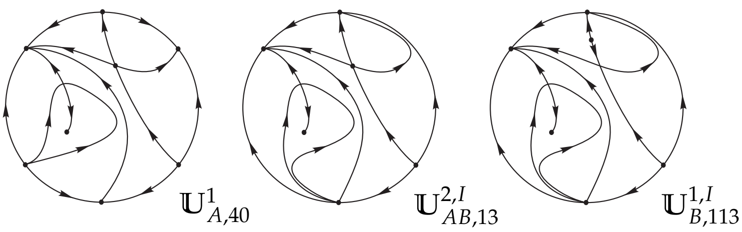


Figure 3.35: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,13}$.

Phase portrait $\mathbb{U}^1_{A,41}$ has phase portrait $\mathbb{U}^2_{AB,32}$ as an evolution (see Figure 3.36). After bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,41}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,14}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,113}$ of *codimension one**, see Figure 3.37. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,14}$ unfolds in $\mathbb{U}^1_{A,6}$.

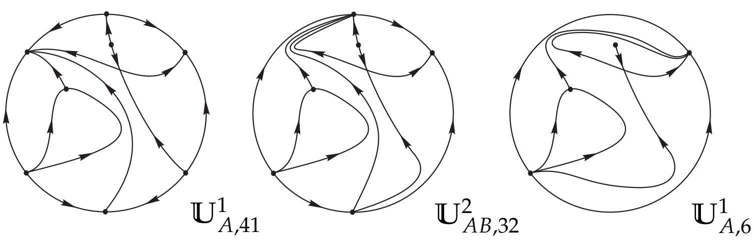


Figure 3.36: Unstable system $\mathbb{U}^2_{AB,32}$.

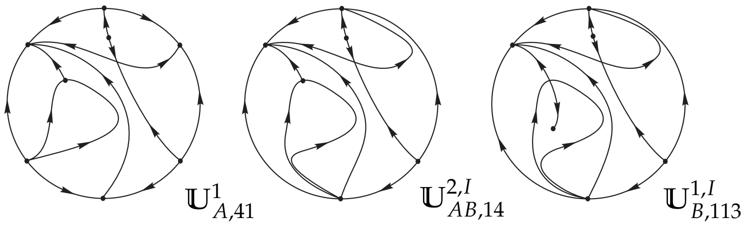


Figure 3.37: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,14}$.

Phase portrait $\mathbb{U}^1_{A,42}$ has phase portrait $\mathbb{U}^2_{AB,33}$ as an evolution (see Figure 3.38). After bifurcation we get phase portrait $\mathbb{U}^1_{A,9}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,42}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,15}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,114}$ of *codimension one**, see Figure 3.39. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,15}$ unfolds in $\mathbb{U}^1_{A,9}$.

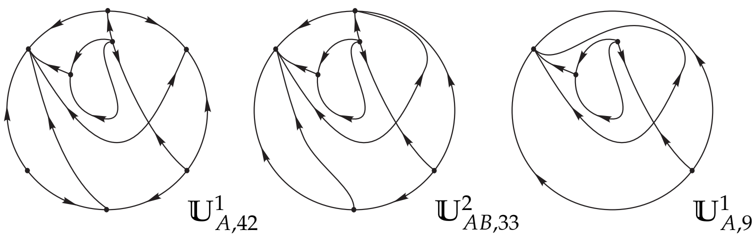


Figure 3.38: Unstable system $\mathbb{U}^2_{AB,33}$.

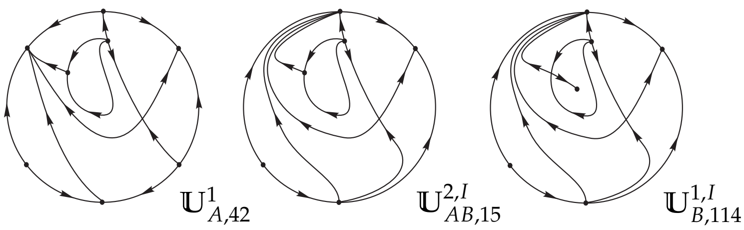


Figure 3.39: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,15}$.

Phase portrait $\mathbb{U}^1_{A,43}$ has phase portrait $\mathbb{U}^2_{AB,34}$ as an evolution (see Figure 3.40). After bifurcation we get phase portrait $\mathbb{U}^1_{A,10}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,43}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,16}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,114}$ of *codimension one**, see Figure 3.41. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,16}$ unfolds in $\mathbb{U}^1_{A,10}$.

Phase portrait $\mathbb{U}^1_{A,44}$ has phase portrait $\mathbb{U}^2_{AB,35}$ as an evolution (see Figure 3.42). After bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,44}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,17}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,115}$ of *codimension one**, see Figure 3.43. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,17}$ unfolds in $\mathbb{U}^1_{A,6}$.

Phase portrait $\mathbb{U}^1_{A,45}$ has phase portrait $\mathbb{U}^2_{AB,36}$ as an evolution (see Figure 3.44). After

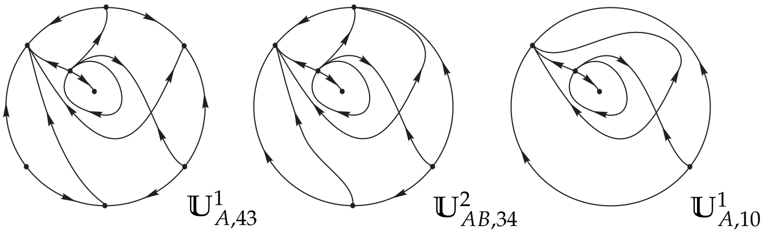


Figure 3.40: Unstable system $\mathbb{U}^2_{AB,34}$.

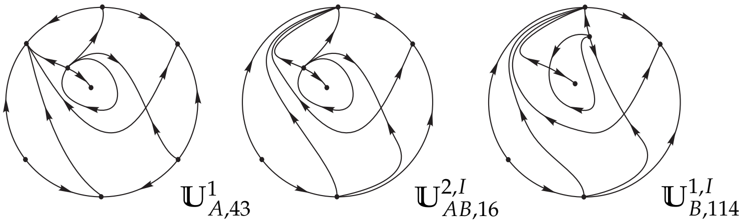


Figure 3.41: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,16}$.

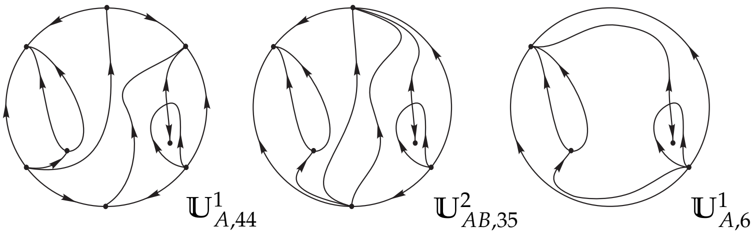


Figure 3.42: Unstable system $\mathbb{U}^2_{AB,35}$.

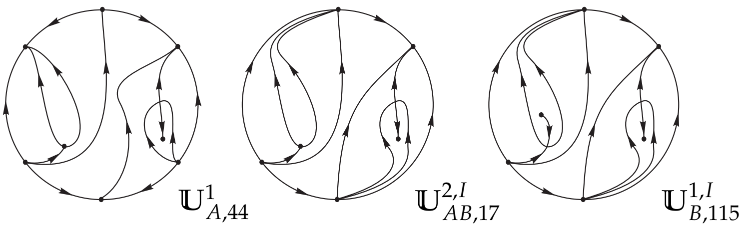


Figure 3.43: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,17}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,45}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,18}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,116}$ of *codimension one**, see Figure 3.45. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,18}$ unfolds in $\mathbb{U}^1_{A,6}$.

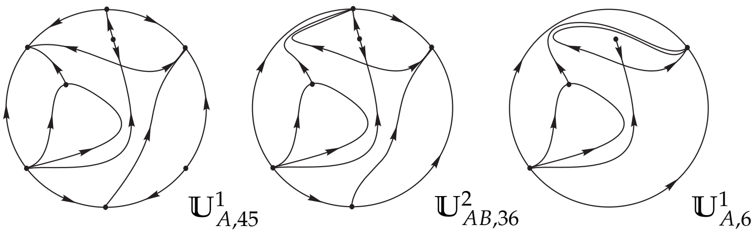
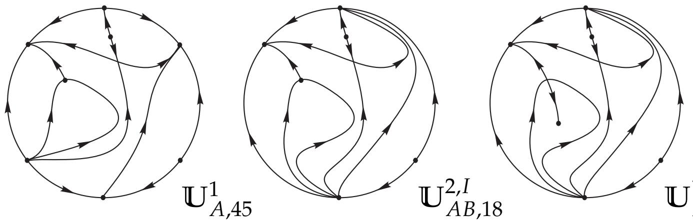


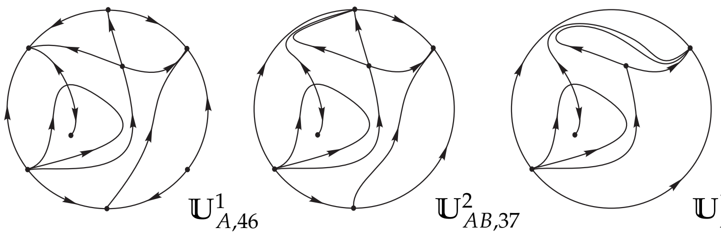
Figure 3.44: Unstable system $\mathbb{U}^2_{AB,36}$.



$\mathbb{U}^1_{A,45}$ $\mathbb{U}^{2,I}_{AB,18}$ $\mathbb{U}^{1,I}_{B,116}$

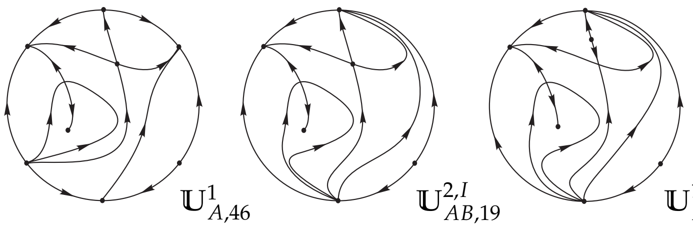
Figure 3.45: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,18}$.

Phase portrait $\mathbb{U}^1_{A,46}$ has phase portrait $\mathbb{U}^2_{AB,37}$ as an evolution (see Figure 3.46). After bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,46}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,19}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,116}$ of *codimension one**, see Figure 3.47. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,19}$ unfolds in $\mathbb{U}^1_{A,6}$.



$\mathbb{U}^1_{A,46}$ $\mathbb{U}^2_{AB,37}$ $\mathbb{U}^1_{A,6}$

Figure 3.46: Unstable system $\mathbb{U}^2_{AB,37}$.



$\mathbb{U}^1_{A,46}$ $\mathbb{U}^{2,I}_{AB,19}$ $\mathbb{U}^{1,I}_{B,116}$

Figure 3.47: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,19}$.

Phase portrait $\mathbb{U}^1_{A,47}$ has phase portraits $\mathbb{U}^2_{AB,38}$ and $\mathbb{U}^2_{AB,39}$ as evolution (see Figure 3.48). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,48}$ has phase portraits $\mathbb{U}^2_{AB,40}$ and $\mathbb{U}^2_{AB,41}$ as evolution (see Figure 3.49). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,50}$ has phase portraits $\mathbb{U}^2_{AB,42}$ and $\mathbb{U}^2_{AB,43}$ as evolution (see Figure 3.50). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,51}$ has phase portraits $\mathbb{U}^2_{AB,44}$ and $\mathbb{U}^2_{AB,45}$ as evolution (see Figure 3.51). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,52}$ has phase portraits $\mathbb{U}^2_{AB,46}$ and $\mathbb{U}^2_{AB,47}$ as evolution (see Figure 3.52). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node

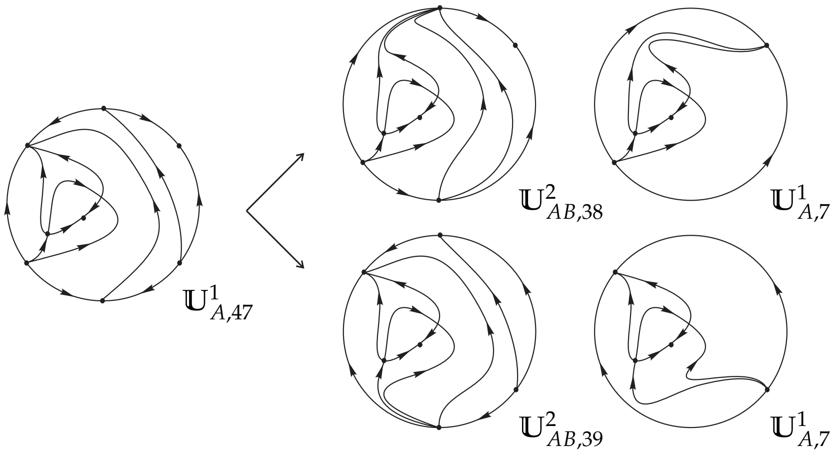


Figure 3.48: Unstable systems $\mathbb{U}^2_{AB,38}$ and $\mathbb{U}^2_{AB,39}$.

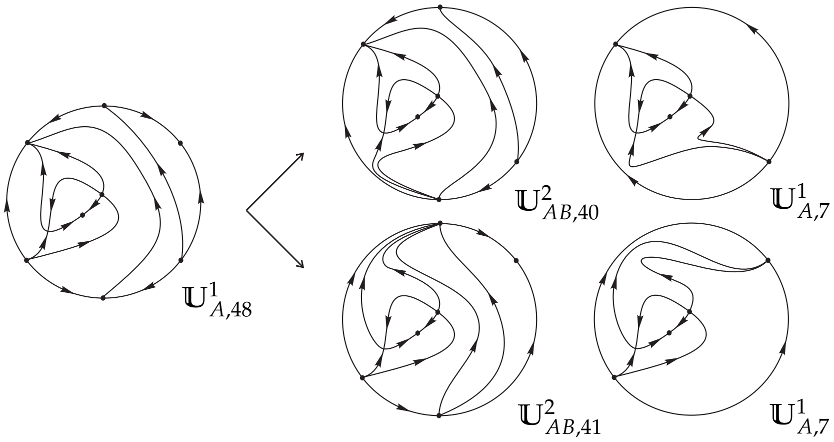


Figure 3.49: Unstable systems $\mathbb{U}^2_{AB,40}$ and $\mathbb{U}^2_{AB,41}$.

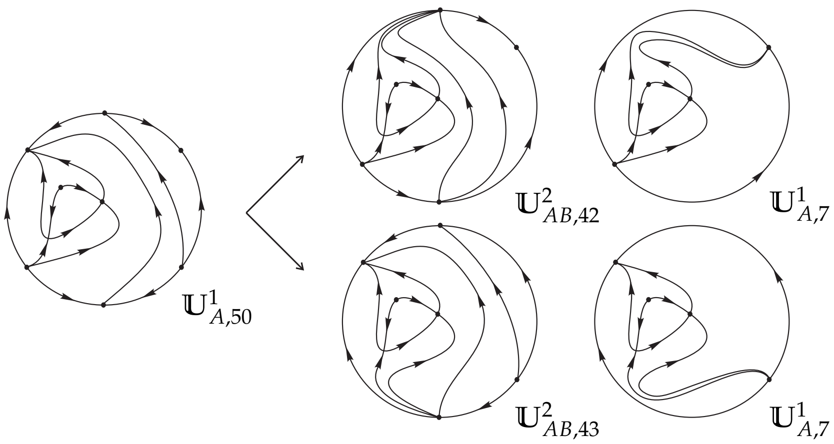


Figure 3.50: Unstable systems $\mathbb{U}^2_{AB,42}$ and $\mathbb{U}^2_{AB,43}$.

$\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,53}$ has phase portraits $\mathbb{U}^2_{AB,48}$ and $\mathbb{U}^2_{AB,49}$ as evolution (see Figure 3.53). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,54}$ has phase portrait $\mathbb{U}^2_{AB,50}$ as an evolution (see Figure 3.54). After

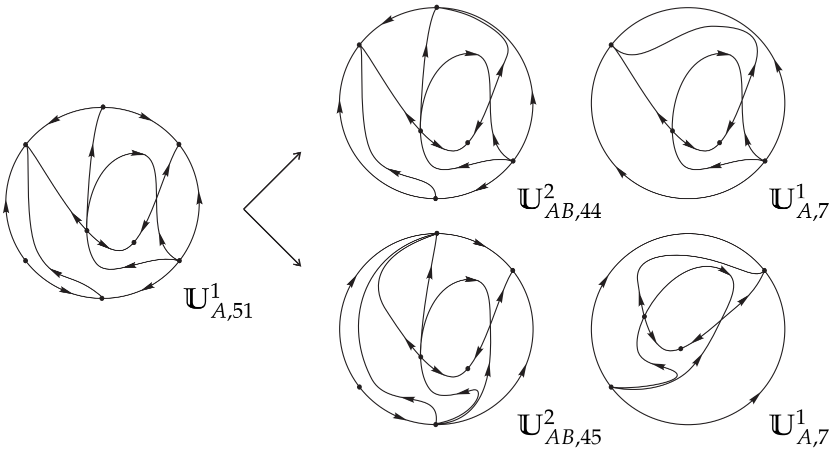


Figure 3.51: Unstable systems $\mathbb{U}^2_{AB,44}$ and $\mathbb{U}^2_{AB,45}$.

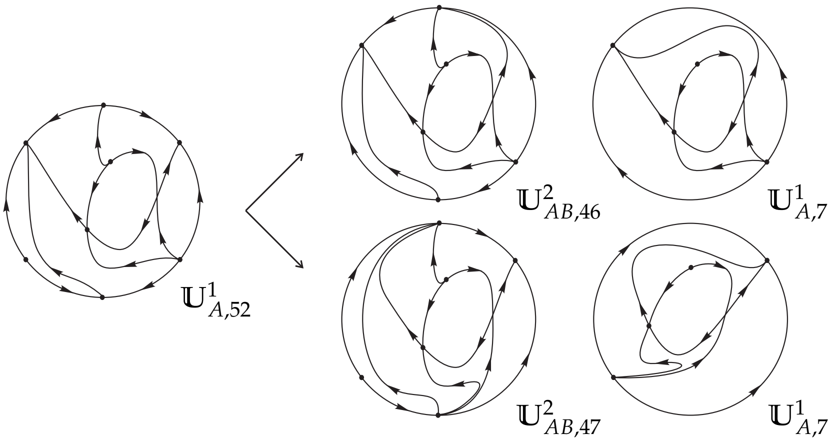


Figure 3.52: Unstable systems $\mathbb{U}^2_{AB,46}$ and $\mathbb{U}^2_{AB,47}$.

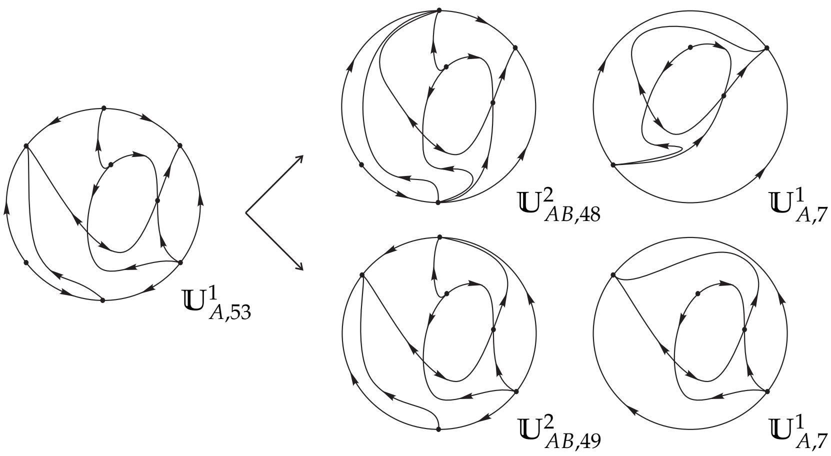


Figure 3.53: Unstable systems $\mathbb{U}^2_{AB,48}$ and $\mathbb{U}^2_{AB,49}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,54}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,20}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,117}$ of *codimension one**, see

Figure 3.55. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,20}$ unfolds in $\mathbb{U}^{1}_{A,6}$.

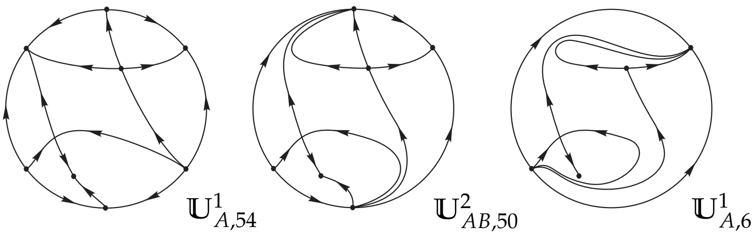


Figure 3.54: Unstable system $\mathbb{U}^{2}_{AB,50}$.

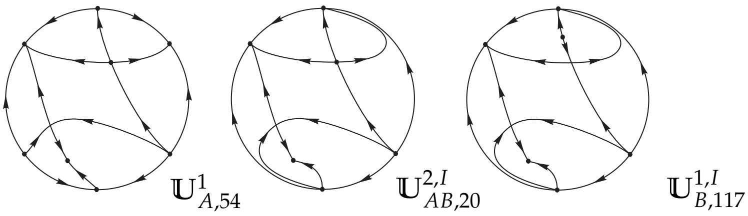


Figure 3.55: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,20}$.

Phase portrait $\mathbb{U}^{1}_{A,55}$ has phase portraits $\mathbb{U}^{2}_{AB,51}$ and $\mathbb{U}^{2}_{AB,52}$ as evolution (see Figure 3.56). After bifurcation we get phase portrait $\mathbb{U}^{1}_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

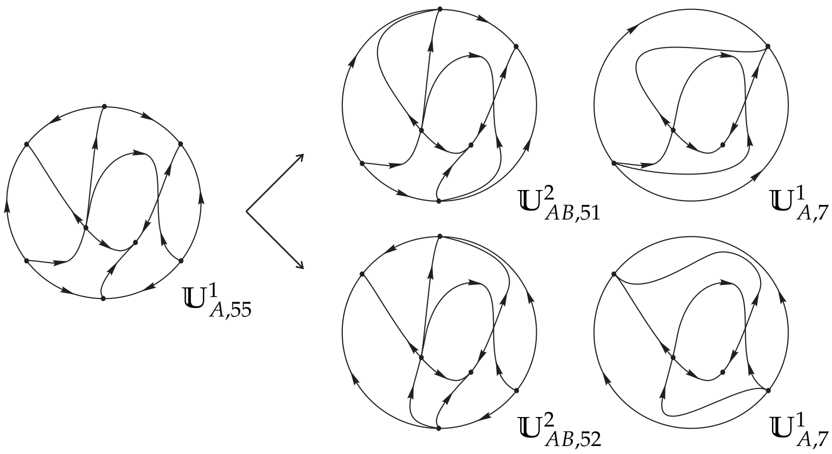


Figure 3.56: Unstable systems $\mathbb{U}^{2}_{AB,51}$ and $\mathbb{U}^{2}_{AB,52}$.

Phase portrait $\mathbb{U}^{1}_{A,56}$ has phase portrait $\mathbb{U}^{2}_{AB,53}$ as an evolution (see Figure 3.57). After bifurcation we get phase portrait $\mathbb{U}^{1}_{A,11}$, modulo limit cycle, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^{1}_{A,56}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,21}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,4}$ of *codimension one**, see Figure 3.58. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,21}$ also unfolds in an impossible phase portrait because after bifurcation we would get a limit cycle surrounding more than one finite singular points, and this is not possible in quadratic systems (see Lemma 3.14 from [6]).

Phase portrait $\mathbb{U}^{1}_{A,57}$ has phase portrait $\mathbb{U}^{2}_{AB,54}$ as an evolution (see Figure 3.59). After bifurcation we get phase portrait $\mathbb{U}^{1}_{A,12}$, modulo limit cycle, by making the infinite saddle-node

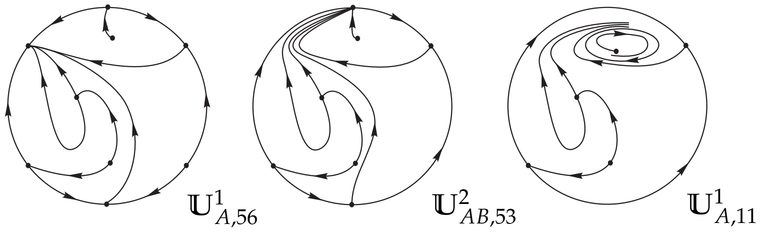


Figure 3.57: Unstable system $\mathbb{U}^2_{AB,53}$.

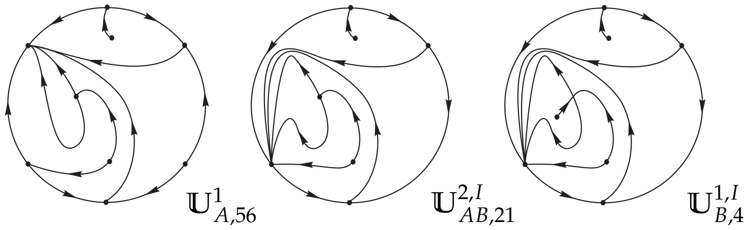


Figure 3.58: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,21}$.

$\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,57}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,22}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,4}$ of *codimension one**, see Figure 3.60. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,22}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

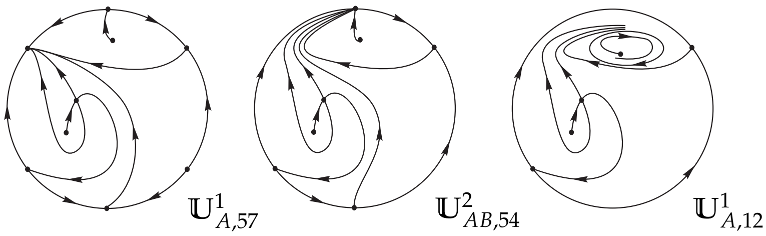


Figure 3.59: Unstable system $\mathbb{U}^2_{AB,54}$.

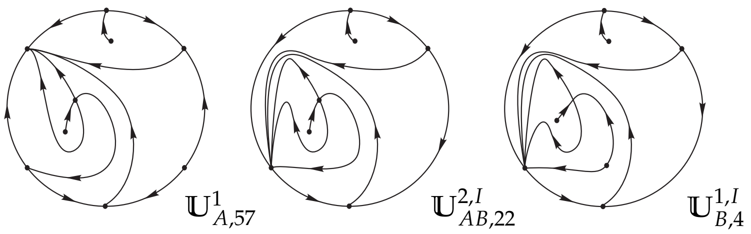


Figure 3.60: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,22}$.

Phase portrait $\mathbb{U}^1_{A,58}$ has phase portrait $\mathbb{U}^2_{AB,55}$ as an evolution (see Figure 3.61). After bifurcation we get phase portrait $\mathbb{U}^1_{A,12}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,58}$ has a second phase portrait which is not presented since it is topologically equivalent to $\mathbb{U}^2_{AB,55}$.

Phase portrait $\mathbb{U}^1_{A,59}$ has phase portraits $\mathbb{U}^2_{AB,56}$ and $\mathbb{U}^2_{AB,57}$ as evolution (see Figure 3.62). After bifurcation we get phase portrait $\mathbb{U}^1_{A,13}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

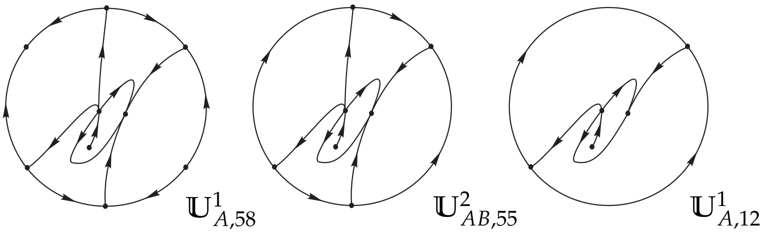


Figure 3.61: Unstable system $\mathbb{U}^2_{AB,55}$.

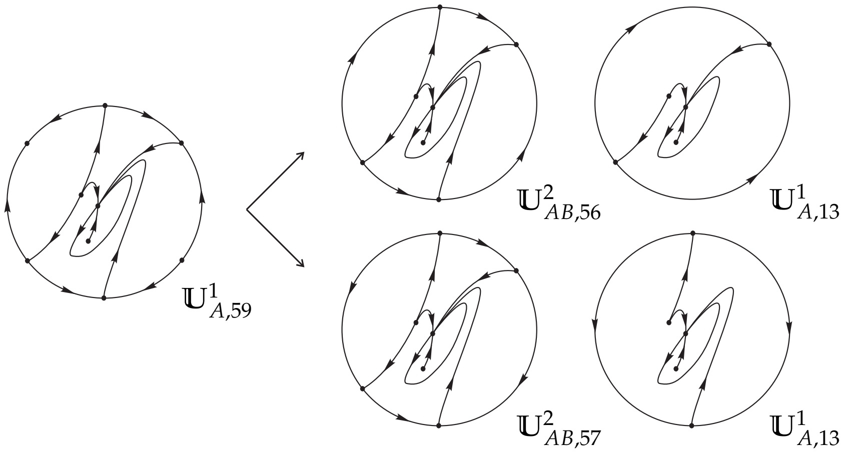


Figure 3.62: Unstable systems $\mathbb{U}^2_{AB,56}$ and $\mathbb{U}^2_{AB,57}$.

Phase portrait $\mathbb{U}^1_{A,60}$ has phase portrait $\mathbb{U}^2_{AB,58}$ as an evolution (see Figure 3.63). After bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,60}$ has a second phase portrait which is not presented since it is topologically equivalent to $\mathbb{U}^2_{AB,58}$.

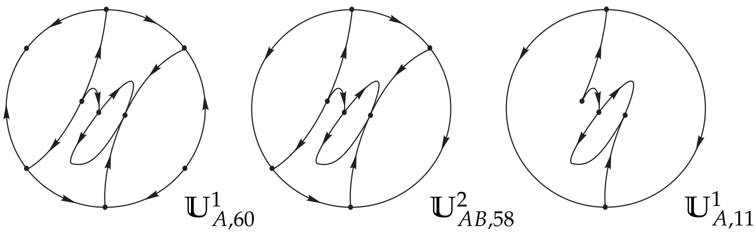


Figure 3.63: Unstable system $\mathbb{U}^2_{AB,58}$.

Phase portrait $\mathbb{U}^1_{A,61}$ has phase portraits $\mathbb{U}^2_{AB,59}$ and $\mathbb{U}^2_{AB,60}$ as evolution (see Figure 3.64). After bifurcation we get phase portraits $\mathbb{U}^1_{A,11}$ and $\mathbb{U}^1_{A,12}$, respectively, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,62}$ has phase portrait $\mathbb{U}^2_{AB,61}$ as an evolution (see Figure 3.65). After bifurcation we get phase portrait $\mathbb{U}^1_{A,13}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,62}$ has a second phase portrait which is not presented since it is topologically equivalent to $\mathbb{U}^2_{AB,61}$.

Phase portrait $\mathbb{U}^1_{A,63}$ has phase portrait $\mathbb{U}^2_{AB,62}$ as an evolution (see Figure 3.66). After bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$, modulo limit cycle, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,63}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,23}$ as an evolution. By

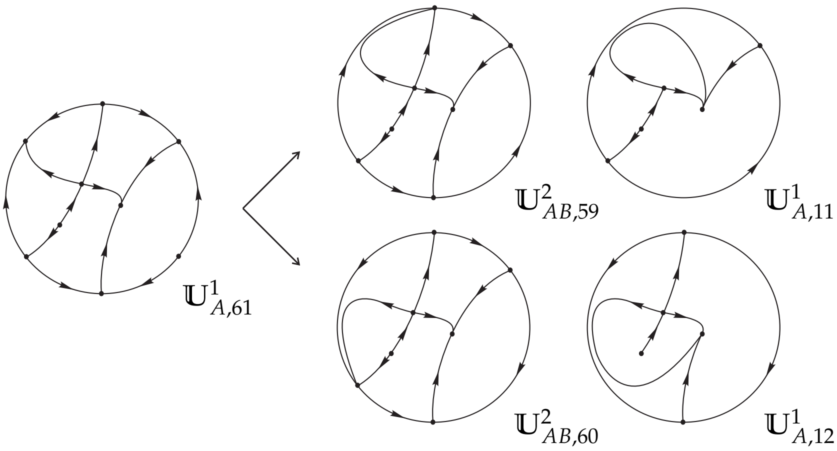


Figure 3.64: Unstable systems $\mathbb{U}^2_{AB,59}$ and $\mathbb{U}^2_{AB,60}$.

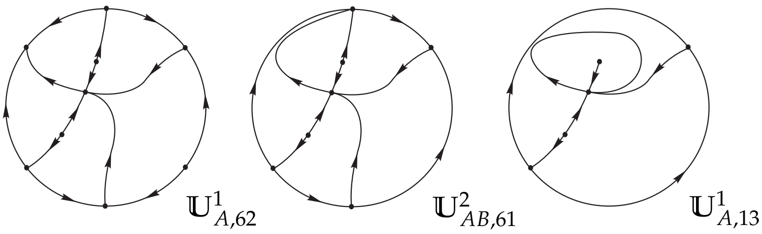


Figure 3.65: Unstable system $\mathbb{U}^2_{AB,61}$.

Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,5}$ of *codimension one**, see Figure 3.67. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,23}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

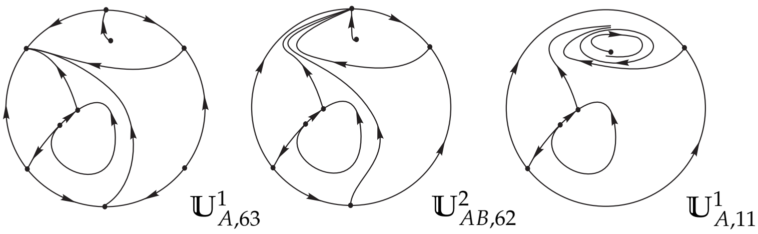


Figure 3.66: Unstable system $\mathbb{U}^2_{AB,62}$.

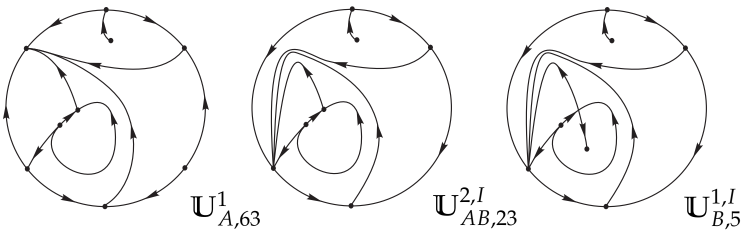


Figure 3.67: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,23}$.

Phase portrait $\mathbb{U}^1_{A,64}$ has phase portrait $\mathbb{U}^2_{AB,63}$ as an evolution (see Figure 3.68). After bifurcation we get phase portrait $\mathbb{U}^1_{A,13}$, modulo limit cycle, by making the infinite saddle-node

$\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,64}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,24}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,5}$ of *codimension one**, see Figure 3.69. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,24}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

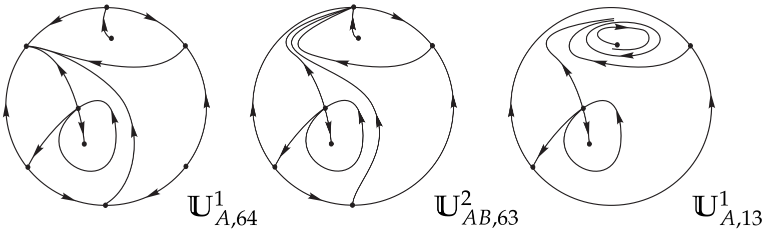


Figure 3.68: Unstable system $\mathbb{U}^2_{AB,63}$.

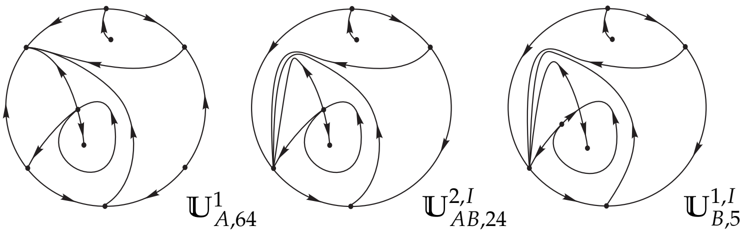


Figure 3.69: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,24}$.

Phase portrait $\mathbb{U}^1_{A,65}$ has phase portrait $\mathbb{U}^2_{AB,64}$ as an evolution (see Figure 3.70). After bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,65}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,25}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,6}$ of *codimension one**, see Figure 3.71. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,25}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

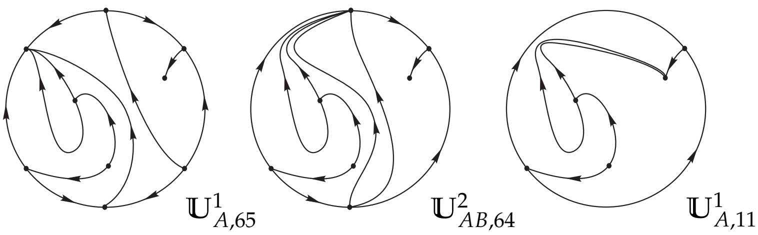


Figure 3.70: Unstable system $\mathbb{U}^2_{AB,64}$.

Phase portrait $\mathbb{U}^1_{A,66}$ has phase portrait $\mathbb{U}^2_{AB,65}$ as an evolution (see Figure 3.72). After bifurcation we get phase portrait $\mathbb{U}^1_{A,12}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,66}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,26}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,6}$ of *codimension one**, see Figure 3.73. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,26}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

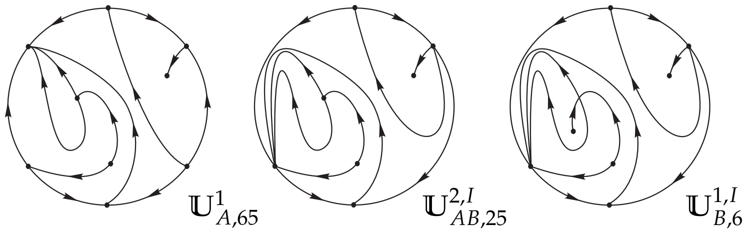


Figure 3.71: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,25}$.

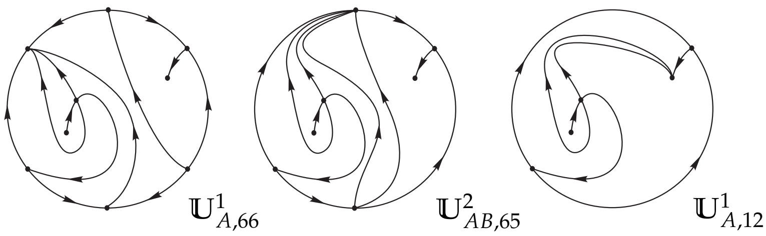


Figure 3.72: Unstable system $\mathbb{U}^2_{AB,65}$.

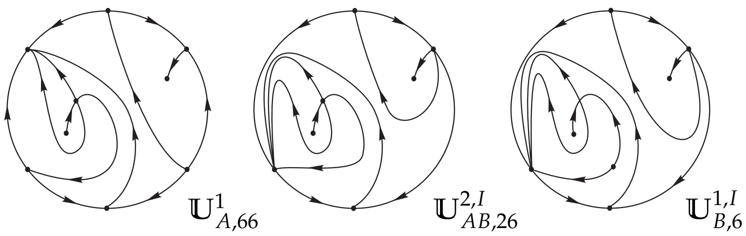


Figure 3.73: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,26}$.

Phase portrait $\mathbb{U}^1_{A,67}$ has phase portraits $\mathbb{U}^2_{AB,66}$ and $\mathbb{U}^2_{AB,67}$ as evolution (see Figure 3.74). After bifurcation we get phase portraits $\mathbb{U}^1_{A,11}$ and $\mathbb{U}^1_{A,13}$ (being this last one modulo limit cycles), respectively, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

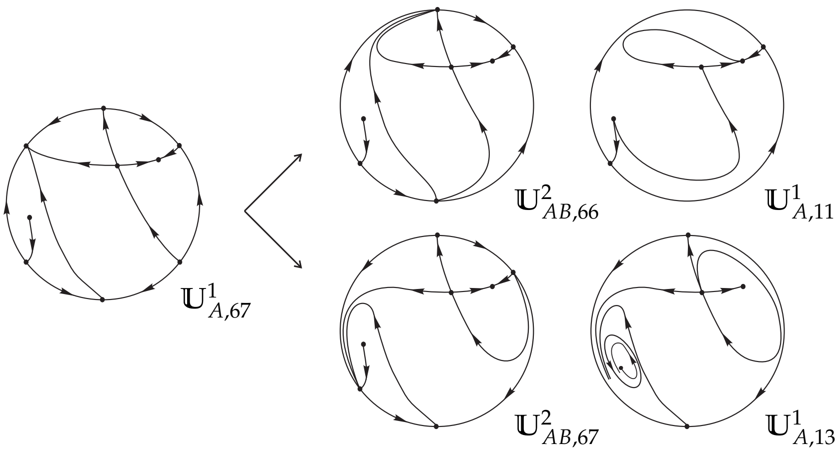


Figure 3.74: Unstable systems $\mathbb{U}^2_{AB,66}$ and $\mathbb{U}^2_{AB,67}$.

Phase portrait $\mathbb{U}^1_{A,68}$ has phase portraits $\mathbb{U}^2_{AB,68}$ and $\mathbb{U}^2_{AB,69}$ as evolution (see Figure 3.75). After bifurcation we get phase portraits $\mathbb{U}^1_{A,11}$ (modulo limit cycles) and $\mathbb{U}^1_{A,13}$, respectively, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,69}$ has phase portrait $\mathbb{U}^2_{AB,70}$ as an evolution (see Figure 3.76). After

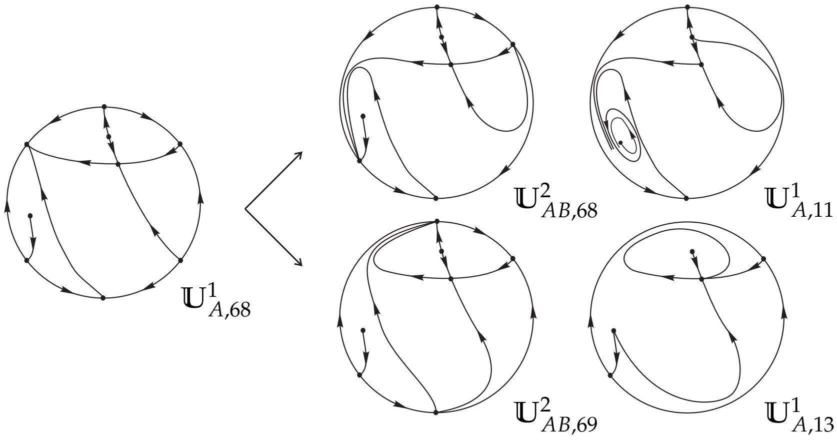


Figure 3.75: Unstable systems $\mathbb{U}^2_{AB,68}$ and $\mathbb{U}^2_{AB,69}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,69}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,27}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,7}$ of *codimension one**, see Figure 3.77. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,27}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

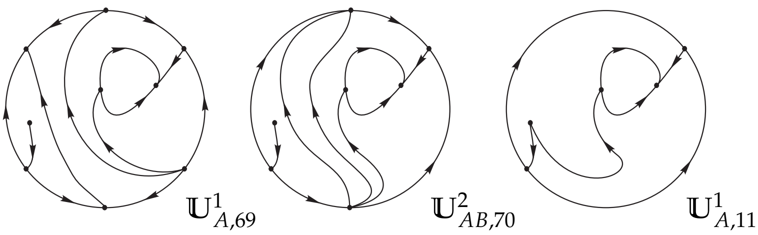


Figure 3.76: Unstable system $\mathbb{U}^2_{AB,70}$.

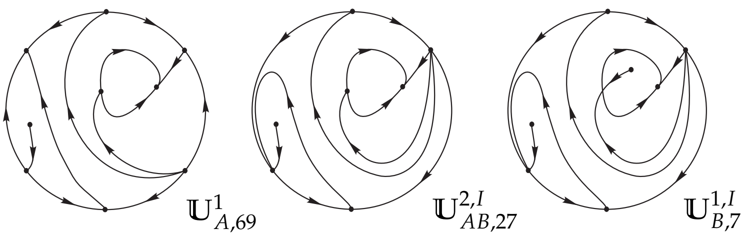


Figure 3.77: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,27}$.

Phase portrait $\mathbb{U}^1_{A,70}$ has phase portrait $\mathbb{U}^2_{AB,71}$ as an evolution (see Figure 3.78). After bifurcation we get phase portrait $\mathbb{U}^1_{A,13}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,70}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,28}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,7}$ of *codimension one**, see Figure 3.79. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,28}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

Therefore, we have just finished obtaining all the 71 topologically potential phase portraits

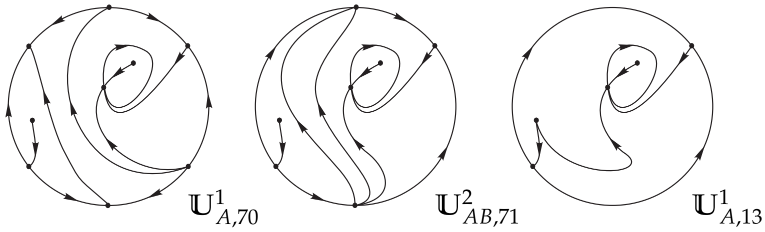


Figure 3.78: Unstable system $\mathbb{U}^2_{AB,71}$.

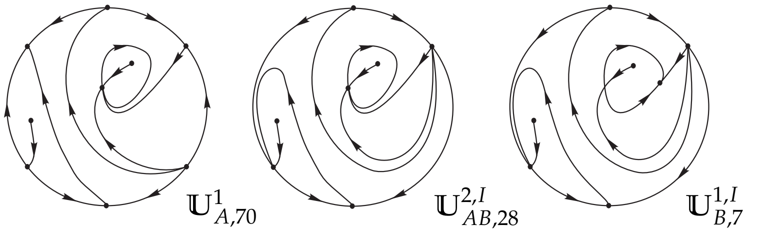


Figure 3.79: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,28}$.

of *codimension two** from the set (AB) presented in Figures 1.1 to 1.3.

Now we explain how one can obtain these 71 phase portraits by starting the study from the set (B). Let us consider all the 40 realizable structurally unstable quadratic vector fields of *codimension one** from the set (B). In order to obtain a phase portrait of *codimension two** belonging to the set (AB) starting from a phase portrait of *codimension one** of the set (B), we keep the existing infinite saddle-node $\overline{\binom{0}{2}}SN$ and by using Theorem 2.6 we build a finite saddle-node $\overline{sn}_{(2)}$ by the coalescence of a finite saddle with a finite node. On the other hand, from the phase portraits of *codimension two** from the set (AB), there exist two ways of obtaining phase portraits of *codimension one** also belonging to the set (B) after perturbation: splitting $\overline{sn}_{(2)}$ into a saddle and a node, or moving it to complex singularities (see Remark 3.2).

Remark 3.2. We recall that, in quadratic differential systems, the finite singular points are zeroes of a polynomial of degree four. Supposing that we have a singular point of multiplicity two, then the remaining singular points are zeroes of a quadratic polynomial. Therefore, these other two points can be two simple singular points, a double point (a saddle-node) or two complex conjugate singular points.

According to these facts, if a phase portrait does not possess finite singularities (for instance, $\mathbb{U}^1_{B,1}$ and $\mathbb{U}^1_{B,2}$) or if it possesses only two finite antisaddles (as for instance $\mathbb{U}^1_{B,29}$ to $\mathbb{U}^1_{B,32}$), it is not possible to obtain a phase portrait from it which belongs to the set (AB).

The main goal of this section is to obtain all the topologically potential phase portraits from the set (AB) and then prove their realization or show that they are not possible. So we have to be sure that no other phase portrait can be found if one does some evolution in all elements of the set (B) in order to obtain a phase portrait belonging to the set (AB). We point out that we have done this verification, i.e. we have also considered each element from the set (B) and produced a coalescence (when it was possible) of a finite saddle with a finite node and we also have obtained the 71 topologically potential phase portraits of *codimension two** from the set (AB) presented in Figures 1.1 to 1.3. In what follows we show the result (modulo limit cycles) of this study. We point out that we will not give all the details of this study. We will not even mention anything about why there are no more potential cases to be considered an evolution of a *codimension one** phase portrait, since we believe that this can be easily verified by the reader. Additionally, we will present pictures only of the impossible phase portraits obtained in order to explain their impossibility and we will not mention anything about phase portraits which are topologically equivalent to phase portraits already obtained.

It is important to remark that the realizable phase portraits that we will obtain from the set (B) to the set (AB) will coincide exactly with those ones previously found. However, the non-realizable ones that we will find from (B) will be different from those ones coming from (A). The reason is that the arguments used to prove the impossibility of those coming from (A) were precisely that they would bifurcate in some impossible from (B) and now, they will be those ones that bifurcate in some impossible from (A).

In Table 3.1 we present the study of phase portraits $\mathbb{U}^1_{B,3}$ to $\mathbb{U}^1_{B,11}$. In the first column we present the corresponding phase portrait from the set (B), in the second column we indicate its corresponding phase portrait belonging to the set (AB) i.e. after producing a finite saddle-node $\overline{sn}_{(2)}$, and in the third column we show the corresponding phase portrait after we make this finite saddle-node $\overline{sn}_{(2)}$ disappear.

phase portrait from the set (B)	**phase portrait from the set (AB)**	**phase portrait from the set (B)**
$\mathbb{U}^1_{B,3}$	$\mathbb{U}^2_{AB,1}$	$\mathbb{U}^1_{B,1}$
$\mathbb{U}^1_{B,4}$	$\mathbb{U}^2_{AB,2}$	$\mathbb{U}^1_{B,2}$
$\mathbb{U}^1_{B,5}$	$\mathbb{U}^2_{AB,3}$	$\mathbb{U}^1_{B,1}$
$\mathbb{U}^1_{B,6}$	$\mathbb{U}^2_{AB,4}$	$\mathbb{U}^1_{B,2}$
$\mathbb{U}^1_{B,7}$	$\mathbb{U}^2_{AB,6}$	$\mathbb{U}^1_{B,2}$
$\mathbb{U}^1_{B,8}$	$\mathbb{U}^2_{AB,5}$	$\mathbb{U}^1_{B,1}$
$\mathbb{U}^1_{B,9}$	$\mathbb{U}^2_{AB,7}$ $\mathbb{U}^2_{AB,9}$ $\mathbb{U}^2_{AB,11}$	$\mathbb{U}^1_{B,8}$
$\mathbb{U}^1_{B,10}$	$\mathbb{U}^2_{AB,8}$ $\mathbb{U}^2_{AB,10}$ $\mathbb{U}^2_{AB,12}$	$\mathbb{U}^1_{B,7}$
$\mathbb{U}^1_{B,11}$	$\mathbb{U}^2_{AB,13}$ $\mathbb{U}^2_{AB,14}$	$\mathbb{U}^1_{B,4}$ $\mathbb{U}^1_{B,7}$

Table 3.1: Phase portraits from the set (AB) obtained from evolution of elements of the set (B).

Phase portrait $\mathbb{U}^1_{B,12}$ has phase portraits $\mathbb{U}^2_{AB,15}$ and $\mathbb{U}^2_{AB,16}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{B,3}$ (for both cases) by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,12}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,29}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}} SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,1}$ of *codimension one**, see Figure 3.80. We point out that, in the set (B), the corresponding unfolding of $\mathbb{U}^{2,I}_{AB,29}$ does not exist, since if such a phase portrait does exist, it would be an evolution of the impossible phase portrait $\mathbb{I}_{9,1}$ (see Figure 4.4 from [6]), which contradicts Theorem 2.11.

In Table 3.2 we present the study of phase portraits $\mathbb{U}^1_{B,13}$ to $\mathbb{U}^1_{B,15}$. In the first column we present the corresponding phase portrait from the set (B), in the second column we indicate its corresponding phase portrait belonging to the set (AB) i.e. after producing a finite saddle-

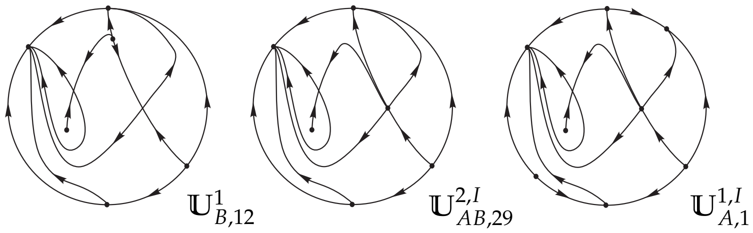


Figure 3.80: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,29}$.

node $\overline{sn}_{(2)}$, and in the third column we show the corresponding phase portrait after we make this finite saddle-node $\overline{sn}_{(2)}$ disappear.

phase portrait from the set (B)	**phase portrait from the set (AB)**	**phase portrait from the set (B)**
$\mathbb{U}^1_{B,13}$	$\mathbb{U}^2_{AB,17}$ $\mathbb{U}^2_{AB,18}$	$\mathbb{U}^1_{B,6}$ $\mathbb{U}^1_{B,7}$
$\mathbb{U}^1_{B,14}$	$\mathbb{U}^2_{AB,19}$ $\mathbb{U}^2_{AB,20}$ $\mathbb{U}^2_{AB,21}$	$\mathbb{U}^1_{B,3}$
$\mathbb{U}^1_{B,15}$	$\mathbb{U}^2_{AB,23}$ $\mathbb{U}^2_{AB,22}$	$\mathbb{U}^1_{B,3}$ $\mathbb{U}^1_{B,5}$

Table 3.2: Phase portraits from the set (AB) obtained from evolution of elements of the set (B).

Phase portrait $\mathbb{U}^1_{B,16}$ has phase portraits $\mathbb{U}^2_{AB,29}$, $\mathbb{U}^2_{AB,25}$, and $\mathbb{U}^2_{AB,26}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,5}$, $\mathbb{U}^1_{B,8}$ and $\mathbb{U}^1_{B,8}$ (being this last one modulo limit cycle), respectively, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,16}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,30}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,103}$ of *codimension one**, see Figure 3.81. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,30}$ unfolds in $\mathbb{U}^1_{B,8}$ (modulo limit cycles).

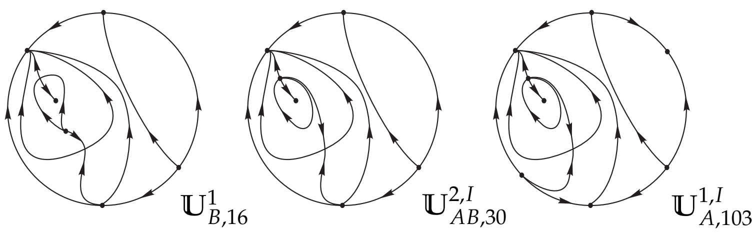


Figure 3.81: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,30}$.

Phase portrait $\mathbb{U}^1_{B,17}$ has phase portraits $\mathbb{U}^2_{AB,28}$, $\mathbb{U}^2_{AB,24}$, and $\mathbb{U}^2_{AB,27}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,6}$, $\mathbb{U}^1_{B,7}$ and $\mathbb{U}^1_{B,7}$ (being this last one modulo limit cycle), respectively, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,17}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,31}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,103}$ of *codimension one**, see Figure 3.82. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,31}$ unfolds in $\mathbb{U}^{1}_{B,7}$ (modulo limit cycles).

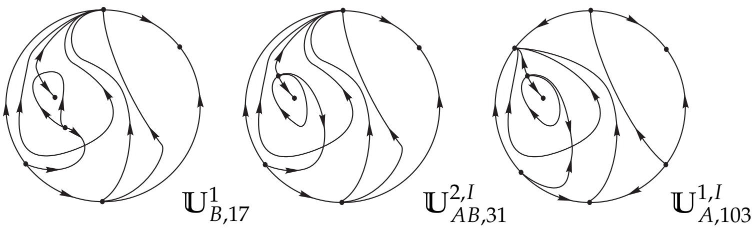


Figure 3.82: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,31}$.

Phase portrait $\mathbb{U}^{1}_{B,18}$ has phase portrait $\mathbb{U}^{2}_{AB,30}$ as an evolution and after bifurcation we get phase portrait $\mathbb{U}^{1}_{B,7}$, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^{1}_{B,18}$ has a second phase portrait as an evolution which is topologically equivalent to $\mathbb{U}^{2}_{AB,30}$.

Phase portrait $\mathbb{U}^{1}_{B,19}$ has phase portraits $\mathbb{U}^{2}_{AB,32}$ and $\mathbb{U}^{2}_{AB,31}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^{1}_{B,4}$ and $\mathbb{U}^{1}_{B,6}$, respective, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^{1}_{B,20}$ has phase portraits $\mathbb{U}^{2}_{AB,33}$ and $\mathbb{U}^{2}_{AB,34}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^{1}_{B,3}$, in both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^{1}_{B,20}$ has the impossible phase portraits $\mathbb{U}^{2,I}_{AB,32}$ and $\mathbb{U}^{2,I}_{AB,33}$ as evolution. By Theorem 2.11 such phase portraits are impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portraits $\mathbb{U}^{1,I}_{A,2}$ and $\mathbb{U}^{1,I}_{A,104}$, respectively, of *codimension one**, see Figure 3.83. We point out that, in the set (B), the corresponding unfolding of $\mathbb{U}^{2,I}_{AB,32}$ does not exist (by the exactly same reason that we have discussed in $\mathbb{U}^{2,I}_{AB,29}$) and the corresponding unfolding of $\mathbb{U}^{2,I}_{AB,33}$ is $\mathbb{U}^{1}_{B,3}$ (modulo limit cycles).

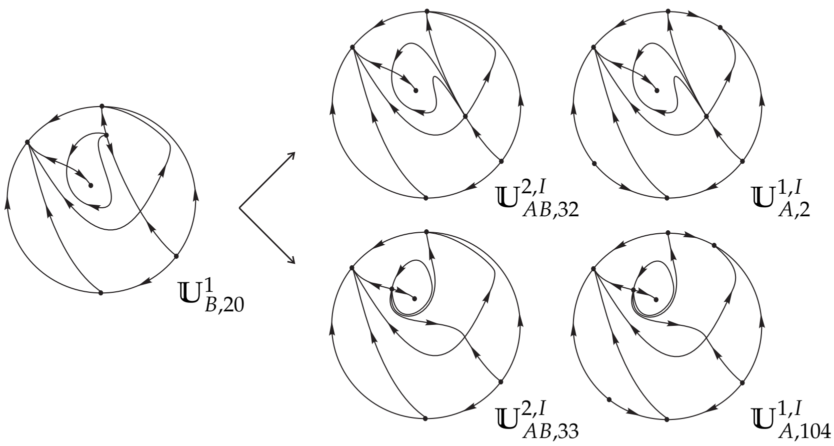
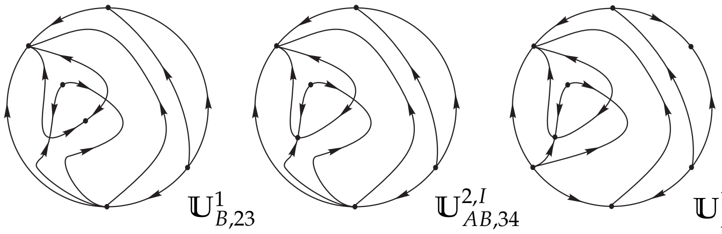


Figure 3.83: Impossible unstable phase portraits $\mathbb{U}^{2,I}_{AB,32}$ and $\mathbb{U}^{2,I}_{AB,33}$.

Phase portrait $\mathbb{U}^{1}_{B,21}$ has phase portrait $\mathbb{U}^{2}_{AB,35}$ as an evolution and after bifurcation we get phase portrait $\mathbb{U}^{1}_{B,6}$, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^{1}_{B,21}$ has a second phase portrait as an evolution which is topologically equivalent to $\mathbb{U}^{2}_{AB,35}$.

Phase portrait $\mathbb{U}^{1}_{B,22}$ has phase portraits $\mathbb{U}^{2}_{AB,36}$ and $\mathbb{U}^{2}_{AB,37}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^{1}_{B,3}$ and $\mathbb{U}^{1}_{B,8}$, respective, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

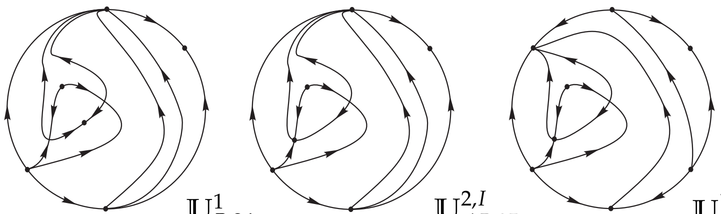
Phase portrait $\mathbb{U}^1_{B,23}$ has phase portraits $\mathbb{U}^2_{AB,39}$, $\mathbb{U}^2_{AB,40}$, and $\mathbb{U}^2_{AB,43}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,5}$ (for the two first cases) and $\mathbb{U}^1_{B,8}$ (for the third case), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,23}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,34}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,49}$ of *codimension one**, see Figure 3.84. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,34}$ unfolds in $\mathbb{U}^1_{B,8}$.



$\mathbb{U}^1_{B,23}$ $\mathbb{U}^{2,I}_{AB,34}$ $\mathbb{U}^{1,I}_{A,49}$

Figure 3.84: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,34}$.

Phase portrait $\mathbb{U}^1_{B,24}$ has phase portraits $\mathbb{U}^2_{AB,38}$, $\mathbb{U}^2_{AB,41}$, and $\mathbb{U}^2_{AB,42}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,6}$ (for the two first cases) and $\mathbb{U}^1_{B,7}$ (for the third case), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,24}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,35}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,49}$ of *codimension one**, see Figure 3.85. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,35}$ unfolds in $\mathbb{U}^1_{B,7}$.



$\mathbb{U}^1_{B,24}$ $\mathbb{U}^{2,I}_{AB,35}$ $\mathbb{U}^{1,I}_{A,49}$

Figure 3.85: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,35}$.

Phase portrait $\mathbb{U}^1_{B,25}$ has phase portraits $\mathbb{U}^2_{AB,46}$, $\mathbb{U}^2_{AB,49}$, and $\mathbb{U}^2_{AB,44}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,3}$ (for the two first cases) and $\mathbb{U}^1_{B,8}$ (for the third case), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,25}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,36}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,3}$ of *codimension one**, see Figure 3.86. We point out that, in the set (B), the corresponding unfolding of $\mathbb{U}^{2,I}_{AB,36}$ does not exist (by the exactly same reason that we have discussed in $\mathbb{U}^{2,I}_{AB,29}$).

Phase portrait $\mathbb{U}^1_{B,26}$ has phase portraits $\mathbb{U}^2_{AB,47}$, $\mathbb{U}^2_{AB,48}$, and $\mathbb{U}^2_{AB,45}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,4}$ (for the two first cases) and $\mathbb{U}^1_{B,7}$ (for the third case), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,26}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,37}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node

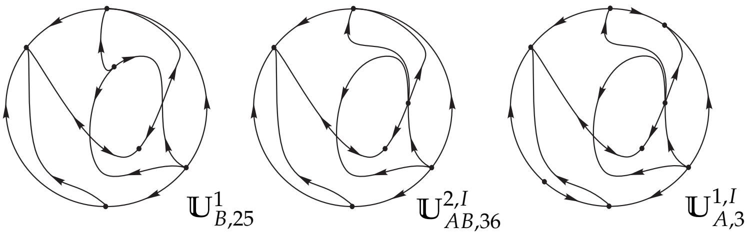


Figure 3.86: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,36}$.

we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,3}$ of *codimension one**, see Figure 3.87. We point out that, in the set (B), the corresponding unfolding of $\mathbb{U}^{2,I}_{AB,37}$ does not exist (by the exactly same reason that we have discussed in $\mathbb{U}^{2,I}_{AB,29}$).

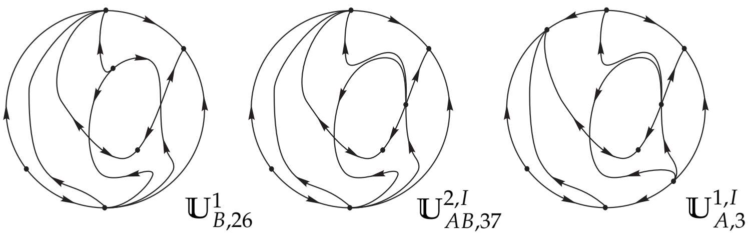


Figure 3.87: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,37}$.

In Table 3.3 we present the study (modulo limit cycles) of phase portraits $\mathbb{U}^1_{B,27}$ to $\mathbb{U}^1_{B,35}$. In the first column we present the corresponding phase portrait from the set (B), in the second column we indicate its corresponding phase portrait belonging to the set (AB) i.e. after producing a finite saddle-node $\overline{sn}_{(2)}$, and in the third column we show the corresponding phase portrait after we make this finite saddle-node $\overline{sn}_{(2)}$ disappear.

phase portrait from the set (*B*)	**phase portrait from the set (*AB*)**	**phase portrait from the set (*B*)**
$\mathbb{U}^1_{B,27}$	$\mathbb{U}^2_{AB,50}$	$\mathbb{U}^1_{B,4}$
$\mathbb{U}^1_{B,28}$	$\mathbb{U}^2_{AB,51}$ $\mathbb{U}^2_{AB,52}$	$\mathbb{U}^1_{B,3}$ $\mathbb{U}^1_{B,4}$
$\mathbb{U}^1_{B,33}$	$\mathbb{U}^2_{AB,53}$ $\mathbb{U}^2_{AB,54}$	$\mathbb{U}^1_{B,29}$
$\mathbb{U}^1_{B,34}$	$\mathbb{U}^2_{AB,55}$ $\mathbb{U}^2_{AB,56}$ $\mathbb{U}^2_{AB,57}$ $\mathbb{U}^2_{AB,58}$	$\mathbb{U}^1_{B,32}$
$\mathbb{U}^1_{B,35}$	$\mathbb{U}^2_{AB,61}$ $\mathbb{U}^2_{AB,59}$ $\mathbb{U}^2_{AB,60}$	$\mathbb{U}^1_{B,29}$ $\mathbb{U}^1_{B,32}$

Table 3.3: Phase portraits from the set (AB) obtained from evolution of elements of the set (B).

Phase portrait $\mathbb{U}^1_{B,36}$ has phase portraits $\mathbb{U}^2_{AB,62}$ and $\mathbb{U}^2_{AB,63}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{B,29}$, for both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,36}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,38}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,105}$ of *codimension one**, see Figure 3.88. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,38}$ unfolds in $\mathbb{U}^1_{B,29}$ (modulo limit cycles).

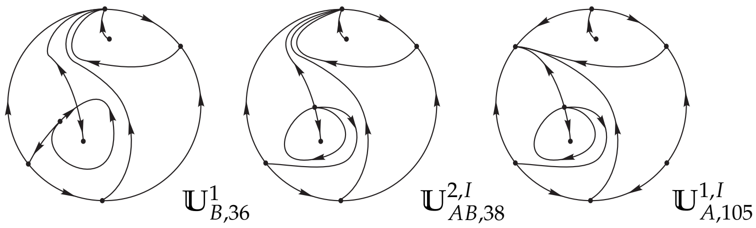


Figure 3.88: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,38}$.

Phase portrait $\mathbb{U}^1_{B,37}$ has phase portraits $\mathbb{U}^2_{AB,64}$ and $\mathbb{U}^2_{AB,65}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{B,31}$, for both cases, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^1_{B,38}$ has phase portraits $\mathbb{U}^2_{AB,68}$ and $\mathbb{U}^2_{AB,67}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,29}$ and $\mathbb{U}^1_{B,30}$, respective, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^1_{B,39}$ has phase portraits $\mathbb{U}^2_{AB,69}$ and $\mathbb{U}^2_{AB,66}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,29}$ and $\mathbb{U}^1_{B,31}$, respective, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^1_{B,40}$ has phase portraits $\mathbb{U}^2_{AB,70}$ and $\mathbb{U}^2_{AB,71}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{B,31}$, for both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,40}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,39}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,106}$ of *codimension one**, see Figure 3.89. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,39}$ unfolds in $\mathbb{U}^1_{B,31}$ (modulo limit cycles).

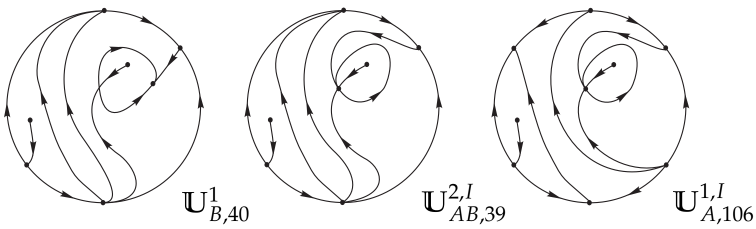


Figure 3.89: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,39}$.

3.2 The realization of the potential phase portraits

In the previous subsection we have produced all the topologically potential phase portraits for structurally unstable quadratic systems of *codimension two** belonging to the set $\sum_2^2(AB)$. And from them, we have discarded 33 which are not realizable due to their respective unfoldings of *codimension one** being impossible.

In this subsection we aim to give specific examples for the remaining 71 different topological classes of structurally unstable quadratic systems of *codimension two** belonging to the set $\sum_2^2(AB)$ and presented in Figures 1.1 to 1.3.

In [2] the authors showed that for each structurally stable phase portrait with limit cycles there exists a realizable structurally stable phase portrait without limit cycles so that modulo limit cycles they are equivalent. On the contrary, due to the large number of cases, in [6] the authors did not follow the same procedure for the realizable structurally unstable phase portraits of *codimension one**. Since this present paper is directly derived from this second study, here we have found examples of *codimension two** phase portraits with no evidence of limit cycles, but we have not proved the absence of the infinitesimal ones (i.e. the ones born by Hopf-bifurcation).

In [14] the authors classified, with respect to a specific normal form, the set of all real quadratic polynomial differential systems with a finite semi-elemental saddle-node $\overline{sn}_{(2)}$ located at the origin of the plane and an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ (obtained by the coalescence of an infinite saddle with an infinite node) located in the bisector of first and third quadrants. In [10] the authors show that phase portrait V_{171} from [14] is not topologically equivalent to V_{170} (i.e. the equivalence presented in Table 65 from the mentioned paper is not correct) and in [10] the authors present the correct picture of phase portrait V_{171}.

Remark 3.3. The study of a bifurcation diagram of a certain family of quadratic systems produces not only the class of phase portraits that we look for, but also all of those of their closure according to the normal form that we consider. Even though the study is mainly algebraic, analytic and numerical tools are also required. This implies that these studies may be not complete and subject to the existence of possible "islands" which could contain an undetected phase portrait. The border of that "island" could mean the connection of two separatrices, and its interior could contain a different phase portrait from the ones stated in the main theorem. In [14] the authors studied a bifurcation diagram in which the most generic phase portraits correspond to elements of the set (AB). In Section 7 of that paper the authors said that the bifurcation diagram they obtained is completely coherent, i.e. by taking any two points in the parameter space and joining them by a continuous curve, along this curve the changes in phase portraits that occur when crossing the different bifurcation surfaces could be completely explained. Nevertheless, at that moment, the authors could not be sure that the bifurcation diagram was the complete bifurcation diagram for the family they consider in their paper, due to the possibility of "islands" inside the bifurcation diagram. The topological study that we do in this paper solves partially this problem, since we prove that all the realizable phase portraits of class (AB) do really exist, and no other topological possibility does. However, the possible existence of "islands" in the bifurcation diagram still persists since they can be related to double limit cycles, as discussed in Section 7 of [14].

By using the phase portraits of generic regions of the bifurcation diagram from [14] plus the correct V_{171} presented in [10] we realize all the 71 unstable systems of *codimension two** of the set (AB), i.e. we can give concrete examples of all structurally unstable phase portraits from the set (AB).

Consider systems (2.1), which were studied in [14] and describe quadratic systems having a finite semi-elemental saddle-node $\overline{sn}_{(2)}$ and an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ located in the endpoints of the bisector of the first and third quadrants.

In Tables 3.4 and 3.6 we present one representative from each generic region of the bifurcation diagram of [14] (as described before) corresponding to each phase portrait of *codimension* *two** from the set (AB) and, therefore, we conclude the proof of Theorem 1.6.

Cod 2*	[14]	g	h	ł	n
$\mathbb{U}^2_{AB,1}$	V_{23}	1	0	1/2	10
$\mathbb{U}^2_{AB,2}$	V_{84}	1	91/100	1	2304/625
$\mathbb{U}^2_{AB,3}$	V_{22}	1	0	9/10	10
$\mathbb{U}^2_{AB,4}$	V_{85}	1	22/25	1	2304/625
$\mathbb{U}^2_{AB,5}$	V_{20}	1	0	18	10
$\mathbb{U}^2_{AB,6}$	V_{21}	1	−2	1	10
$\mathbb{U}^2_{AB,7}$	V_{1}	1	−21/5	18	10
$\mathbb{U}^2_{AB,8}$	V_{2}	1	−5	10	10
$\mathbb{U}^2_{AB,9}$	V_{190}	1	3/5	−33/10	−1
$\mathbb{U}^2_{AB,10}$	V_{191}	1	3/5	−3	−1
$\mathbb{U}^2_{AB,11}$	V_{25}	1	173/80	6	10
$\mathbb{U}^2_{AB,12}$	V_{31}	1	112/25	6	30
$\mathbb{U}^2_{AB,13}$	V_{9}	1	−5	11/10	10
$\mathbb{U}^2_{AB,14}$	V_{121}	1	−9999/100000	4/25	81/100
$\mathbb{U}^2_{AB,15}$	V_{147}	1	−6/5	5	−1
$\mathbb{U}^2_{AB,16}$	V_{66}	1	5	−15	10
$\mathbb{U}^2_{AB,17}$	V_{7}	1	−9/2	13/5	10
$\mathbb{U}^2_{AB,18}$	V_{136}	1	−59999/100000	7/10	4/25
$\mathbb{U}^2_{AB,19}$	V_{64}	1	11/5	−4	10
$\mathbb{U}^2_{AB,20}$	V_{145}	1	−4/5	5	−1
$\mathbb{U}^2_{AB,21}$	V_{13}	1	−5	1/2	10
$\mathbb{U}^2_{AB,22}$	V_{83}	1	9201/10000	−15	2304/625
$\mathbb{U}^2_{AB,23}$	V_{10}	1	−5	7/10	10
$\mathbb{U}^2_{AB,24}$	V_{141}	1	−69/100	601/1000	9/100
$\mathbb{U}^2_{AB,25}$	V_{144}	1	−7999/10000	6397/10000	1/25
$\mathbb{U}^2_{AB,26}$	V_{172}	1	−1/10	−3	−1
$\mathbb{U}^2_{AB,27}$	V_{173}	1	−7/100	−31/20	−1
$\mathbb{U}^2_{AB,28}$	V_{41}	1	44773/10000	11/5	30
$\mathbb{U}^2_{AB,29}$	V_{69}	1	11/5	6	10

Table 3.4: Correspondence between *codimension two** phase portraits of the set (AB) and phase portraits from generic regions of the bifurcation diagram presented in [14]. In the first column we present the *codimension two** phase portraits from the set (AB) in the present paper, in the second column we show the corresponding phase portraits from [14] given by normal form (2.1), and in the other columns we present the values of the parameters g, h, ł, and n of (2.1) which realizes such phase portrait (remember that the correct phase portrait V_{171} is presented in [10]).

Cod 2*	**[14]**	**g**	**h**	**ł**	**n**
$\mathbb{U}^2_{AB,30}$	V_{15}	1	$-21/5$	3	10
$\mathbb{U}^2_{AB,31}$	V_{114}	1	$-211/2000$	$9549/50000$	$4/5$
$\mathbb{U}^2_{AB,32}$	V_{109}	1	$-41/400$	$99999/100000$	$4/5$
$\mathbb{U}^2_{AB,33}$	V_{154}	1	$-7/5$	$8/25$	-1
$\mathbb{U}^2_{AB,34}$	V_{102}	1	$481/2000$	-10	1
$\mathbb{U}^2_{AB,35}$	V_{129}	1	$-5499/10000$	$3/4$	$81/400$
$\mathbb{U}^2_{AB,36}$	V_{108}	1	$-41/400$	$11/10$	$4/5$
$\mathbb{U}^2_{AB,37}$	V_{78}	1	$9201/10000$	-50	$2304/625$
$\mathbb{U}^2_{AB,38}$	V_{42}	1	$44777/10000$	$203/100$	30
$\mathbb{U}^2_{AB,39}$	V_{71}	1	$223/100$	6	10
$\mathbb{U}^2_{AB,40}$	V_{170}	1	$-9/50$	-3	-1
$\mathbb{U}^2_{AB,41}$	V_{171}	1	$-3/40$	$-3/2$	-1
$\mathbb{U}^2_{AB,42}$	V_{142}	1	$-69/100$	$6007/10000$	$9/100$
$\mathbb{U}^2_{AB,43}$	V_{143}	1	$-7999/10000$	$27/50$	$1/25$
$\mathbb{U}^2_{AB,44}$	V_{104}	1	$573/1250$	-8	$19/10$
$\mathbb{U}^2_{AB,45}$	V_{123}	1	$-39/400$	$1/100$	$81/100$
$\mathbb{U}^2_{AB,46}$	V_{155}	1	$-7/5$	$3/10$	-1
$\mathbb{U}^2_{AB,47}$	V_{165}	1	$-1/5$	$-13/10$	-1
$\mathbb{U}^2_{AB,48}$	V_{37}	1	3	$11/10$	10
$\mathbb{U}^2_{AB,49}$	V_{44}	1	$22/5$	2	10
$\mathbb{U}^2_{AB,50}$	V_{110}	1	$-41/400$	$9/10$	$4/5$
$\mathbb{U}^2_{AB,51}$	V_{46}	1	$11/5$	$9/10$	10
$\mathbb{U}^2_{AB,52}$	V_{49}	1	$23/5$	$9/10$	10
$\mathbb{U}^2_{AB,53}$	V_{6}	1	-5	3	10
$\mathbb{U}^2_{AB,54}$	V_{189}	1	$37/50$	$-147/100$	-1
$\mathbb{U}^2_{AB,55}$	V_{61}	1	$4501/1000$	-1	10
$\mathbb{U}^2_{AB,56}$	V_{53}	1	6	$-1/10000$	10
$\mathbb{U}^2_{AB,57}$	V_{107}	1	$9/25$	$-1/2$	$41/25$
$\mathbb{U}^2_{AB,58}$	V_{149}	1	$-11/10$	$3/2$	-1
$\mathbb{U}^2_{AB,59}$	V_{62}	1	3	-1	10
$\mathbb{U}^2_{AB,60}$	V_{198}	1	$-2/5$	$11/10$	-1
$\mathbb{U}^2_{AB,61}$	V_{51}	1	6	$1/5$	10
$\mathbb{U}^2_{AB,62}$	V_{138}	1	$-3/5$	$7/10$	$9/100$
$\mathbb{U}^2_{AB,63}$	V_{177}	1	$3/100$	$-9/10$	-1
$\mathbb{U}^2_{AB,64}$	V_{3}	1	-5	6	10
$\mathbb{U}^2_{AB,65}$	V_{192}	1	$3/5$	$-123/50$	-1
$\mathbb{U}^2_{AB,66}$	V_{122}	1	$-39/400$	$31/1000$	$81/100$
$\mathbb{U}^2_{AB,67}$	V_{169}	1	$-1/5$	$-7/10$	-1
$\mathbb{U}^2_{AB,68}$	V_{113}	1	$-39/400$	$1/10$	$81/100$

Table 3.5: Continuation of Table 3.4.

Cod 2*	**[14]**	**g**	**h**	**ℓ**	**n**
$\mathbb{U}^2_{AB,69}$	V_{166}	1	$-1/5$	$-53/50$	-1
$\mathbb{U}^2_{AB,70}$	V_{140}	1	$-69/100$	$63/100$	$9/100$
$\mathbb{U}^2_{AB,71}$	V_{174}	1	$-41/1000$	$-133/100$	-1

Table 3.6: Continuation of Table 3.5.

4 Proof of Theorem 1.7

In this section we present the proof of Theorem 1.7. The procedure is the same as used in the previous section. In Subsection 4.1 we obtain all the topologically potential phase portraits possessing the saddle-nodes $\overline{sn}_{(2)}$ and $\overline{\binom{1}{1}}SN$ (we have 45 phase portraits) and we prove that five of them are impossible. In Subsection 4.2 we show the realization of each one of the remaining 40 phase portraits.

4.1 The topologically potential phase portraits

The main goal of this subsection is to obtain all the topologically potential phase portraits from the set (AC).

As we said before, inside the set (AC), the unstable objects of *codimension two** that we are considering in this paper belong to the set of saddle-nodes $\left\{\overline{sn}_{(2)}+\overline{\binom{1}{1}}SN\right\}$. Considering all the different ways of obtaining phase portraits belonging to the set (AC) of *codimension two**, we have to consider all the possible ways of coalescing specific singular points in both sets (A) and (C). However, as the sets (AC) and (CA) are the same (i.e. their elements are obtained independently of the order of evolution in elements of the sets (A) or (C)), it is necessary to consider only all the possible ways of obtaining an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ in each element from the set (A) (phase portraits possessing a finite saddle-node $\overline{sn}_{(2)}$). Anyway, in order to make things clear, in page 77 we discuss briefly how should we perform if we start by considering the set (C).

In order to obtain phase portraits from the set (AC) by starting our study from the set (A), we have to consider Theorem 2.9 and also Lemma 3.26 from [6] (regarding phase portraits from the set (C)) which we state as follows.

Lemma 4.1. *Assume that a codimension one* polynomial vector field X has an infinite singular point p being a saddle-node of multiplicity two with $\rho_0 = (\partial P/\partial x + \partial Q/\partial y)_p \neq 0$ and second eigenvalue equal to zero.*

(a) Any perturbation of X in a sufficiently small neighborhood of this point will produce a structurally stable system (with one infinite saddle and one finite node, or vice versa) or a system topologically equivalent to X.

(b) Both possibilities of structurally stable systems are realizable.

(c) If the saddle-node is the only unstable object in the region of definition and we consider the perturbation which leaves a saddle and a node in a small neighborhood, then the node is ω-limit or α-limit (depending on its stability) of at least one of the separatrices of the saddle.

(d) In the case that after bifurcation the node remains at infinity and the saddle moves to the finite plane, then the separatrices of this new saddle have their α- and ω-limits fixed according to next rule:

(1) *The separatrix γ that corresponds to the one of the saddle-node different from the infinity line must maintain the same α- or ω-limit set.*

(2) *The separatrix (belonging to the same eigenspace of γ) which appears after bifurcation must go to the node that remains at infinity, and this will be the only separatrix which can arrive to this node in this side of the infinity.*

(3) *The two separatrices which were the infinite line in the unstable phase portrait, and that now are two separatrices of the saddle drawn on the finite plane, must end at the same infinite node where they ended before the bifurcation (if a node was adjacent to the saddle-node) or in the same α- or ω-limit point of the finite separatrix of the adjacent infinite saddle. In case that the saddle-node is the only infinite singular point, then both separatrices go to the symmetric point which will remain as a node.*

Here we consider all 69 realizable structurally unstable quadratic vector fields of *codimension one** from the set (A). In order to obtain a phase portrait of *codimension two** belonging to the set (AC) starting from a phase portrait of *codimension one** of the set (A), we keep the existing finite saddle-node and using Lemma 4.1 we build an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ by the coalescence of a finite node (respectively, finite saddle) with an infinite saddle (respectively, infinite node). As we said before, we point out that the finite singularity that coalesces with an infinite singularity cannot be the finite saddle-node since then what we would obtain at infinity would not be a saddle-node of type $\overline{\binom{1}{1}}SN$ but a multiplicity three singularity. Even though this is also a *codimension two** case and somehow can be considered inside the set (AC), we have preferred to put it into the set (CC) where two possibilities will be needed to be studied: either two finite singularities coalescing with different infinite singularities, or two finite singularities coalescing with the same infinite singularity. On the other hand, from the phase portraits of *codimension two** from the set (AC), one can obtain phase portraits of *codimension one** also belonging to the set (A) after perturbation by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$ into a finite saddle (respectively, finite node) and an infinite node (respectively, infinite saddle). More precisely, after bifurcation the point that has arrived to infinity remains there with the same local behavior, and the one which was at infinity moves into the real plane at the other side of the infinity line.

As in the previous section, in what follows we denote by $\mathbb{U}^2_{AC,k}$, where $\mathbb{U}^2_{AC}$ stands for structurally unstable quadratic vector field of *codimension two** from the set (AC) and $k \in \{1, \dots, 40\}$. The impossible phase portraits will be denoted by $\mathbb{U}^{2,I}_{AC,j}$, where $\mathbb{U}^{2,I}_{AC}$ stands for *Impossible of codimension two** *from the set (AC)* and $j \in \mathbb{N}$.

We point out that in this study we do not present phase portraits which are topologically equivalent to phase portraits already obtained. Additionally, as we explained clearly about how we obtain an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ from a phase portrait from the set (A), we will not mention anything about why we do not have no more possibilities (of obtaining an infinite saddle-node of type $\overline{\binom{1}{1}}SN$) beyond those ones that we will present.

Phase portrait $\mathbb{U}^1_{A,1}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution, since $\mathbb{U}^1_{A,1}$ has only the finite saddle-node $\overline{sn}_{(2)}$ and only the infinite node.

Phase portrait $\mathbb{U}^1_{A,2}$ has phase portrait $\mathbb{U}^2_{AC,1}$ as an evolution (see Figure 4.1). After bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,3}$ has phase portrait $\mathbb{U}^2_{AC,2}$ as an evolution (see Figure 4.2). After bifur-

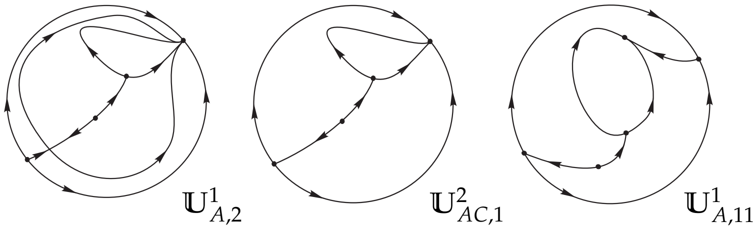


Figure 4.1: Unstable system $\mathbb{U}^2_{AC,1}$.

cation we get phase portrait $\mathbb{U}^1_{A,12}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

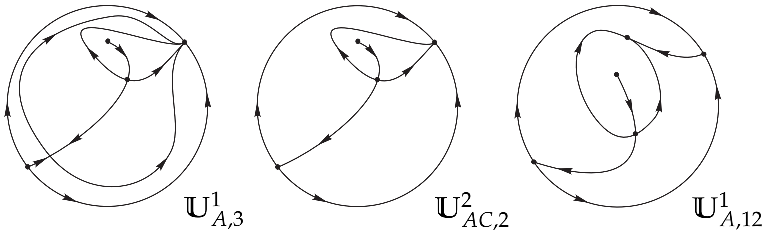


Figure 4.2: Unstable system $\mathbb{U}^2_{AC,2}$.

Phase portrait $\mathbb{U}^1_{A,4}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution. In fact, such a phase portrait possesses only an infinite node which receives four separatrices from finite singularities. Then by item $(d)-(2)$ of Lemma 4.1 the finite saddle cannot reach the infinite node. We point out that this same situation happens in many other phase portraits, such as in $\mathbb{U}^1_{A,5}$ to $\mathbb{U}^1_{A,8}$. Because it is quite simple to detect this phenomena, when we deal again with this situation we will skip all the details.

Phase portrait $\mathbb{U}^1_{A,9}$ has phase portrait $\mathbb{U}^2_{AC,3}$ as an evolution (see Figure 4.3). After bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

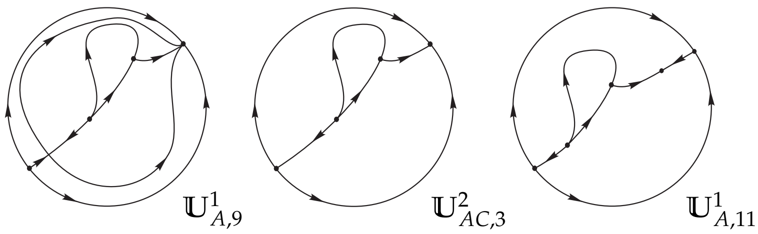


Figure 4.3: Unstable system $\mathbb{U}^2_{AC,3}$.

Phase portrait $\mathbb{U}^1_{A,10}$ has phase portrait $\mathbb{U}^2_{AC,4}$ as an evolution (see Figure 4.4). After bifurcation we get phase portrait $\mathbb{U}^1_{A,13}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

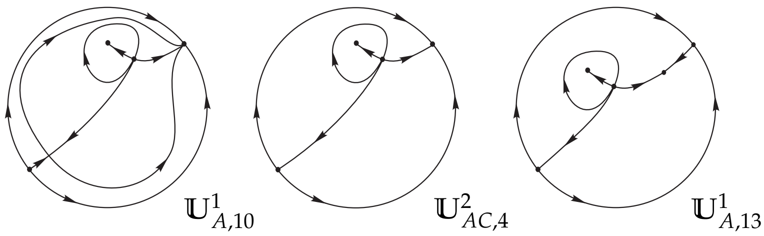


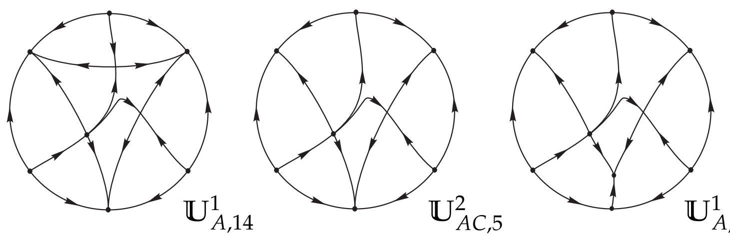
Figure 4.4: Unstable system $\mathbb{U}^2_{AC,4}$.

It is quite common that a given phase portrait of a certain codimension K be an unfolding of topologically distinct phase portraits of codimension $K+1$ (modulo limit cycles). This situation appears in this study. In the first column of Table 4.1 we present the phase portrait of the set (A), in the second column we indicate the corresponding phase portrait belonging to the set (AC), and in the third column we show the respective phase portrait after bifurcation. We point out that it is not necessary to present any explanation for the phase portraits present in the first column, since their corresponding elements from the third column already appeared and were explained before.

phase portrait from the set (A)	**phase portrait from the set (AC)**	**phase portrait from the set (A)**
$\mathbb{U}^1_{A,11}$	$\mathbb{U}^2_{AC,1}$ $\mathbb{U}^2_{AC,3}$	$\mathbb{U}^1_{A,2}$ $\mathbb{U}^1_{A,9}$
$\mathbb{U}^1_{A,12}$	$\mathbb{U}^2_{AC,2}$	$\mathbb{U}^1_{A,3}$
$\mathbb{U}^1_{A,13}$	$\mathbb{U}^2_{AC,4}$	$\mathbb{U}^1_{A,10}$

Table 4.1: Phase portraits from the set (AC) obtained from evolution of some elements of the set (A).

Phase portrait $\mathbb{U}^1_{A,14}$ has phase portrait $\mathbb{U}^2_{AC,5}$ as an evolution (see Figure 4.5). After bifurcation we get phase portrait $\mathbb{U}^1_{A,55}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.



$\mathbb{U}^1_{A,14}$ $\mathbb{U}^2_{AC,5}$ $\mathbb{U}^1_{A,55}$

Figure 4.5: Unstable system $\mathbb{U}^2_{AC,5}$.

Phase portrait $\mathbb{U}^1_{A,15}$ has phase portraits $\mathbb{U}^2_{AC,6}$ and $\mathbb{U}^2_{AC,7}$ as evolution (see Figure 4.6). After bifurcation we get phase portraits $\mathbb{U}^1_{A,32}$ and $\mathbb{U}^1_{A,53}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,16}$ has phase portraits $\mathbb{U}^2_{AC,8}$, $\mathbb{U}^2_{AC,9}$, and $\mathbb{U}^2_{AC,10}$ as evolution (see Figure 4.7). After bifurcation we get phase portraits $\mathbb{U}^1_{A,33}$, $\mathbb{U}^1_{A,52}$, and $\mathbb{U}^1_{A,54}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,17}$ has phase portraits $\mathbb{U}^2_{AC,11}$, $\mathbb{U}^2_{AC,12}$, and $\mathbb{U}^2_{AC,13}$ as evolution (see Figure 4.8). After bifurcation we get phase portraits $\mathbb{U}^1_{A,35}$, $\mathbb{U}^1_{A,41}$, and $\mathbb{U}^1_{A,42}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,18}$ has phase portraits $\mathbb{U}^2_{AC,14}$, $\mathbb{U}^2_{AC,15}$, and $\mathbb{U}^2_{AC,16}$ as evolution (see Figure 4.9). After bifurcation we get phase portraits $\mathbb{U}^1_{A,25}$, $\mathbb{U}^1_{A,27}$, and $\mathbb{U}^1_{A,45}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portraits $\mathbb{U}^1_{A,19}$, $\mathbb{U}^1_{A,20}$, and $\mathbb{U}^1_{A,21}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution since each one of them has only the finite saddle-node $\overline{sn}_{(2)}$.

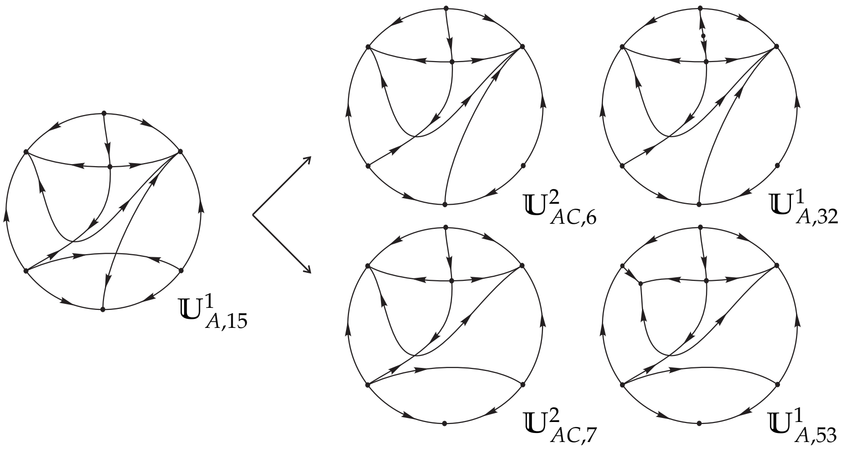


Figure 4.6: Unstable systems $\mathbb{U}^2_{AC,6}$ and $\mathbb{U}^2_{AC,7}$.

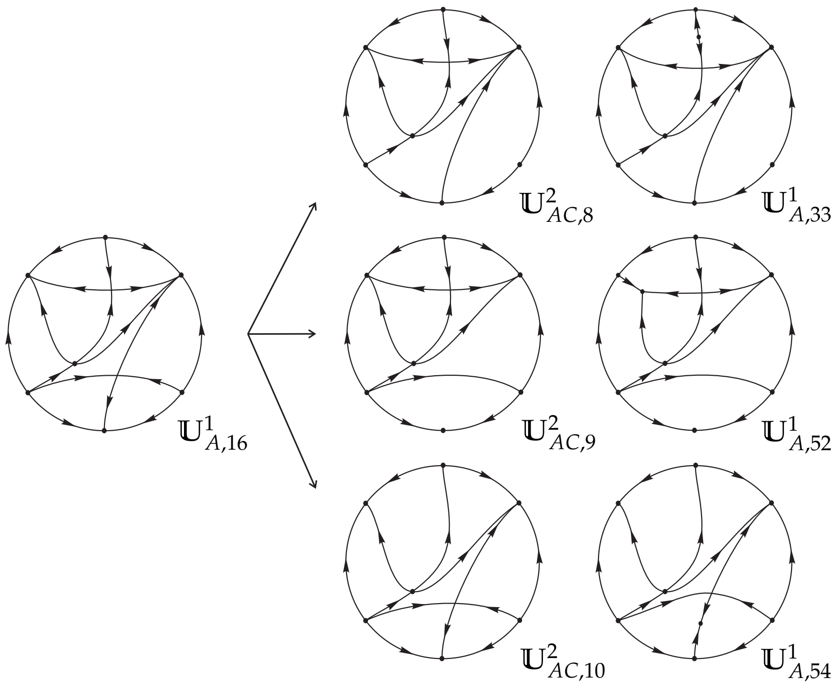


Figure 4.7: Unstable systems $\mathbb{U}^2_{AC,8}$, $\mathbb{U}^2_{AC,9}$, and $\mathbb{U}^2_{AC,10}$.

Phase portrait $\mathbb{U}^1_{A,22}$ has phase portrait $\mathbb{U}^2_{AC,17}$ as an evolution (see Figure 4.10). After bifurcation we get phase portrait $\mathbb{U}^1_{A,65}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,23}$ has phase portrait $\mathbb{U}^2_{AC,18}$ as an evolution (see Figure 4.11). After bifurcation we get phase portrait $\mathbb{U}^1_{A,66}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,24}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution since the finite saddle cannot reach the infinite node (by item $(d)-(2)$ of Lemma 4.1) and the finite node cannot reach the infinite saddle (because this elemental antisaddle is surrounded by the separatrices of the finite saddle).

Phase portrait $\mathbb{U}^1_{A,25}$ has three phase portraits as evolution.

1. $\mathbb{U}^2_{AC,19}$, see Figure 4.12, and after bifurcation we get phase portrait $\mathbb{U}^1_{A,56}$;

2. $\mathbb{U}^2_{AC,14}$, and its study was done when we spoke about $\mathbb{U}^1_{A,18}$;

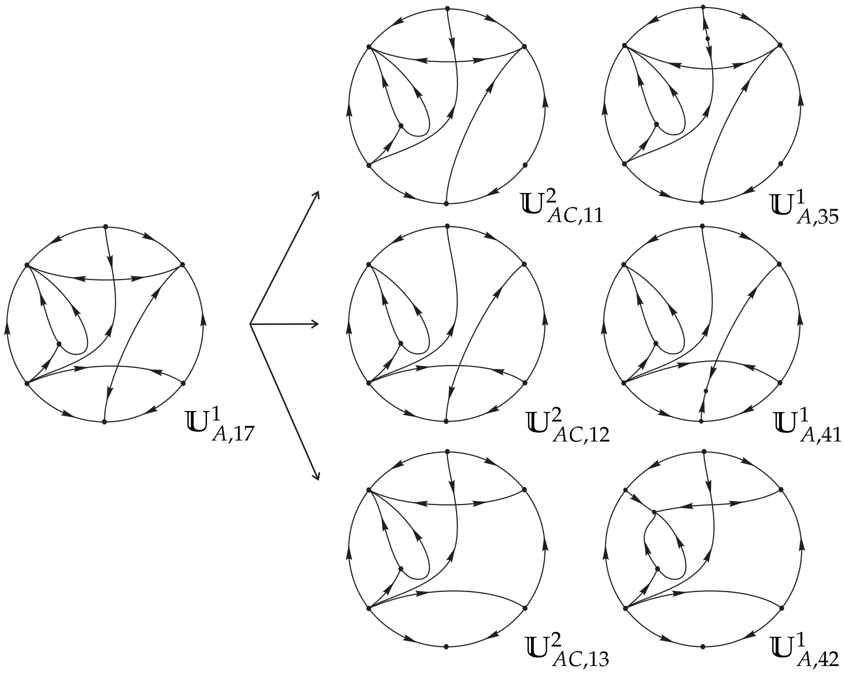


Figure 4.8: Unstable systems $\mathbb{U}^2_{AC,11}$, $\mathbb{U}^2_{AC,12}$, and $\mathbb{U}^2_{AC,13}$.

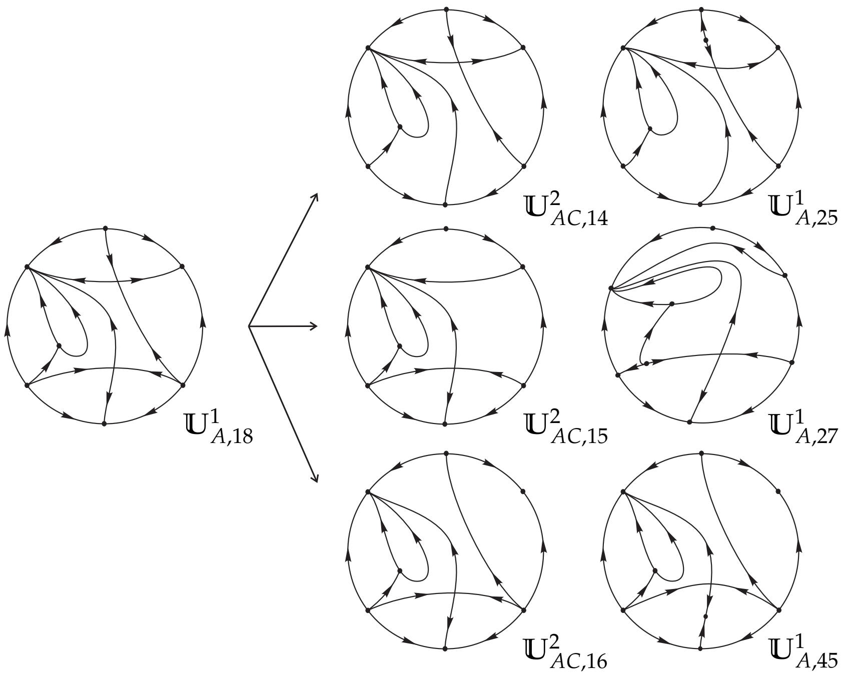


Figure 4.9: Unstable systems $\mathbb{U}^2_{AC,14}$, $\mathbb{U}^2_{AC,15}$, and $\mathbb{U}^2_{AC,16}$.

3. impossible phase portrait $\mathbb{U}^{2,I}_{AC,1}$. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{C,8}$ of *codimension one**, see Figure 4.13. We point out that, in the set (A), the corresponding unfolding of $\mathbb{U}^{2,I}_{AC,1}$ does not exist, since if such a phase portrait does exist, it would be an evolution of the impossible phase portrait $\mathbb{I}_{12,3}$ (see Figure 4.4 from [6]), which contradicts Theorem 2.11.

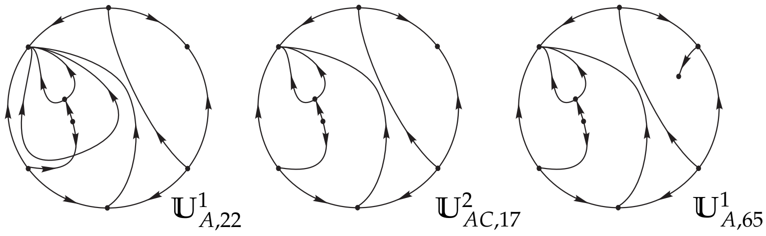


Figure 4.10: Unstable system $\mathbb{U}^2_{AC,17}$.

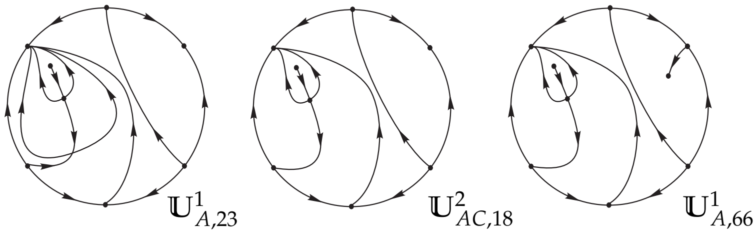


Figure 4.11: Unstable system $\mathbb{U}^2_{AC,18}$.

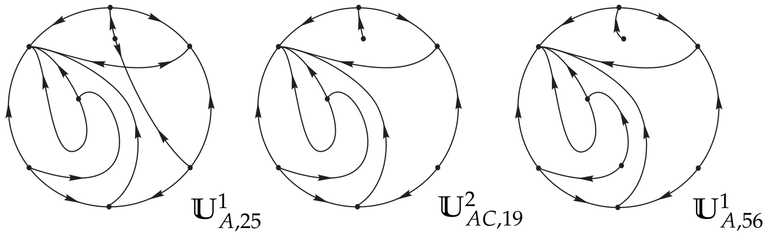


Figure 4.12: Unstable system $\mathbb{U}^2_{AC,19}$.

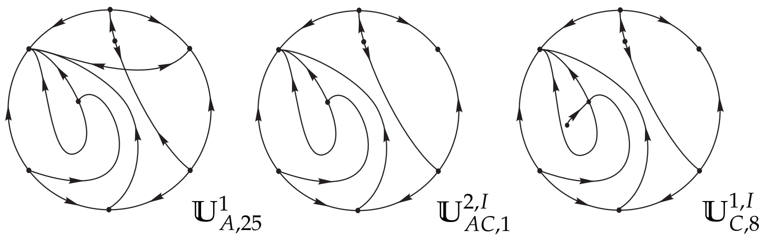


Figure 4.13: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,1}$.

Phase portrait $\mathbb{U}^1_{A,26}$ has phase portrait $\mathbb{U}^2_{AC,20}$ as an evolution (see Figure 4.14). After bifurcation we get phase portrait $\mathbb{U}^1_{A,67}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

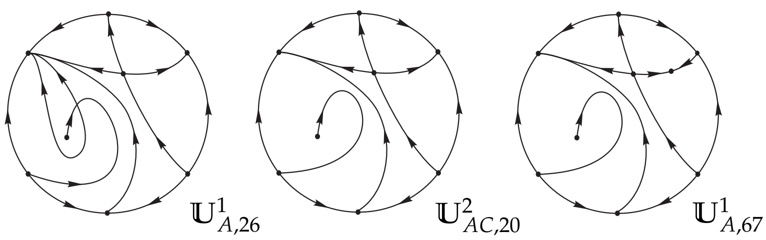


Figure 4.14: Unstable system $\mathbb{U}^2_{AC,20}$.

Phase portrait $\mathbb{U}^1_{A,27}$ has phase portraits $\mathbb{U}^2_{AC,21}$ and $\mathbb{U}^2_{AC,22}$ as evolution (see Figure 4.15). After bifurcation we get phase portraits $\mathbb{U}^1_{A,56}$ and $\mathbb{U}^1_{A,60}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover, $\mathbb{U}^1_{A,27}$ also has $\mathbb{U}^2_{AC,15}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,18}$.

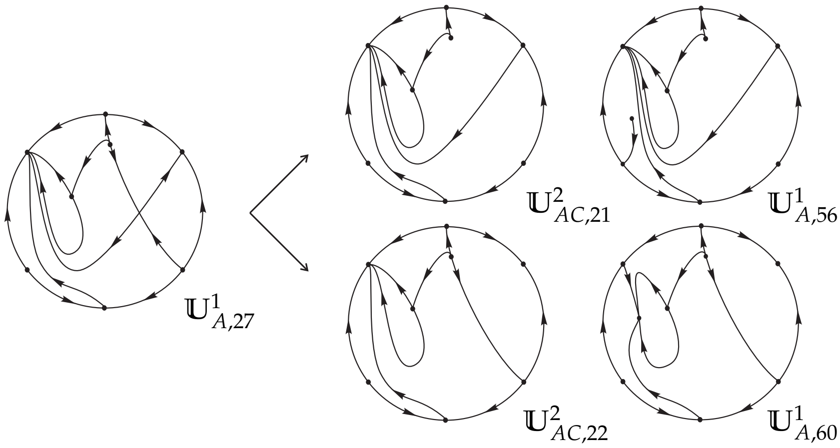


Figure 4.15: Unstable systems $\mathbb{U}^2_{AC,21}$ and $\mathbb{U}^2_{AC,22}$.

Phase portrait $\mathbb{U}^1_{A,28}$ has phase portraits $\mathbb{U}^2_{AC,23}$ and $\mathbb{U}^2_{AC,24}$ as evolution (see Figure 4.16). After bifurcation we get phase portraits $\mathbb{U}^1_{A,57}$ and $\mathbb{U}^1_{A,58}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

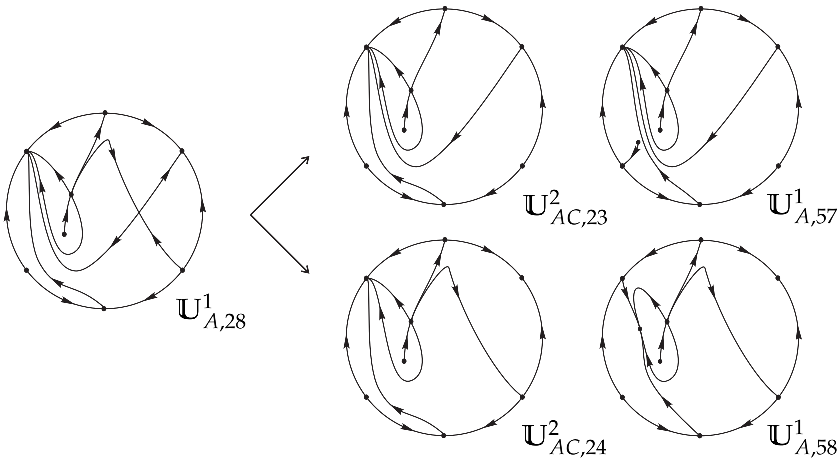


Figure 4.16: Unstable systems $\mathbb{U}^2_{AC,23}$ and $\mathbb{U}^2_{AC,24}$.

Phase portrait $\mathbb{U}^1_{A,29}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution since the finite saddle cannot reach the infinite node (by item $(d)-(2)$ of Lemma 4.1), the finite node cannot reach the infinite saddle (because this elemental antisaddle is surrounded by the separatrices of the finite saddle) and the finite saddle-node cannot go to infinity (as we have discussed during the analysis of $\mathbb{U}^1_{A,1}$).

Phase portrait $\mathbb{U}^1_{A,30}$ has phase portrait $\mathbb{U}^2_{AC,25}$ as an evolution (see Figure 4.17). After bifurcation we get phase portrait $\mathbb{U}^1_{A,69}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,31}$ has phase portrait $\mathbb{U}^2_{AC,26}$ as an evolution (see Figure 4.18). After bifurcation we get phase portrait $\mathbb{U}^1_{A,61}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,32}$ has phase portrait $\mathbb{U}^2_{AC,27}$ as an evolution (see Figure 4.19). After bifurcation we get phase portrait $\mathbb{U}^1_{A,61}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover,

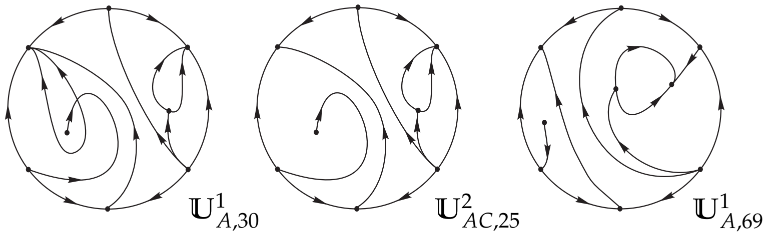


Figure 4.17: Unstable system $\mathbb{U}^2_{AC,25}$.

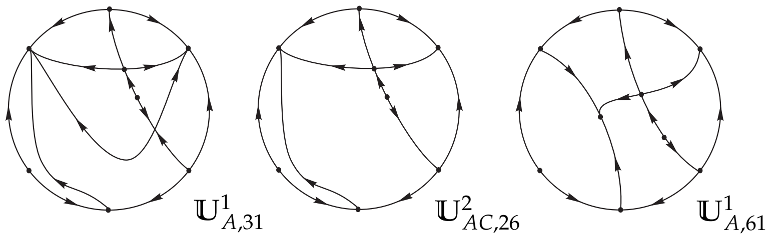


Figure 4.18: Unstable system $\mathbb{U}^2_{AC,26}$.

$\mathbb{U}^1_{A,32}$ also has $\mathbb{U}^2_{AC,6}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,15}$.

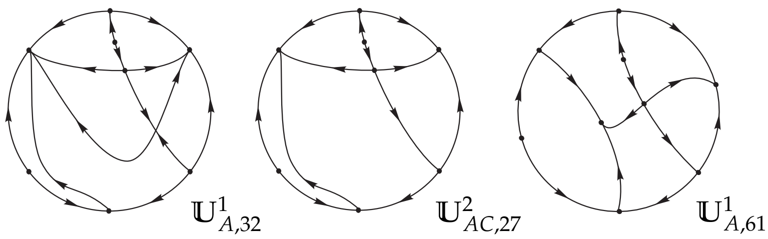


Figure 4.19: Unstable system $\mathbb{U}^2_{AC,27}$.

Phase portrait $\mathbb{U}^1_{A,33}$ has phase portrait $\mathbb{U}^2_{AC,8}$ as an evolution and this last one was mentioned before during the study of $\mathbb{U}^1_{A,16}$.

Phase portrait $\mathbb{U}^1_{A,34}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution, we can conclude this fact by using the same arguments as used for $\mathbb{U}^1_{A,29}$.

Phase portrait $\mathbb{U}^1_{A,35}$ has phase portrait $\mathbb{U}^2_{AC,11}$ as an evolution and this last one was mentioned before during the study of $\mathbb{U}^1_{A,17}$.

Phase portrait $\mathbb{U}^1_{A,36}$ has phase portrait $\mathbb{U}^2_{AC,28}$ as an evolution (see Figure 4.20). After bifurcation we get phase portrait $\mathbb{U}^1_{A,69}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

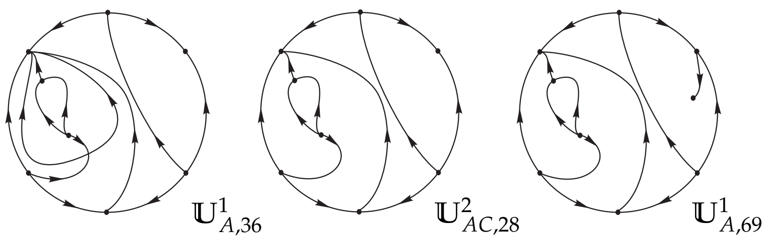


Figure 4.20: Unstable system $\mathbb{U}^2_{AC,28}$.

Phase portrait $\mathbb{U}^1_{A,37}$ has phase portrait $\mathbb{U}^2_{AC,29}$ as an evolution (see Figure 4.21). After bifurcation we get phase portrait $\mathbb{U}^1_{A,70}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

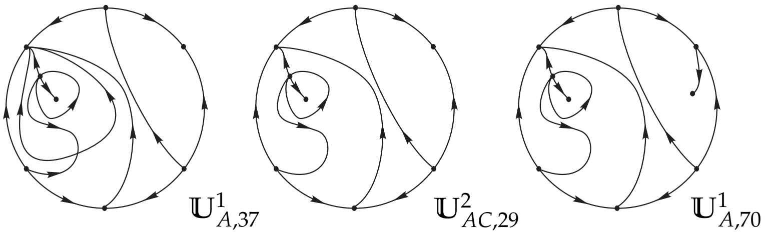


Figure 4.21: Unstable system $\mathbb{U}^2_{AC,29}$.

Phase portrait $\mathbb{U}^1_{A,38}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution.

Phase portrait $\mathbb{U}^1_{A,39}$ has phase portrait $\mathbb{U}^2_{AC,30}$ as an evolution (see Figure 4.22). After bifurcation we get phase portrait $\mathbb{U}^1_{A,65}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

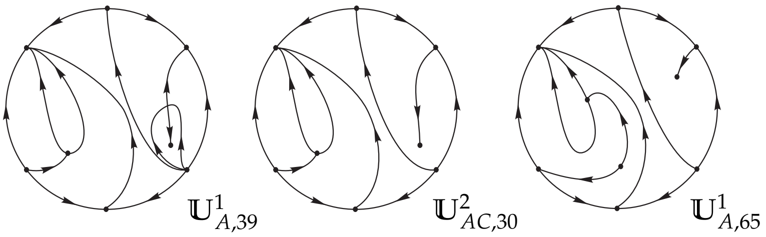


Figure 4.22: Unstable system $\mathbb{U}^2_{AC,30}$.

Phase portrait $\mathbb{U}^1_{A,40}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution.

Phase portrait $\mathbb{U}^1_{A,41}$ has three phase portraits as evolution.

1. $\mathbb{U}^2_{AC,31}$, see Figure 4.23, and after bifurcation we get phase portrait $\mathbb{U}^1_{A,63}$;

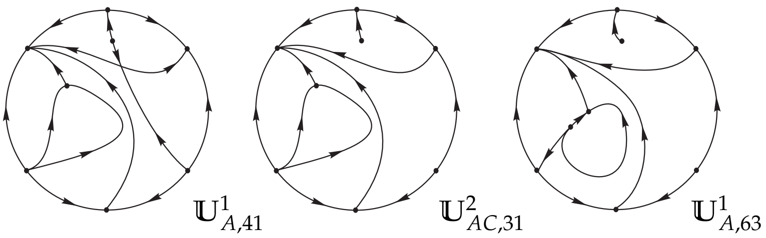


Figure 4.23: Unstable system $\mathbb{U}^2_{AC,31}$.

2. $\mathbb{U}^2_{AC,12}$, and its study was done when we spoke about $\mathbb{U}^1_{A,17}$;

3. impossible phase portrait $\mathbb{U}^{2,I}_{AC,2}$. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{C,9}$ of *codimension one**, see Figure 4.24. We point out that, in the set (A), the corresponding unfolding of $\mathbb{U}^{2,I}_{AC,2}$ does not exist, since if such a phase portrait does exist, it would be an evolution of the impossible phase portrait $\mathbb{I}_{12,2}$ (see Figure 4.4 from [6]), which contradicts Theorem 2.11.

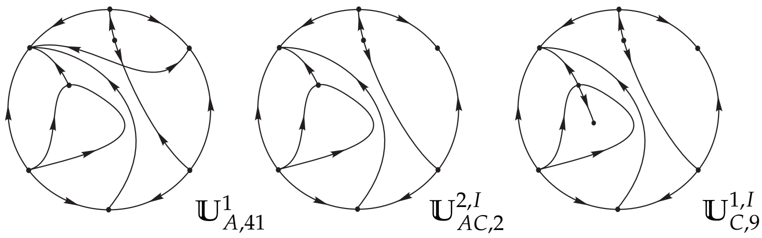


Figure 4.24: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,2}$.

Phase portrait $\mathbb{U}^1_{A,42}$ has phase portraits $\mathbb{U}^2_{AC,32}$ and $\mathbb{U}^2_{AC,33}$ as evolution (see Figure 4.25). After bifurcation we get phase portraits $\mathbb{U}^1_{A,60}$ and $\mathbb{U}^1_{A,63}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover, $\mathbb{U}^1_{A,42}$ also has $\mathbb{U}^2_{AC,13}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,17}$.

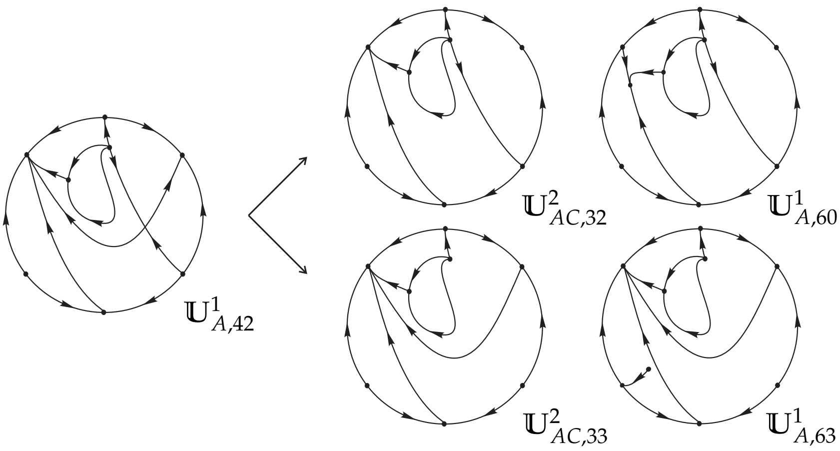


Figure 4.25: Unstable systems $\mathbb{U}^2_{AC,32}$ and $\mathbb{U}^2_{AC,33}$.

Phase portrait $\mathbb{U}^1_{A,43}$ has phase portraits $\mathbb{U}^2_{AC,34}$ and $\mathbb{U}^2_{AC,35}$ as evolution (see Figure 4.26). After bifurcation we get phase portraits $\mathbb{U}^1_{A,59}$ and $\mathbb{U}^1_{A,64}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

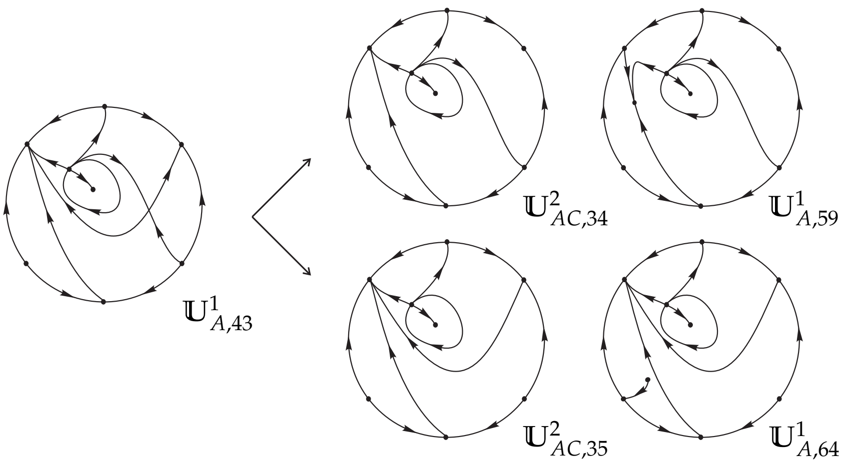


Figure 4.26: Unstable systems $\mathbb{U}^2_{AC,34}$ and $\mathbb{U}^2_{AC,35}$.

Phase portrait $\mathbb{U}^1_{A,44}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution.

Phase portrait $\mathbb{U}^1_{A,45}$ has phase portrait $\mathbb{U}^2_{AC,16}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,18}$.

Phase portraits $\mathbb{U}^1_{A,46}$ to $\mathbb{U}^1_{A,48}$ and also $\mathbb{U}^1_{A,50}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution.

Phase portrait $\mathbb{U}^1_{A,51}$ has phase portrait $\mathbb{U}^2_{AC,36}$ as an evolution (see Figure 4.27). After bifurcation we get phase portrait $\mathbb{U}^1_{A,67}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

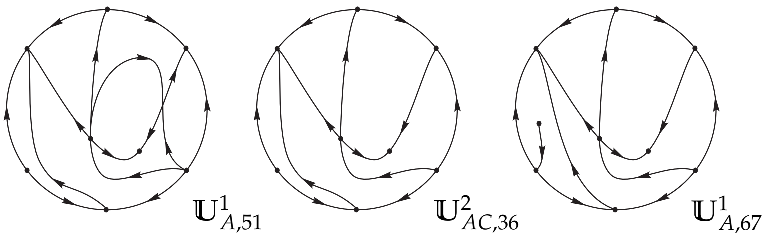


Figure 4.27: Unstable system $\mathbb{U}^2_{AC,36}$.

Phase portrait $\mathbb{U}^1_{A,52}$ has phase portrait $\mathbb{U}^2_{AC,37}$ as an evolution (see Figure 4.28). After bifurcation we get phase portrait $\mathbb{U}^1_{A,68}$, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover, $\mathbb{U}^1_{A,52}$ also has $\mathbb{U}^2_{AC,9}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,16}$.

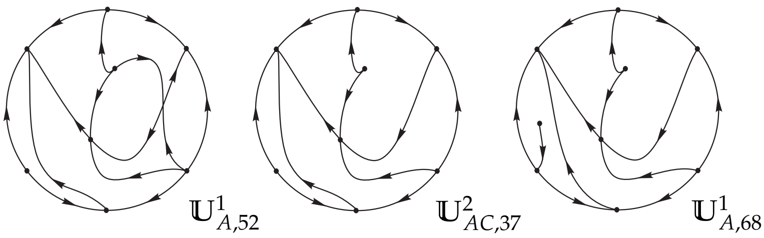


Figure 4.28: Unstable system $\mathbb{U}^2_{AC,37}$.

Phase portrait $\mathbb{U}^1_{A,53}$ has phase portrait $\mathbb{U}^2_{AC,7}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,15}$.

Phase portrait $\mathbb{U}^1_{A,54}$ has phase portrait $\mathbb{U}^2_{AC,38}$ as an evolution (see Figure 4.29). After bifurcation we get phase portrait $\mathbb{U}^1_{A,68}$, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover, $\mathbb{U}^1_{A,54}$ also has $\mathbb{U}^2_{AC,10}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,16}$.

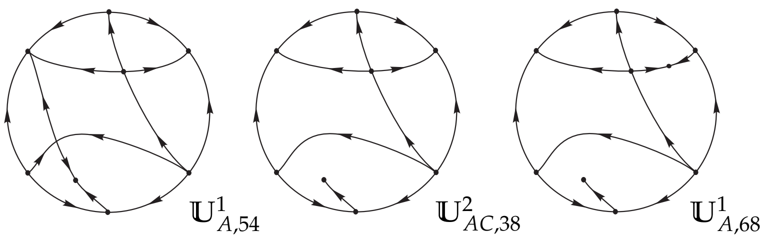


Figure 4.29: Unstable system $\mathbb{U}^2_{AC,38}$.

Phase portrait $\mathbb{U}^1_{A,55}$ has phase portraits $\mathbb{U}^2_{AC,39}$ and $\mathbb{U}^2_{AC,40}$ as evolution (see Figure 4.30). After bifurcation we get phase portraits $\mathbb{U}^1_{A,61}$ and $\mathbb{U}^1_{A,62}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover, $\mathbb{U}^1_{A,55}$ also has $\mathbb{U}^2_{AC,5}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,14}$.

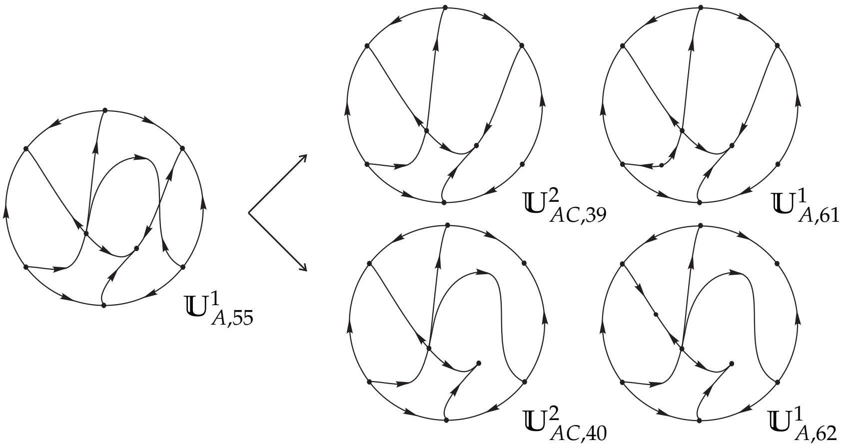


Figure 4.30: Unstable systems $\mathbb{U}^2_{AC,39}$ and $\mathbb{U}^2_{AC,40}$.

Phase portrait $\mathbb{U}^1_{A,56}$ has phase portraits $\mathbb{U}^2_{AC,19}$ and $\mathbb{U}^2_{AC,21}$ as evolution. These two phase portraits were obtained during the study of $\mathbb{U}^1_{A,25}$ and $\mathbb{U}^1_{A,27}$, respectively.

Phase portrait $\mathbb{U}^1_{A,57}$ has phase portrait $\mathbb{U}^2_{AC,23}$ as an evolution and this last one was obtained during the study of $\mathbb{U}^1_{A,28}$.

Phase portrait $\mathbb{U}^1_{A,58}$ has phase portrait $\mathbb{U}^2_{AC,24}$ as an evolution and this last one was obtained during the study of $\mathbb{U}^1_{A,28}$. Moreover, $\mathbb{U}^1_{A,58}$ has a second phase portrait which is topologically equivalent to $\mathbb{U}^2_{AC,24}$.

Phase portrait $\mathbb{U}^1_{A,59}$ has phase portrait $\mathbb{U}^2_{AC,34}$ as an evolution and this last one was obtained during the study of $\mathbb{U}^1_{A,43}$. Moreover, $\mathbb{U}^1_{A,59}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AC,3}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the obtained infinite saddle-node $\overline{\binom{1}{1}}SN$ into a finite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,104}$ of *codimension one**, see Figure 4.31. We observe that, in the set (C), $\mathbb{U}^{2,I}_{AC,3}$ unfolds in $\mathbb{U}^1_{C,17}$ (modulo limit cycles).

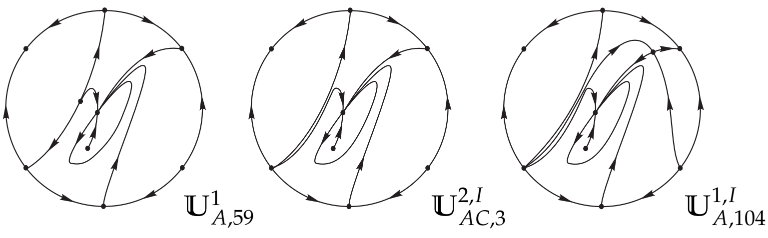


Figure 4.31: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,3}$.

In the first column of Table 4.2 we present the remaining phase portraits of the set (A), in the second column we indicate its corresponding phase portrait belonging to the set (AC), and in the third column we show the corresponding phase portrait after bifurcation. We point out that it is not necessary to present any explanation for the phase portraits present in the first column, since their corresponding elements from the third column already appeared and were explained before.

Therefore, we have just finished obtaining all the 40 topologically potential phase portraits of *codimension two** from the set (AC) presented in Figures 1.4 and 1.5.

phase portrait from the set (A)	phase portrait from the set (AC)	phase portrait from the set (A)
$\mathbb{U}^1_{A,60}$	$\mathbb{U}^2_{AC,22}$ $\mathbb{U}^2_{AC,32}$	$\mathbb{U}^1_{A,27}$ $\mathbb{U}^1_{A,42}$
$\mathbb{U}^1_{A,61}$	$\mathbb{U}^2_{AC,26}$ $\mathbb{U}^2_{AC,27}$ $\mathbb{U}^2_{AC,39}$	$\mathbb{U}^1_{A,31}$ $\mathbb{U}^1_{A,32}$ $\mathbb{U}^1_{A,55}$
$\mathbb{U}^1_{A,62}$	$\mathbb{U}^2_{AC,40}$	$\mathbb{U}^1_{A,55}$
$\mathbb{U}^1_{A,63}$	$\mathbb{U}^2_{AC,31}$ $\mathbb{U}^2_{AC,33}$	$\mathbb{U}^1_{A,41}$ $\mathbb{U}^1_{A,42}$
$\mathbb{U}^1_{A,64}$	$\mathbb{U}^2_{AC,35}$	$\mathbb{U}^1_{A,43}$
$\mathbb{U}^1_{A,65}$	$\mathbb{U}^2_{AC,17}$ $\mathbb{U}^2_{AC,30}$	$\mathbb{U}^1_{A,22}$ $\mathbb{U}^1_{A,39}$
$\mathbb{U}^1_{A,66}$	$\mathbb{U}^2_{AC,18}$	$\mathbb{U}^1_{A,23}$
$\mathbb{U}^1_{A,67}$	$\mathbb{U}^2_{AC,20}$ $\mathbb{U}^2_{AC,36}$	$\mathbb{U}^1_{A,26}$ $\mathbb{U}^1_{A,51}$
$\mathbb{U}^1_{A,68}$	$\mathbb{U}^2_{AC,37}$ $\mathbb{U}^2_{AC,38}$	$\mathbb{U}^1_{A,52}$ $\mathbb{U}^1_{A,54}$
$\mathbb{U}^1_{A,69}$	$\mathbb{U}^2_{AC,25}$ $\mathbb{U}^2_{AC,28}$	$\mathbb{U}^1_{A,30}$ $\mathbb{U}^1_{A,36}$
$\mathbb{U}^1_{A,70}$	$\mathbb{U}^2_{AC,29}$	$\mathbb{U}^1_{A,37}$

Table 4.2: Phase portraits from the set (AC) obtained from evolution of some elements of the set (A).

Now we explain how one can obtain these 40 phase portraits by starting the study from the set (C). Let us consider all the 32 realizable structurally unstable quadratic vector fields of *codimension one** from the set (C). In order to obtain a phase portrait of *codimension two** belonging to the set (AC) starting from a phase portrait of *codimension one** of the set (C), we keep the existing infinite saddle-node $\overline{\binom{1}{1}}SN$ and by using Theorem 2.6 we build a finite saddle-node $\overline{sn}_{(2)}$ by the coalescence of a finite node with a finite saddle. On the other hand, from the phase portraits of *codimension two** from the set (AC), there exist two ways of obtaining phase portraits of *codimension one** also belonging to the set (C) after perturbation: splitting $\overline{sn}_{(2)}$ into a saddle and a node, or moving it to complex singularities (remember Remark 3.2).

According to these facts, if a phase portrait possesses only a finite saddle-node, as $\mathbb{U}^1_{C,1}$ for instance, it is not possible to obtain a phase portrait from it which belongs to the set (AC). Moreover, in some cases when one makes the finite saddle-node disappear, it is possible to find a phase portrait possessing a limit cycle, as happens for instance with phase portrait $\mathbb{U}^1_{C,3}$ (see Figure 4.32). In such a figure we present the two potential phase portraits which can be obtained by forming a finite saddle-node and then by making it disappear. Indeed, phase portrait $\mathbb{U}^1_{C,3}$ has phase portraits $\mathbb{U}^2_{AC,3}$ and $\mathbb{U}^2_{AC,4}$ as evolution, respectively, by the coalescence of the finite saddle with each one of the two finite nodes. After bifurcation, by making the finite saddle-node disappear, from $\mathbb{U}^2_{AC,3}$ we get $\mathbb{U}^1_{C,1}$ and from $\mathbb{U}^2_{AC,4}$ we obtain $\mathbb{U}^1_{C,1}$, being this last one with a limit cycle. However, as our classification of phase portraits is always done modulo limit cycles we simply say that in this case from $\mathbb{U}^2_{AC,4}$ we have $\mathbb{U}^1_{C,1}$. This situation also happens when we perform analogous studies of phase portraits $\mathbb{U}^1_{C,20}$, $\mathbb{U}^1_{C,24}$, and $\mathbb{U}^1_{C,31}$, as we will see in the sequence.

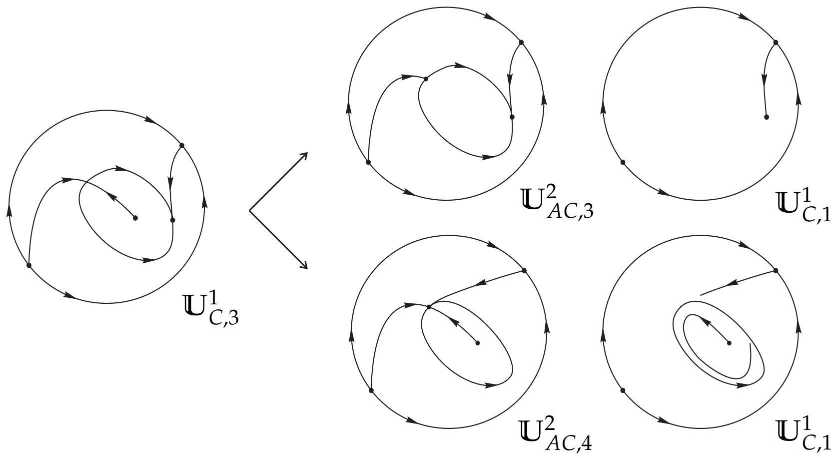


Figure 4.32: Unstable systems $\mathbb{U}^2_{AC,3}$ and $\mathbb{U}^2_{AC,4}$ from phase portrait $\mathbb{U}^1_{C,3}$.

The main goal of this section is to obtain all the topologically potential phase portraits from the set (AC) and then prove their realization or show that they are not possible. So we have to be sure that no other phase portrait can be found if one does some evolution in all elements of the set (C) in order to obtain a phase portrait belonging to the set (AC). We point out that we have done this verification, i.e. we have also considered each element from the set (C) and produced a coalescence (when it was possible) of a finite node with a finite saddle and we also have obtained the 40 topologically potential phase portraits of *codimension two** from the set (AC) presented in Figures 1.4 and 1.5. Moreover, doing this verification we have not found the impossible phase portraits $\mathbb{U}^{2,I}_{AC,1}$ and $\mathbb{U}^{2,I}_{AC,2}$ (this was expected since the corresponding unfoldings of *codimension one** are impossible in the set (C)). In Table 4.3 we present the study of phase portraits $\mathbb{U}^1_{C,2}$ to $\mathbb{U}^1_{C,19}$. In the first column of the mentioned table we present the phase portrait of the set (C), in the second column we indicate its corresponding phase portrait belonging to the set (AC) i.e. after producing a finite saddle-node $\overline{sn}_{(2)}$, and in the third column we show the corresponding phase portrait after we make this finite saddle-node $\overline{sn}_{(2)}$ disappear. Note that the sequence of indexes in the first column is not consecutive since in some phase portraits from the set (C) it is not possible to produce a finite saddle-node $\overline{sn}_{(2)}$ and then it is not possible to obtain a phase portrait belonging to the set (AC).

Phase portrait $\mathbb{U}^1_{C,20}$ has phase portraits $\mathbb{U}^2_{AC,32}$ and $\mathbb{U}^2_{AC,34}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{C,17}$ for both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, phase portrait $\mathbb{U}^1_{C,20}$ also has a phase portrait which is topologically equivalent to impossible phase portrait $\mathbb{U}^{2,I}_{AC,3}$, obtained before during the study of phase portrait $\mathbb{U}^{1,I}_{A,59}$. Again, by Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{1}{1}}SN$ into a finite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,104}$ of *codimension one**, see Figure 4.33. Also, in the set (C), $\mathbb{U}^{2,I}_{AC,3}$ unfolds in $\mathbb{U}^1_{C,17}$ (modulo limit cycles).

Phase portrait $\mathbb{U}^1_{C,21}$ has phase portraits $\mathbb{U}^2_{AC,22}$ and $\mathbb{U}^2_{AC,24}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{C,17}$ for both cases, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^1_{C,22}$ has phase portraits $\mathbb{U}^2_{AC,40}$ and $\mathbb{U}^2_{AC,39}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{C,15}$ and $\mathbb{U}^1_{C,17}$, respectively, by making the finite saddle-node $\overline{sn}_{(2)}$

phase portrait from the set (C)	**phase portrait from the set (AC)**	**phase portrait from the set (C)**
$\mathbb{U}^1_{C,2}$	$\mathbb{U}^2_{AC,1}$ $\mathbb{U}^2_{AC,2}$	$\mathbb{U}^1_{C,1}$
$\mathbb{U}^1_{C,3}$	$\mathbb{U}^2_{AC,3}$ $\mathbb{U}^2_{AC,4}$	$\mathbb{U}^1_{C,1}$
$\mathbb{U}^1_{C,5}$	$\mathbb{U}^2_{AC,14}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,6}$	$\mathbb{U}^2_{AC,15}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,7}$	$\mathbb{U}^2_{AC,6}$ $\mathbb{U}^2_{AC,8}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,8}$	$\mathbb{U}^2_{AC,11}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,9}$	$\mathbb{U}^2_{AC,12}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,10}$	$\mathbb{U}^2_{AC,13}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,11}$	$\mathbb{U}^2_{AC,16}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,12}$	$\mathbb{U}^2_{AC,7}$ $\mathbb{U}^2_{AC,9}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,13}$	$\mathbb{U}^2_{AC,10}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,14}$	$\mathbb{U}^2_{AC,5}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,18}$	$\mathbb{U}^2_{AC,21}$ $\mathbb{U}^2_{AC,23}$	$\mathbb{U}^1_{C,15}$
$\mathbb{U}^1_{C,19}$	$\mathbb{U}^2_{AC,19}$	$\mathbb{U}^1_{C,15}$

Table 4.3: Phase portraits from the set (AC) obtained from evolution of elements of the set (C).

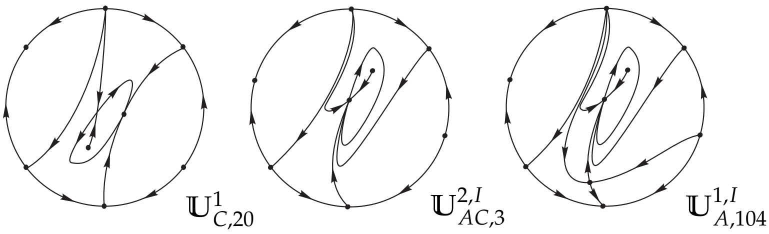


Figure 4.33: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,3}$ (see again Figure 4.31).

disappear.

Phase portrait $\mathbb{U}^1_{C,23}$ has phase portraits $\mathbb{U}^2_{AC,26}$ and $\mathbb{U}^2_{AC,27}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{C,17}$ for both cases, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^1_{C,24}$ has phase portraits $\mathbb{U}^2_{AC,33}$ and $\mathbb{U}^2_{AC,35}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{C,15}$ for both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, phase portrait $\mathbb{U}^1_{C,24}$ also has the impossible phase portrait $\mathbb{U}^{2,I}_{AC,4}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{1}{1}}SN$ into a finite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,104}$ of *codimension one**, see Figure 4.34. We observe that, in the set (C), $\mathbb{U}^{2,I}_{AC,4}$ unfolds in $\mathbb{U}^1_{C,15}$ (modulo limit cycles).

In Table 4.4 we present the study of phase portraits $\mathbb{U}^1_{C,25}$ to $\mathbb{U}^1_{C,30}$ and we follow the same

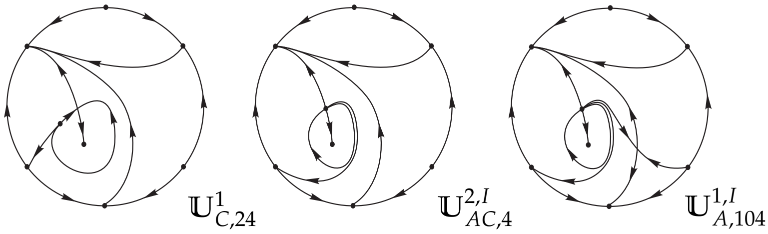


Figure 4.34: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,4}$.

pattern used in Table 4.3.

phase portrait from the set (C)	**phase portrait from the set (AC)**	**phase portrait from the set (C)**
$\mathbb{U}^1_{C,25}$	$\mathbb{U}^2_{AC,31}$	$\mathbb{U}^1_{C,15}$
$\mathbb{U}^1_{C,26}$	$\mathbb{U}^2_{AC,17}$ $\mathbb{U}^2_{AC,18}$	$\mathbb{U}^1_{C,16}$
$\mathbb{U}^1_{C,27}$	$\mathbb{U}^2_{AC,30}$	$\mathbb{U}^1_{C,16}$
$\mathbb{U}^1_{C,28}$	$\mathbb{U}^2_{AC,38}$	$\mathbb{U}^1_{C,15}$
$\mathbb{U}^1_{C,29}$	$\mathbb{U}^2_{AC,20}$	$\mathbb{U}^1_{C,16}$
$\mathbb{U}^1_{C,30}$	$\mathbb{U}^2_{AC,37}$ $\mathbb{U}^2_{AC,36}$	$\mathbb{U}^1_{C,15}$ $\mathbb{U}^1_{C,16}$

Table 4.4: Phase portraits from the set (AC) obtained from evolution of elements of the set (C).

Phase portrait $\mathbb{U}^1_{C,31}$ has phase portraits $\mathbb{U}^2_{AC,28}$ and $\mathbb{U}^2_{AC,29}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{C,16}$ for both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, phase portrait $\mathbb{U}^1_{C,31}$ also has the impossible phase portrait $\mathbb{U}^{2,I}_{AC,5}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{1}{1}}SN$ into an infinite saddle and a finite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,106}$ of *codimension one**, see Figure 4.35. We observe that, in the set (C), $\mathbb{U}^{2,I}_{AC,5}$ unfolds in $\mathbb{U}^1_{C,16}$ (modulo limit cycles).

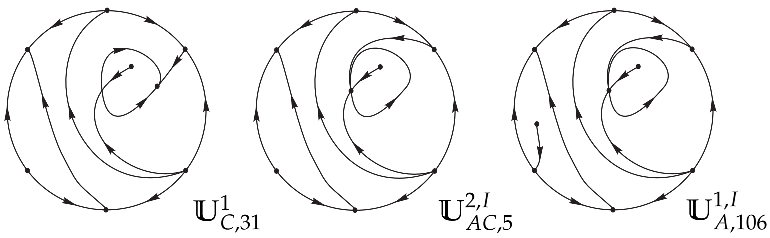


Figure 4.35: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,5}$.

4.2 The realization of the potential phase portraits

In the previous subsection we have produced all the 42 topologically potential phase portraits for structurally unstable quadratic systems of *codimension two** belonging to the set $\sum_2^2(AC)$.

And from them, we have already discarded two which are not realizable due to their respective unfoldings of *codimension one** being impossible.

In this subsection we aim to give specific examples for the 40 different topological classes of structurally unstable quadratic systems of *codimension two** belonging to the set $\sum_2^2(AC)$ and presented in Figures 1.4 and 1.5. As in the previous section (see page 61), we point out that we have found examples with no evidence of limit cycles, but we have not proved the absence of infinitesimal ones.

In [10] the authors classified, with respect to a specific normal form, the set of all real quadratic polynomial differential systems with a finite semi-elemental saddle-node $\overline{sn}_{(2)}$ located at the origin of the plane and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ obtained by the coalescence of a finite antisaddle (respectively, finite saddle) with an infinite saddle (respectively, infinite node).

As we have discussed in the previous section, the study of a bifurcation diagram of a certain family of quadratic systems, produces not only the class of phase portraits looked for, but also all those of their closure according to the normal form used. Even though the study is mainly algebraic, often, also analytic and numerical tools are required. This makes that these studies may be not complete and subject to the existence of possible "islands" which contain an undetected phase portrait. The border of that "island" could mean the connection of two separatrices, and the interior contain a different phase portrait from the ones stated in the theorem. The topological study that we do in this paper solves partially this problem, since we prove that all the realizable phase portraits of class (AC) do really exist, and no other topological possibility does. However, the possible existence of "islands" in the bifurcation diagram still persists since they can be related with double limit cycles, as discussed in Section 6 of [10].

By using the phase portraits of generic regions of the bifurcation diagram of the mentioned paper we realize all the 40 unstable systems of *codimension two** of the set (AC), i.e. we can give concrete examples of all structurally unstable phase portraits from the set (AC).

Consider systems (2.4). Such a normal form was studied in [10] and it describes quadratic polynomial differential systems which have a finite semi-elemental saddle-node $\overline{sn}_{(2)}$, a finite elemental singularity and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$.

In Tables 4.5 and 4.6 we present one representative from each generic region of the bifurcation diagram of [10] corresponding to each phase portrait of *codimension two** from the set (AC) and, therefore, we conclude the proof of Theorem 1.7.

Cod 2*	**[10]**	**c**	**e**	**h**	**m**
$\mathbb{U}^2_{AC,1}$	V_{38}	-10	30	1	4
$\mathbb{U}^2_{AC,2}$	V_{1}	6	$81/2$	1	4
$\mathbb{U}^2_{AC,3}$	V_{33}	-7	$5/2$	1	4
$\mathbb{U}^2_{AC,4}$	V_{53}	2	$47/50$	1	$37/100$
$\mathbb{U}^2_{AC,5}$	V_{13}	-1	-10	1	4
$\mathbb{U}^2_{AC,6}$	V_{4}	7	15	1	4
$\mathbb{U}^2_{AC,7}$	V_{21}	$-9/4$	-10	1	4
$\mathbb{U}^2_{AC,8}$	V_{92}	-3	$7/2$	1	$-6/5$
$\mathbb{U}^2_{AC,9}$	V_{10}	$1/2$	$-11/2$	1	4
$\mathbb{U}^2_{AC,10}$	V_{63}	$-2/5$	$1/50$	1	$-1/4$
$\mathbb{U}^2_{AC,11}$	V_{95}	-3	$31/10$	1	$-6/5$
$\mathbb{U}^2_{AC,12}$	V_{73}	$-19/10$	$17/20$	1	$-3/4$
$\mathbb{U}^2_{AC,13}$	V_{8}	$3/2$	$-9/2$	1	4
$\mathbb{U}^2_{AC,14}$	V_{93}	-1	$11/10$	1	$-6/5$
$\mathbb{U}^2_{AC,15}$	V_{6}	$24/5$	$-4/5$	1	4
$\mathbb{U}^2_{AC,16}$	V_{68}	-3	$2/5$	1	$-1/4$
$\mathbb{U}^2_{AC,17}$	V_{39}	-25	30	1	4
$\mathbb{U}^2_{AC,18}$	V_{3}	$45/2$	98	1	4
$\mathbb{U}^2_{AC,19}$	V_{62}	$-1/40$	$1/50$	1	$-1/4$
$\mathbb{U}^2_{AC,20}$	V_{80}	$-6/5$	$1207/1000$	1	-1
$\mathbb{U}^2_{AC,21}$	V_{81}	$29/50$	$-3/5$	1	$-6/5$
$\mathbb{U}^2_{AC,22}$	V_{36}	-1	4	1	4
$\mathbb{U}^2_{AC,23}$	V_{23}	$-9/2$	-17	1	4
$\mathbb{U}^2_{AC,24}$	V_{112}	$1/2$	42	1	-10
$\mathbb{U}^2_{AC,25}$	V_{77}	$-5/4$	$629/500$	1	$-49/50$
$\mathbb{U}^2_{AC,26}$	V_{90}	$-9/5$	$881/400$	1	$-6/5$
$\mathbb{U}^2_{AC,27}$	V_{2}	1	7	1	4
$\mathbb{U}^2_{AC,28}$	V_{35}	$-1747/50$	30	1	4
$\mathbb{U}^2_{AC,29}$	V_{49}	10	$5156/625$	1	$51/100$
$\mathbb{U}^2_{AC,30}$	V_{65}	$-23/50$	$1151/10000$	1	$-1/4$
$\mathbb{U}^2_{AC,31}$	V_{59}	$-1/50$	$1/40$	1	$-1/4$
$\mathbb{U}^2_{AC,32}$	V_{29}	$-3/2$	$1/2$	1	4
$\mathbb{U}^2_{AC,33}$	V_{82}	$1341/2000$	$-3/5$	1	$-6/5$
$\mathbb{U}^2_{AC,34}$	V_{102}	$1/100$	$31/10$	1	$-5/2$

Table 4.5: Correspondence between *codimension two** phase portraits of the set (AC) and phase portraits from Figures 1 and 2 in [10]. In the first column we present the *codimension two** phase portraits from the set (AC) in the present paper, in the second column we show the corresponding phase portraits from Figures 1 and 2 in [10] given by normal form (2.4), and in the other columns we present the values of the parameters c, e, h, and m of (2.4) which realizes such phase portrait.

Cod 2*	**[10]**	**c**	**e**	**h**	**m**
$\mathbb{U}^2_{AC,35}$	V_{26}	$-687/50$	-17	1	4
$\mathbb{U}^2_{AC,36}$	V_{20}	$-21/10$	$-41/5$	1	4
$\mathbb{U}^2_{AC,37}$	V_{51}	10	$151/20$	1	$3/4$
$\mathbb{U}^2_{AC,38}$	V_{71}	$-1/10000$	$3/125$	1	$-1/4$
$\mathbb{U}^2_{AC,39}$	V_{14}	$-3/2$	-4	1	4
$\mathbb{U}^2_{AC,40}$	V_{55}	$1/100$	$1/100$	1	$-1/4$

Table 4.6: Continuation of Table 4.5.

5 Graphics and limit cycles

Even though the goal of this paper deals little with graphics and limit cycles, there is no doubt that these are two of the most important elements in qualitative theory of ordinary differential equations.

Limit cycles are the most elusive phenomena in phase portraits. They may appear either by a bifurcation of a weak focus (Hopf bifurcation), by a bifurcation of a graphic, or by a bifurcation of a multiple limit cycle, and only the first case can be fully algebraically controlled. The other cases are generically nonalgebraic. In fact, weak foci can be considered among graphics, since they can be seen as graphics reduced to a single point.

Our goal to find all the topologically different phase portraits modulo limit cycles bypasses this big problem, but it is not an irrelevant goal. Whenever the mathematical community finally gets the complete set of phase portraits of quadratic systems (or whatever other family), the subset of the phase portraits modulo limit cycles will be the base for such a classification. It is expected to obtain more than one thousand (maybe even up to 2000) different phase portraits of quadratic systems modulo limit cycles. For quite many of them it will be trivial to determine that they will not have limit cycles (in the case they do not have a finite antisaddle). But for all the others, it will be necessary to determine exactly how many different phase portraits can be obtained from that skeleton by adding limit cycles. Up to now and up to our knowledge, there are very few nontrivial skeletons of phase portraits which could theoretically have limit cycles, and for which the absence of limit cycles has been proved. To be more precise, we are only completely sure of one of them, namely the structurally stable phase portrait $S^2_{7,1}$. This phase portrait was obtained in [2] and was conjectured by statistical tools to be incompatible with limit cycles in [4] and this conjecture was proved in [5]. Also in [4] some other phase portraits are conjectured (by statistical data) to be incompatible with limit cycles, but no proof is available yet. Apart from these last ones, other candidates can be found in Class I of [37]. In that paper the authors produce three normal forms (denoted by I, II and III) and they prove that any system with limit cycle can be transformed in an element of them. The three classes have no intersection since they deal with the number of finite singularities that have gone to infinity ($\geq$ 2, 1 and 0, respectively). And in [37] it is also proved that systems from Class I have at most one limit cycle. There is still no conclusive study of phase portraits from Class I, but some phase portraits of this class have already been found having one limit cycle and some others with no limit cycle (see [15, 23, 34]). For the cases with limit cycle, it is closed the fact that such phase portraits can have at most one limit cycle, and if a conclusive study is done and results are confirmed, the cases with no limit cycle would add to the phase portrait $S^2_{7,1}$ as skeletons of phase portraits without limit cycles. For all other skeletons of phase portraits found up to now, there is not a single proof determining which is the maximum number of limit cycles that each one may have. There are many other papers related to the maximum number of limit cycles, but they are mostly linked to a certain normal form. Most of them simply prove that a specific normal form may have just one limit cycle. But this does not imply that the skeletons of phase portraits obtained in that normal form may have more limit cycles in the entire classification.

Up to now, it is known that there are examples of phase portraits of quadratic systems with four limit cycles distributed into two nests around two foci, more precisely, three limit cycles in one nest and the fourth limit cycle in the other nest. And even though it is conjectured that the effective maximum is four with the distribution just mentioned, there is still no conclusive proof. The phase portraits for which there are examples with four limit cycles belong to three skeletons of phase portraits, namely, the structurally stables $\mathbb{S}^2_{4,1}$ and $\mathbb{S}^2_{11,2}$ from [2], and the *codimension one** $\mathbb{U}^1_{B,31}$ from [6]. The proof that they may have at least four limit cycles appears in several papers since they appear in classifications with a weak focus of order three, already having a limit cycle around a strong focus.

But not even if the maximum bound was four, we would not be close to obtain all the phase portraits of quadratic systems. Any of the three skeletons mentioned before may have the topologically different configurations $(0,0)$, $(1,0)$, $(2,0)$, $(3,0)$, $(1,1)$, $(2,1)$, and $(3,1)$. That is, seven different configurations. But even that is not a criterion (that is, multiply the number of skeletons by 7) to obtain a simple upper bound for the total number of phase portraits. There are phase portraits like $\mathbb{S}^2_{5,1}$ from [2] which has three finite antisaddles. One of them receives (or emits) a single separatrix, the second one receives (or emits) exactly two separatrices, and the third one receives (or emits) exactly three separatrices. So, the fact that a limit cycle could be surrounding any of the three antisaddles would generate a topologically different phase portrait. And in case there were two nests of limit cycles, and assuming that they could have up to four limit cycles, the number of cases would increase up to 25 possibilities. But from these 25 possibilities, up to now only six have been confirmed to exist. We are collecting a large database and recording the maximum number of limit cycles found in each one of the skeletons classified up to now.

With all these facts we want to remark that the topological classification of phase portraits modulo limit cycles is important since it produces a complete set of skeletons from which all the complete set of phase portraits must be located. For each particular skeleton, it must be studied if it contains none, one, two or up to three antisaddles around which the limit cycles may be located. If there is a complete collection of phase portraits modulo limit cycles, and if an upper bound of limit cycles is found, it will give a quite rough upper bound for the number of different phase portraits. But the real number will need a deeper study case by case. Nowadays, the moment that we could have a complete topological classification is quite far away. However, the topological classification modulo limit cycles is within reach, although they are not easily reachable yet.

Let us now talk about graphics. Graphics are also very important because they can become the bifurcation edge which leads to the birth of limit cycles. There has been a lot of literature related to graphics, and one of the most relevant papers is [19] where the authors list a set of 121 different graphics whose finite cyclicity needs to be proved in order to prove the finiteness part of Hilbert 16th problem for quadratic systems. The graphics in this list can be of different types. Many of them imply the connection of one (or more) couple of separatrices, finite or infinite. Other graphics are formed simply because a separatrix arrives to the nodal part of a saddle-node (finite or infinite) or an even more degenerate singularity in coexistence with other properties of the phase portrait. Unfortunately, most of these graphics cannot be detected by means of algebraic tools. In many studies of families of systems where a complete bifurcation is given in the parameter space, after all the algebraic bifurcations are given, the use of continuity and coherence arguments allows the detection of some other nonalgebraic bifurcations where these graphics appear.

Our methodical study of phase portraits of quadratic systems modulo limit cycles started with codimension zero (structurally stable) [2] and of course these phase portraits cannot have any graphic at all. The second step was the classification of codimension-one phase portraits (modulo limit cycles), and in that study we could start finding some graphics, but not too many. Precisely, we found graphic (F_2^1) from [19] in $\mathbb{U}^1_{A,37}$, $\mathbb{U}^1_{A,43}$, $\mathbb{U}^1_{A,64}$, and $\mathbb{U}^1_{A,70}$. This graphic is formed simply by one finite saddle-node which sends its center manifold (separatrix of zero eigenvalue) to its own nodal part. We also have graphic (I_{19}^2) from [19] in $\mathbb{U}^1_{B,29}$, $\mathbb{U}^1_{B,30}$ (twice), $\mathbb{U}^1_{B,33}$, $\mathbb{U}^1_{B,36}$, and $\mathbb{U}^1_{B,38}$. This graphic is formed by one elemental infinite saddle which sends one of its separatrices to the nodal part of an infinite adjacent saddle-node formed by the coalescence of two infinite singularities. There are no graphics in the set (C) of codimension-one phase portraits (modulo limit cycles, see page 4). Finally, in the set (D) (see again page 4) we found the graphics (F_1^1), (H_1^1), and (I_1^2) from [19]. The first one is just a loop of a finite elemental saddle, the second one is a separatrix connection between opposite infinite elemental saddles, and the third one is a separatrix connection between adjacent infinite elemental saddles. The loop is present in $\mathbb{U}^1_{D,1}$, $\mathbb{U}^1_{D,6}$, $\mathbb{U}^1_{D,7}$, $\mathbb{U}^1_{D,8}$, $\mathbb{U}^1_{D,9}$, $\mathbb{U}^1_{D,12}$, $\mathbb{U}^1_{D,19}$, $\mathbb{U}^1_{D,20}$, $\mathbb{U}^1_{D,22}$, $\mathbb{U}^1_{D,23}$, $\mathbb{U}^1_{D,30}$, $\mathbb{U}^1_{D,31}$, $\mathbb{U}^1_{D,32}$, $\mathbb{U}^1_{D,46}$, $\mathbb{U}^1_{D,47}$, $\mathbb{U}^1_{D,48}$, $\mathbb{U}^1_{D,49}$, $\mathbb{U}^1_{D,50}$, $\mathbb{U}^1_{D,51}$, $\mathbb{U}^1_{D,52}$, $\mathbb{U}^1_{D,53}$, and $\mathbb{U}^1_{D,54}$. The second graphic appears in $\mathbb{U}^1_{D,10}$ and $\mathbb{U}^1_{D,11}$. And the third one can be seen in $\mathbb{U}^1_{D,28}$, $\mathbb{U}^1_{D,29}$, $\mathbb{U}^1_{D,37}$, $\mathbb{U}^1_{D,38}$, and $\mathbb{U}^1_{D,39}$. No other graphic from these last five may appear, since all the remaining 116 imply higher codimension.

Thus, in our current study of phase portraits of *codimension two** with a finite saddle-node and an infinite saddle-node, the only graphics that we can see will be those ones which are inherited from the respective phase portraits of *codimension one** already having a graphic. No new graphic may appear from the consolidation of the two different instabilities we mix here. In the studies of the sets (AD), (BD), and (CD) we will start incorporating more graphics from [19], since we will find, for example, saddle-nodes forming a loop instead of an elemental saddle. Also the set (DD) will provide graphics with two separatrix connections. Anyway, the graphics will appear in larger numbers when *codimension three** is studied.

There is another important fact, related to stability and graphics, to comment about the classification that we are working with. As mentioned in Section 1, in [6] it is claimed that there are at least 204 structurally unstable phase portraits of *codimension one** and at most 211. Two papers have found two mistakes in that book and the newly proved numbers are 202 and 209, respectively. The seven cases that have not been found correspond to cases which are conjectured as impossible and some arguments are given to support that conjecture. We point out that all the seven cases conjectured impossible contain a graphic, more precisely the polycycles (F_2^1) or (H_1^1). These phase portraits consist in an skeleton of separatrices which depending on the stability of the focus inside the polycycle (compared to other stabilities outside it) may lead or not to a realizable phase portrait. That is, they lead to a phase portrait which is already known to exist, or lead to a phase portrait which (up to our knowledge) never appeared before in any paper. The normal techniques which have allowed us to prove the impossibility of hundreds of phase portraits are useless in these seven cases. All we can say about these seven phase portraits is that in case they exist, some perturbations from them would produce phase portraits with a limit cycle that we have not found anywhere. Using the tools of perturbations related to stability that we use in this paper, we may claim that if one of those phase portraits with a limit cycle could be proven impossible, then the related unstable phase portrait with a polycycle would be also impossible. However, the opposite is not true. If the phase portrait with a limit cycle does exist, it is not sure that the related unstable phase portrait with a polycycle may exist. There is the possibility that by means of a rotated vector field one could pass from one to the other, but it is not guaranteed.

So, we see once more the importance of graphics and limit cycles in the classification of phase portraits. The fact that we talk so little about limit cycles is simply because we want to do the classification modulo limit cycles in order to have a good base upon which we or others may add the limit cycles. And the fact that we talk so little about graphics is because at the level of codimension that we are in this stage, there appear very few of the 121 graphics described in [19].

Acknowledgements

We thank the reviewers of this paper for the valuable suggestions for our text and for having indicated the manuscript for publication. The first author is partially supported by a MEC/FEDER grant number MTM 2016-77278-P and by a CICYT grant number 2017 SGR 1617. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brazil (CAPES) - Finance Code 001 (the second author is partially supported by this grant). The third author is partially supported by CAPES, by FAPESP Processo no. 2018/21320-7, and by FAPESP Processo no. 2019/21181-0.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **36**, 1–28; https://doi.org/10.14232/ejqtde.2021.1.36 www.math.u-szeged.hu/ejqtde/

Asymptotic phase for flows with exponentially stable partially hyperbolic invariant manifolds

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Received 25 August 2020, appeared 16 April 2021
Communicated by Sergei Trofimchuk

Abstract. We consider an autonomous system admitting an invariant manifold $\mathcal{M}$. The following questions are discussed: (i) what are the conditions ensuring exponential stability of the invariant manifold? (ii) does every motion attracting by $\mathcal{M}$ tend to some motion on $\mathcal{M}$ (i.e. have an asymptotic phase)? (iii) what is the geometrical structure of the set formed by orbits approaching a given orbit? We get an answer to (i) in terms of Lyapunov functions omitting the assumption that the normal bundle of $\mathcal{M}$ is trivial. An affirmative answer to (ii) is obtained for invariant manifold $\mathcal{M}$ with partially hyperbolic structure of tangent bundle. In this case, the existence of asymptotic phase is obtained under new conditions involving contraction rates of the linearized flow in normal and tangential to $\mathcal{M}$ directions. To answer the question (iii), we show that a neighborhood of $\mathcal{M}$ has a structure of invariant foliation each leaf of which corresponds to motions with common asymptotic phase. In contrast to theory of cascades, our technique exploits the classical Lyapunov–Perron method of integral equations.

Keywords: invariant manifold, exponential stability, asymptotic phase, partially hyperbolic dynamical system

2020 Mathematics Subject Classification: 34C45, 34D35, 37D10, 37D30

1 Introduction

It is well known that, under quite general conditions, motions of dissipative dynamical system evolve towards attracting invariant sets. One may reasonably expect that the behavior of system on attracting set adequately displays main asymptotic properties of system motions in the whole phase space. It is important to note that in many cases the dimension of attracting set such as, e.g., fixed point, limit cycle, invariant torus, strange or chaotic attractor, is essentially lower than the dimension of the total phase space. This circumstance can help us to simplify the qualitative analysis of the system under consideration.

Nevertheless we should keep in mind that there are cases where no motion starting outside the attracting invariant set exhibits the same long time behavior as a motion on the set. As an

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example, consider the polynomial planar system

$$\begin{aligned} \dot{x} &= x(1-x^2-y^2)^3 - y(1+x^2+y^2), \\ \dot{y} &= x(1+x^2+y^2) + y(1-x^2-y^2)^3 \end{aligned}$$

which in polar coordinates $(\varphi|\bmod 2\pi, r)$ takes the form

$$\begin{aligned} \dot{\varphi} &= 1+r^2, \\ \dot{r} &= r\left(1-r^2\right)^3. \end{aligned}$$

The limit cycle of the system given by $r=1$ attracts all the orbits except the equilibrium $(0,0)$. Let $\varphi(t;\varphi_0,r_0)$ be the φ-coordinate of the motion starting at point $(r_0\cos\varphi_0, r_0\sin\varphi_0)$. Obviously, $\varphi(t;\varphi_*,1)=2t+\varphi_*$, but if $r_0\notin\{0,1\}$, then it is not hard to show that

$$\lim_{t\to\infty}|\varphi(t;\varphi_0,r_0)-\varphi(t;\varphi_*,1)|=\infty \qquad \forall\,\{\varphi_0,\varphi_*\}\subset[0,2\pi),$$

meaning that there is no motion starting outside the cycle and asymptotic to a motion on the cycle (for another examples with non-polynomial planar systems we refer the reader to [11, 14]).

Let $\left\{\chi^t(\cdot):\mathfrak{M}\mapsto\mathfrak{M}\right\}_{t\in\mathbb{R}}$ (resp. $\left\{\chi^t(\cdot):\mathfrak{M}\mapsto\mathfrak{M}\right\}_{t\in\mathbb{Z}}$) be a flow (resp. a cascade) on a metric space $\mathfrak{M}$ with metric $\varrho(\cdot,\cdot)$, and let there exists a χ^t-invariant set $\mathcal{A}\subset\mathfrak{M}$. It is said that a motion $t\mapsto\chi^t(x)$ attracted by $\mathcal{A}$ has an *asymptotic phase* if there exists $z\in\mathcal{A}$ such that

$$\varrho\left(\chi^t(x),\chi^t(z)\right)\to 0, \qquad t\to\infty.$$

The following problem arises: what are the conditions ensuring the existence of asymptotic phase? The answer to this problem is rather important, especially in the case where $\mathcal{A}$ is an attractor with a basin $\mathfrak{B}$. In fact, the existence of asymptotic phase for every $x\in\mathfrak{B}$ guarantees that the flow restricted to attractor $\mathcal{A}$ faithfully describes the long-time behavior of the motions starting in $\mathfrak{B}$.

The above problem was studied in a series of papers. The most complete examination concerns the case where the attracting set is a closed orbit [7, 11, 12, 14, 19, 31]. For more general situation, it is known that if $\mathcal{A}$ is an isolated compact invariant *hyperbolic* set of a cascade, then every motion which is asymptotic to such a set has an asymptotic phase [21, 26]. N. Fenichel [16] established the existence and uniqueness of asymptotic phase for a cascade possessing exponentially stable overflowing invariant manifold with, so-called, expanding structure. A. M. Samojlenko [28] and W. A. Coppel [13] studied the problem for the case of exponentially stable invariant torus. B. Aulbach [4] proved the existence of asymptotic phase for motions approaching a normally hyperbolic invariant manifold under assumption that the latter carries a parallel flow. In [8], A. A. Bogolyubov and Yu. A. Il'in established the existence of asymptotic phase for non-exponentially stable invariant torus under some quite restrictive hypotheses concerning the corresponding system (however the authors do not use the notion of asymptotic phase explicitly).

As was pointed out in [4, 10], standard conditions ensuring the existence of asymptotic phases for motions approaching an invariant set $\mathcal{A}$, involve the requirement that the exponential rate of contraction in the normal to $\mathcal{A}$ direction is greater than that along $\mathcal{A}$ (see, e.g., [6, 16, 28]). Analogous conditions usually appear in the perturbation theory of invariant manifolds (see, e.g. [15, 17, 23, 27, 29] and references therein).

One of the main goals of the present paper is to show that the aforementioned requirement can be weakened in the presence of more accurate information about the character of the flow within the invariant manifold. We consider an autonomous system in $\mathbb{R}^n$ admitting an invariant manifold $\mathcal{M}$ satisfying the following condition of *partial hyperbolicity in the broad sense* [9, 20]: the tangent co-cycle generated by the associated linearized system (system in variations) splits the tangent bundle $T\mathcal{M}$ into a Whitney sum of two invariant sub-bundles V^s and $V^\star$ such that the maximal Lyapunov exponent corresponding to V^s does not exceed some negative number $-\nu$, while the minimal Lyapunov exponent corresponding to $V^\star$ is not less then $-\sigma \in (-\nu, 0)$. (In an important particular case, where the restriction of the flow on $\mathcal{M}$ is an Anosov type dynamical system, the tangent bundle splits into Whitney sum $T\mathcal{M} = V^s \oplus V^c \oplus V^u$ of invariant sub-bundles: stable V^s, center V^c, and unstable V^u. Then $V^\star = V^c \oplus V^u$ and one can consider that $\sigma = 0$.)

It should be stressed that a priori we do not require that $\mathcal{M}$ is a partially hyperbolic set as a subset of the whole space $\mathbb{R}^n$, in particular, the Whitney sum of V^s and normal bundle of $\mathcal{M}$ need not be invariant. Nevertheless, we prove that if the decay rate of solutions of linearized system in normal to $\mathcal{M}$ direction is characterized by a Lyapunov exponent $-\gamma < 0$, then the inequality $\lambda := \min\{\nu, \gamma\} > \sigma$ guarantees both the partial hyperbolicity of $\mathcal{M}$ and the existence of asymptotic phase for all motions starting in a neighborhood of $\mathcal{M}$. Thus, we need not require any additional inequalities involving ν and γ, meaning that our result cover the case $\nu > \gamma$ which, to our knowledge, was excluded in preceding papers concerning the asymptotic phase.

If there holds the inequality $\nu \geq \gamma$, then in contrast to [16], we cannot be sure that the asymptotic phase is unique. The reason lies in the geometrical structure of a neighborhood of $\mathcal{M}$. Namely, let $\mathcal{W}(z)$ be the stable manifold for a point $z \in \mathcal{M}$ [26, p. 88] (i.e. $\mathcal{W}(z)$ is the set of points $x \in \mathbb{R}^n$ such that $\|\chi^t(x) - \chi^t(z)\| = O\left(\mathrm{e}^{-\lambda t}\right)$, $t \to \infty$). In our case, we cannot exclude that $\mathcal{W}(z_1) = \mathcal{W}(z_2)$ for different points $z_1 \neq z_2$. As a consequence, when proving that every motion starting in a neighborhood of the invariant manifold $\mathcal{M}$ has an asymptotic phase, we are not able to apply the theorem on invariance of domain as in [16] . Our proof is based on the Brouwer fixed point theorem.

In contrast to the technique developed for cascades, e.g., in [16, 21–23, 26], our main results concerning theory of asymptotic phase are obtained by exploiting the classical Lyapunov–Perron method of integral equations. With this in mind, and targeting on the rather general readers audience we intentionally provide independent proofs of some facts on the invariant manifolds theory already known to specialists in the field. Hope that this will not cause serious objection from experts on the issue.

The present paper is organized as follows. In Section 2, we consider an autonomous nonlinear system possessing invariant manifold $\mathcal{M}$ and in terms of Lyapunov functions establish conditions ensuring that $\mathcal{M}$ is exponentially stable. In Section 3, we formulate the main conditions concerning the co-cycle $\{X^t\}$ generated by system in variations. These include the aforementioned partial hyperbolicity condition of $\{X^t\}$ on $T\mathcal{M}$ and decay rate condition for $\{X^t\}$ in normal to $\mathcal{M}$ direction. Next we show that there do exists a X^t-invariant splitting of $T\mathbb{R}^n$ along $\mathcal{M}$ into a direct Whitney sum $W \oplus V^*$ of tangent sub-bundle $V^\star \subset T\mathcal{M}$ and a complementary exponentially stable sub-bundle W. Thus, actually, under the conditions imposed, $\mathcal{M}$ turns out to be a partially hyperbolic subset of $\mathbb{R}^n$ in the sense of [20, Definition 2.1, p. 8]. Due to this circumstance, for any orbit $\mathcal{O}(z) \subset \mathcal{M}$, there is a local stable invariant manifold through $\mathcal{O}(z)$ tangent to W along this orbit. Each motion starting at this invariant manifold exponentially approaches a motion on $\mathcal{O}(z)$ as $t \to \infty$ (see Section 4). In Section 5, we prove the main theorem which states that the union of all local stable invariant manifolds form an open neighborhood of $\mathcal{M}$. The global geometrical aspects of the exposed theory and some generalizations are discussed in Sections 6 and 7. Finally, in Section 8, we apply the main theorem to a system defined on cotangent bundle of a compact homogeneous space $\mathrm{SL}(2;\mathbb{R})/\Gamma$.

2 Exponential stability of invariant manifold

Let v be a C^2-vector field in a domain $\mathcal{D}$ of the space $\mathbb{R}^n$ endowed with the standard scalar product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$. Assume that the vector field v is complete, i.e.the corresponding autonomous system

$$\dot{x} = v(x) \tag{2.1}$$

generates the flow$\{\chi^t(\cdot) : \mathcal{D} \mapsto \mathcal{D}\}_{t\in\mathbb{R}}$, and let this system possesses an m-dimensional compact χ^t-invariant C^2-sub-manifold $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathcal{D}$, where $\iota(\cdot) : \mathcal{M} \mapsto \mathbb{R}^n$ stands for an isometric inclusion map.

Introduce some notations. Denote by $N_z\mathcal{M}$ the orthogonal complement of the tangent space $T_z\mathcal{M}$ at $z \in \mathcal{M}$. For the sake of simplifying notations, it will be convenient for us to identify $T_z\mathbb{R}^n$ with $\mathbb{R}^n$and to treat both $T_z\mathcal{M}$ and $N_z\mathcal{M}$ as linear sub-spaces of $\mathbb{R}^n$. Thus, for any given $z \in \mathcal{M}$, we have $T_z\mathbb{R}^n = T_z\mathcal{M} \oplus N_z\mathcal{M}$, and the vector bundle $\coprod_{z\in\mathcal{M}} T_z\mathbb{R}^n$ splits into Whitney sum of the tangent and normal sub-bundles

$$\coprod_{z\in\mathcal{M}} T_z\mathbb{R}^n = T\mathcal{M} \oplus N\mathcal{M}, \qquad T\mathcal{M} := \coprod_{z\in\mathcal{M}} T_z\mathcal{M}, \qquad N\mathcal{M} := \coprod_{z\in\mathcal{M}} N_z\mathcal{M}.$$

Let $\pi : T\mathcal{M} \oplus N\mathcal{M} \mapsto \mathcal{M}$ stands for the natural vector bundle projection. As is well known, there exists sufficiently small $r > 0$ such that the set $N\mathcal{M}_r = \{\xi \in N\mathcal{M} : \|\xi\| < r\}$ can be identified with a tubular neighborhood of $\mathcal{M}$. Namely, the mapping $N\mathcal{M}_r \ni \xi \mapsto z + \xi \in \mathbb{R}^n$, where $z = \pi(\xi)$, define a natural embedding $N\mathcal{M}_r \hookrightarrow \mathbb{R}^n$. Let the vector bundle mappings $P_N : T\mathcal{M} \oplus N\mathcal{M} \mapsto N\mathcal{M}$ and $P_T : T\mathcal{M} \oplus N\mathcal{M} \mapsto T\mathcal{M}$ stand for the orthogonal projections onto $N\mathcal{M}$ and $T\mathcal{M}$ respectively.

There naturally arise problems concerning the behavior of the flow in a neighborhood of $\mathcal{M}$, in particular the stability problem of $\mathcal{M}$. The first step in solving the latter is to study the so-called normal co-cycle generated by the system in variations w.r.t. a given motion $t \mapsto \chi^t(x)$ of a point $x \in \mathcal{D}$

$$\dot{y} = v'\left(\chi^t(x)\right) y. \tag{2.2}$$

As is well known, the group property of the flow, $\chi^{t+\tau}(\cdot) = \chi^t \circ \chi^\tau(\cdot)$ for all $t, \tau \in \mathbb{R}$, implies the co-cycle property of the the corresponding evolution operator

$$X^t(x) := \frac{\partial \chi^t(x)}{\partial x},$$

namely

$$X^{t+\tau}(x) = X^t\left(\chi^\tau(x)\right) X^\tau(x), \qquad X^{-\tau}\left(\chi^\tau(x)\right) = \left[X^\tau(x)\right]^{-1} \qquad \forall t, \tau \in \mathbb{R},\ \forall x \in \mathcal{D}, \tag{2.3}$$

and the χ^t-invariance of $\mathcal{M}$ implies the X^t-equivariance of fibers of vector bundle $T\mathcal{M} \oplus N\mathcal{M}$ and its sub-bundle $T\mathcal{M}$, meaning that for each $z \in \mathcal{M}$ and $t \in \mathbb{R}$ there hold

$$\begin{gathered} X^t(z)\,(T\mathcal{M} \oplus N\mathcal{M})\,|_z = (T\mathcal{M} \oplus N\mathcal{M})\,|_{\chi^t(z)}, \\ X^t(z)\, T_z\mathcal{M} = T_{\chi^t(z)}\mathcal{M}. \end{gathered} \tag{2.4}$$

In other words, the linear co-cycle $\{X^t\}_{t\in\mathbb{R}}$ over the flow $\{\chi^t(\cdot) : \mathcal{M} \mapsto \mathcal{M}\}_{t\in\mathbb{R}}$ defines a one-parameter family of automorphisms both of $T\mathcal{M} \oplus N\mathcal{M}$ and $T\mathcal{M}$. As a result, we obtain

$$X^t P_T = P_T X^t P_T, \qquad P_N X^t = P_N X^t (P_N + P_T) = P_N X^t P_N. \tag{2.5}$$

Note that the fibers of $N\mathcal{M}$ need not be X^t-equivariant. At the same time, the one-parameter family of mappings (the normal co-cycle)

$$X_N^t(z) := P_N X^t(z) : N_z\mathcal{M} \mapsto N_{\chi^t(z)}\mathcal{M}, \qquad t \in \mathbb{R},$$

possesses the required property:

$$\begin{aligned} X_N^{t+s}(z) &= P_N(\chi^{t+s}(z))X^{t+s}(z) = P_N\left(\chi^t \circ \chi^s(z)\right) X^t\left(\chi^s(z)\right) X^s(z) \\ &= P_N\left(\chi^t \circ \chi^s(z)\right) X^t\left(\chi^s(z)\right) P_N(\chi^s(z))X^s(z) = X_N^t(\chi^s(z))X_N^s(z). \end{aligned}$$

One can expect that the invariant manifold $\mathcal{M}$ will be stable provided that $\|X_N^t\|$ tends to zero as $t \to \infty$ sufficiently fast. Following [25, 28, 29], to approve the correctness of such a hypothesis, we shall exploit the apparatus of Lyapunov functions. Proposition 2.1 given below is a direct generalization of results [25] obtained for the case where $\mathcal{M}$ is a torus with trivial normal bundle.

Proposition 2.1. *The following statements are equivalent:*

(i) the integral $\int_0^\infty \|X_N^s(z)\|^2\,\mathrm{d}s$ *is uniformly convergent w.r.t.* z*;*

(ii) there exist positive constants γ *and* c_0 *such that*

$$\|X_N^t(z)\| \le c_0 \mathrm{e}^{-\gamma t} \qquad \forall t \ge 0; \tag{2.6}$$

(iii) there exists a continuous field of positive definite symmetric operators

$$\{S(z) : N_z\mathcal{M} \mapsto N_z\mathcal{M}\}_{z\in\mathcal{M}}$$

such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\rangle = -\|\xi\|^2 \qquad \forall z \in \mathcal{M},\ \forall \xi \in N_z\mathcal{M}. \tag{2.7}$$

Proof. To show that (i)⇒(ii) and (i)⇒(iii), define the continuous field of positive definite symmetric operators on fibers of $N\mathcal{M}$ by

$$\langle S(z)\xi, \xi\rangle := \int_0^\infty \|X_N^s(z)\xi\|^2\,\mathrm{d}s \qquad \forall z \in \mathcal{M},\ \forall \xi \in N_z\mathcal{M}. \tag{2.8}$$

Due to the compactness of $\mathcal{M}$ there are positive constants a and A such that

$$a\|\xi\|^2 \le \langle S(z)\xi, \xi\rangle \le A\|\xi\|^2 \qquad \forall z \in \mathcal{M},\ \forall \xi \in N_z\mathcal{M}. \tag{2.9}$$

Since

$$\begin{aligned} \langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\rangle &= \int_0^\infty \left\|X_N^s(\chi^t(z))X_N^t(z)\xi\right\|^2 \mathrm{d}s \\ &= \int_0^\infty \left\|X_N^{t+s}(z)\xi\right\|^2 \mathrm{d}s = \int_t^\infty \|X_N^s(z)\xi\|^2\,\mathrm{d}s, \end{aligned}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\right\rangle = -\left\|X_N^t(z)\xi\right\|^2 \leq -\frac{1}{A}\left\langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\right\rangle. \tag{2.10}$$

Hence,

$$\left\langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\right\rangle \leq \mathrm{e}^{-t/A}\left\langle S(z)\xi, \xi\right\rangle \qquad \forall t \geq 0,$$

and thus,

$$\left\|X_N^t(z)\xi\right\|^2 \leq \frac{A}{a}\mathrm{e}^{-t/A}\left\|\xi\right\|^2 \qquad \forall t \geq 0.$$

It is obvious, that (2.10) implies (2.7), and (ii)⇒(i).

It remains to show that (iii)⇒(ii). If (2.7) is satisfied, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle S(\chi^t(z))X_N^t(z)\xi, X_N^t(z)\xi\right\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left\langle S(\chi^{t+s}(z))X_N^{t+s}(z)\xi, X_N^{t+s}(z)\xi\right\rangle$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left\langle S(\chi^s\circ\chi^t(z))X_N^s\left(\chi^t(z)\right)X_N^t(z)\xi, X_N^s\left(\chi^t(z)\right)X_N^t(z)\xi\right\rangle = -\left\|X_N^t(z)\xi\right\|^2.$$

This ensures inequality (2.10), which implies (2.6) with $c_0 = A/a$ and $\gamma = 1/A$. □

As in the case where $\mathcal{M}$ is a torus with trivial normal bundle, the additional requirement of continuous differentiability of $S(\cdot)$ together with (2.7) ensures exponential stability of $\mathcal{M}$.

Proposition 2.2. *Let there exist a continuously differentiable field of positive definite symmetric operators*

$$\{S(z) : N_z\mathcal{M} \mapsto N_z\mathcal{M}\}_{z\in\mathcal{M}}$$

satisfying (2.7). Then the invariant manifold $\mathcal{M}$ is exponentially stable.

Proof. Let $x \in N\mathcal{M}_r$. Then there is a unique representation $x = z(x) + \xi(x)$ where $z(x) \in \mathcal{M}$, $\xi(x) \in N_z\mathcal{M}$. Define the function $V(x) := \langle S(z(x))\xi(x), \xi(x)\rangle$. To calculate the derivative $\dot{V}_v(x)$ of this function along the vector $v(x)$, consider a finite open cover $\bigcup_{i=1}^I \mathcal{U}_i$ of $\mathcal{M}$ with the following properties: the restriction of normal bundle to every $\mathcal{U}_i$ is trivial, and there exist compact subsets $\mathcal{K}_i \subset \mathcal{U}_i$, $i = 1, \ldots, I$, such that $\bigcup_{i=1}^I \mathcal{K}_i = \mathcal{M}$.

Let $\mathcal{U}$ stands for one of the sets $\mathcal{U}_1, \ldots, \mathcal{U}_I$ and $\mathcal{K} \in \{\mathcal{K}_1, \ldots, \mathcal{K}_I\}$ be the corresponding compact subset, thus $\mathcal{K} \subset \mathcal{U}$. Then there exist C^1-mappings $\nu_k(\cdot) : \mathcal{U} \mapsto N\mathcal{M}$, $k = 1, \ldots, n-m$, such that for any $z \in \mathcal{U}$ the vectors $\nu_1(z), \ldots, \nu_{n-m}(z)$ form an orthonormal basis of $N_z\mathcal{M}$. Compose the matrix $\mathbf{N}(z)$ of the vectors $\nu_1(z), \ldots, \nu_{n-m}(z)$ as columns and denote by $\mathbf{N}^\top(z)$ the transposed matrix. Then $\mathbf{P}_N(z) := \mathbf{N}(z)\mathbf{N}^\top(z)$ and $\mathbf{P}_T(z) := \mathbf{Id} - \mathbf{P}_N(z)$ are matrices of projections $P_N(z)$ and $P_T(z)$ respectively. Now by means of the diffeomorphism

$$\mathcal{U} \times \mathcal{B}_r^{n-m}(0) \ni (z, p) \mapsto z + \mathbf{N}(z)p \in N\mathcal{M}_r \tag{2.11}$$

where $p := (p_1, \ldots, p_{n-m})$, $\mathcal{B}_r^{n-m}(0) := \{p : \|p\| < r\}$ and r is sufficiently small, we obtain a system on $\mathcal{U} \times B_r^{n-m}$ induced by system (2.1). Namely, we have

$$\left(\mathbf{Id} + [\mathbf{N}(z)p]_z'\right)\dot{z} + \mathbf{N}(z)\,\dot{p} = v\left(z + \mathbf{N}(z)p\right),$$

and taking into account that $v(z) \perp \nu_i(z)$, $i = 1, \ldots, n-m$, the induced system on $\mathcal{U} \times B_r^{n-m}$ takes the form

$$\dot{z} = v\left(z\right) + v_1(z, p), \qquad \dot{p} = \left[\mathbf{A}(z) + \mathbf{A}_1(z, p)\right]p, \tag{2.12}$$

where

$$\mathbf{A}(z) := \mathbf{N}^\top(z)\left[\mathbf{J}(z)\,\mathbf{N}(z) - \dot{\mathbf{N}}_{v(z)}(z)\right], \tag{2.13}$$

$$v_1(z,p) := \left(\mathbf{Id} + [\mathbf{N}(z)p]'_z\right)^{-1}\mathbf{P}_T(z)v\left(z+\mathbf{N}(z)p\right) - v(z),$$

$$A_1(z,p)p := \mathbf{N}^\top(z)\left(v(z+\mathbf{N}(z)p) - v(z) - \mathbf{J}(z)\mathbf{N}(z)p - [\mathbf{N}(z)p]'_z\, v_1(z,p)\right).$$

Here $\mathbf{J}(z)$ is the Jacobi matrix of the mapping $x \mapsto v(x)$ at the point $x = z$, and $\dot{\mathbf{N}}_{v(z)}(z) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{s=0}\mathbf{N}\left(\chi^t(z)\right)$. It is not hard to see that there exists a constant $C > 0$ such that

$$\|v_1(z,p)\| \le C\,\|p\|, \qquad \|A_1(z,p)\| \le C\,\|p\| \qquad \forall z \in \mathcal{K},\ \forall p \in \mathcal{B}^{n-m}_r(0). \tag{2.14}$$

Obviously, since $\mathcal{M}$ is compact, one can choose a common constant C for all $\mathcal{K}_1, \dots, \mathcal{K}_I$.

Note that locally the diffeomorphism (2.11) conjugates the system on $N\mathcal{M}$ generating the normal co-cycle $\{X^t_N\}$ with the system

$$\dot z = v(z), \qquad \dot p = \mathbf{A}(z)p.$$

Hence, for $\xi = \mathbf{N}(z)p$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle S(\chi^t(z))X^t_N(z)\xi, X^t_N(z)\xi\rangle = \left(\langle S(z)\mathbf{N}(z)p, \mathbf{N}(z)p\rangle\right)'_p\mathbf{A}(z)p + \left(\langle S(z)\mathbf{N}(z)p, \mathbf{N}(z)p\rangle\right)'_z v(z) = -\|\xi\|^2,$$

and thus

$$\dot V_v(x) = -\|\xi\|^2 + \left(\langle S(z)\mathbf{N}(z)p, \mathbf{N}(z)p\rangle\right)'_p\mathbf{A}_1(z,p)p + \left(\langle S(z)\mathbf{N}(z)p, \mathbf{N}(z)p\rangle\right)'_z v_1(z,p).$$

Since $\|\xi\| = \|p\|$ and there are positive constants A and a such that $S(z)$ satisfies (2.9), then on account of (2.14) there holds the inequality

$$\dot V_v(x) \le -\frac{1}{2}\|\xi\|^2 \le -\frac{1}{2A}V(x) \qquad \forall x \in N\mathcal{M}_r$$

provided that r is sufficiently small. By means of the last inequality one can show in a standard way that there exists $\delta \in (0,r)$ such that $\|\chi^t(x) - \pi\left(\chi^t(x)\right)\|$ tends to zero with exponential rate as $t \to \infty$ provided that $x \in N\mathcal{M}_\delta$. □

3 Invariant splitting of vector bundle along invariant manifold

Let us agree on the following. Hereinafter, if $\xi \in T\mathcal{M} \oplus N\mathcal{M}$ and $z = \pi(\xi)$, then $X^t\xi := X^t(z)\,\xi$, and $X^tX^\tau\xi := X^t\left(\chi^\tau(z)\right)X^\tau(z)\xi$ for all $t, \tau \in \mathbb{R}$.

Assume that the following conditions are fulfilled:

H1 The tangent bundle $T\mathcal{M}$ splits into a continuous Whitney sum $T\mathcal{M} = V^s \oplus V^\star$ of X^t-invariant vector sub-bundles $V^s = \coprod_{z\in\mathcal{M}} V^s_z$, $V^\star = \coprod_{z\in\mathcal{M}} V^\star_z$ (i.e. fibers of vector bundles $V^s$and $V^\star$are X^t-equivariant), and there exist constants $c_0 \ge 1$, $\nu > 0$, $\sigma \in [0,\nu)$ such that

$$\|X^t\xi\| \le c_0\mathrm{e}^{-\nu t}\,\|\xi\| \qquad \forall t \ge 0,\ \forall \xi \in V^s, \tag{3.1}$$

$$\|X^t\xi\| \le c_0\mathrm{e}^{-\sigma t}\,\|\xi\| \qquad \forall t \le 0,\ \forall \xi \in V^\star. \tag{3.2}$$

H2 There exists $\gamma > \sigma$ such that

$$\left\|P_N X^t P_N\right\| \le c_0 \mathrm{e}^{-\gamma t} \qquad \forall t \ge 0.$$

It should be noted that the last inequality actually matches (2.6) and on account of (2.5) implies

$$\left\|P_N X^t\right\| \le c_0 \mathrm{e}^{-\gamma t} \qquad \forall t \ge 0. \tag{3.3}$$

Besides, (3.2) together with (2.3) implies

$$\left\|X^t \xi\right\| \ge c_0^{-1} \mathrm{e}^{-\sigma t} \left\|\xi\right\| \qquad \forall t \ge 0,\ \forall \xi \in V^\star. \tag{3.4}$$

Note also that the sub-bundle $V^\star$ contains 1-D X^t-invariant sub-bundle $V^c := \{\theta v\}_{\theta \in \mathbb{R}}$ generated by the vector field v. Each solution of (2.2) with initial value in V^c is bounded. An important particular case is when $\mathcal{M}$ is hyperbolic, i.e. there is X^t-invariant splitting $V^\star = V^c \oplus V^u$ such that

$$\left\|X^t \xi\right\| \le c_0 \mathrm{e}^{\nu t} \left\|\xi\right\| \qquad \forall t \le 0,\ \forall \xi \in V^u.$$

In this case we consider that $\sigma = 0$.

Define the natural projections

$$P_s : T\mathcal{M} \mapsto V^s, \qquad P_\star : T\mathcal{M} \mapsto V^\star.$$

Since the splitting $V^s \oplus V^\star$ is X^t-invariant, then

$$X^t P_{s,\star} P_T = P_{s,\star} X^t P_T \qquad \forall t \in \mathbb{R}. \tag{3.5}$$

On account of (2.3) and (3.5), we get

$$X^{t-\tau}\left(\chi^\tau(z)\right) = X^t(z) X^{-\tau}\left(\chi^\tau(z)\right) = \left[X^t(z)\right]\left[X^\tau(z)\right]^{-1} \tag{3.6}$$

and thus,

$$2X^t P_{s,\star}\left[X^\tau\right]^{-1} P_T = \left[X^t\right]\left[X^\tau\right]^{-1} P_{s,\star} P_T = X^{t-\tau} P_{s,\star} P_T. \tag{3.7}$$

Now **H1** yields that there exists a positive constant c_1 such that

$$\begin{aligned} \left\|X^t P_s \left[X^\tau\right]^{-1} P_T\right\| &\le c_1 \mathrm{e}^{-\nu(t-\tau)}, \qquad 0 \le \tau \le t, \\ \left\|X^t P_\star \left[X^\tau\right]^{-1} P_T\right\| &\le c_1 \mathrm{e}^{-\sigma(t-\tau)}, \qquad 0 \le t < \tau \end{aligned} \tag{3.8}$$

In what follows, for any $\xi \in T\mathcal{M} \oplus N\mathcal{M}$, we will use the notations

$$\xi_T := P_T \xi, \qquad \xi_N := P_N \xi, \qquad \xi_{s,\star} := P_{s,\star} P_T \xi.$$

Proposition 3.1. *There exists a continuous X^t-invariant splitting of $T\mathcal{M} \oplus N\mathcal{M}$ into a Whitney sum $W \oplus V^\star$ such that $P_N W = N\mathcal{M}$, and there is a positive constant c such that*

$$\left\|X^t \xi\right\| \le c\mathrm{e}^{-\lambda t} \left\|\xi\right\| \qquad \forall t \ge 0,\ \forall \xi \in W \tag{3.9}$$

where $\lambda := \min\{\nu, \gamma\}$.

Proof. Let us construct a sub-bundle of vectors $\xi \in T\mathcal{M} \oplus N\mathcal{M}$, such that $\|X^t\xi\|$ has a Lyapunov exponent not exceeding $-\lambda$. Since

$$X^t\xi = P_T X^t\xi + P_N X^t\xi,$$

then, on account of (3.3), it remains to deal with $P_T X^t\xi$. Derive an equation for $P_T X^t\xi$. Since

$$P_T X^t\xi = X^t\xi - P_N X^t\xi = X^t\xi - P_N P_N X^t\xi$$

and the map $\mathcal{M} \ni z \mapsto P_N(z)$ is continuously differentiable, then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} P_T X^t\xi = v' X^t\xi - \frac{\mathrm{d}}{\mathrm{d}t}\left(P_N P_N X^t\xi\right) \qquad \Longrightarrow \\
&\frac{\mathrm{d}}{\mathrm{d}t} P_T X^t\xi = v' P_T X^t\xi + v' P_N X^t\xi - \left(P_N' v\right) P_N X^t\xi - P_N \frac{\mathrm{d}}{\mathrm{d}t}\left(P_N X^t\xi\right).
\end{aligned}$$

In view of (2.5), we get

$$P_T \frac{\mathrm{d}}{\mathrm{d}t} P_T X^t\xi = P_T v' P_T X^t\xi + P_T\left(v' - P_N' v\right) P_N X^t\xi_N.$$

Recall that, for a given vector field $\mathbb{R} \ni t \mapsto \eta(t) \in T_{z(t)}\mathcal{M}$ along a curve $z(\cdot) : \mathbb{R} \mapsto \mathcal{M}$ and for any $t \in \mathbb{R}$, the vector $P_T\dot{\eta}(t)$ is nothing else but the covariant derivative $\nabla_{\dot{z}}\eta(t)$ at point $z(t)$. Hence, for every ξ such that $\pi(\xi) = z$, the vector field $\eta(t;\xi) := P_T X^t\xi$ along the curve $t \mapsto \chi^t(z)$ is a unique solution of the initial problem

$$\nabla_{\dot{z}}\eta = P_T v'\left(\chi^t(z)\right)\eta + P_T Q(t)\,\xi_N, \qquad \eta(0) = \xi_T, \tag{3.10}$$

where the vector bundle homomorphism $Q(t)$ is defined by

$$Q(t)\,\xi = \left[v'\left(\chi^t(z)\right) - P_N'\left(\chi^t(z)\right) v\left(\chi^t(z)\right)\right] P_N X^t(z)\,\xi \qquad \forall \xi \in T_z\mathcal{M} \oplus N_z\mathcal{M}. \tag{3.11}$$

It turns out that the set of solutions of problem (3.10), which we are interested in, is given by

$$\eta(t;\xi) = X^t\xi_s + \int_0^\infty \Gamma(t,\tau)\, P_T Q(\tau)\,\xi_N \mathrm{d}\tau \tag{3.12}$$

where $\xi_s \in V^s$ is taken at will and

$$\Gamma(t,\tau) := \begin{cases} X^t P_s\left[X^\tau\right]^{-1}, & 0 \le \tau \le t \\ -X^t P_\star\left[X^\tau\right]^{-1}, & 0 \le t < \tau. \end{cases}$$

In fact, taking into account (3.8), one can choose a constant $c_2 > 0$ such that

$$\begin{aligned}
\left\|X^t P_s\left[X^\tau\right]^{-1} P_T Q(\tau)\right\| &\le c_2 \mathrm{e}^{-\nu t + (\nu-\gamma)\tau}, && 0 \le \tau \le t, \\
\left\|X^t P_\star\left[X^\tau\right]^{-1} P_T Q(\tau)\right\| &\le c_2 \mathrm{e}^{-\sigma t + (\sigma-\gamma)\tau}, && 0 \le t < \tau.
\end{aligned}$$

Hence, there exists a positive constant $c_3 > 0$ such that

$$\|\eta(t;\xi)\| \le \left\|X^t\xi_s\right\| + \int_0^\infty \|\Gamma(t,\tau)\, P_T Q(\tau)\, P_N\|\,\mathrm{d}\tau\, \|\xi\| \le c_3 \mathrm{e}^{-\lambda t}\|\xi\|, \qquad t \ge 0.$$

By means of direct calculations, one can easily verify that $\eta(\cdot;\xi)$ is a unique solution of the initial problem for linear inhomogeneous system

$$\dot{y} = v'\left(\chi^t(z)\right) y + P_T Q(t)\,\xi_N, \qquad y(0) = \xi_T \in T_x\mathcal{M}, \tag{3.13}$$

where

$$\xi_T = \xi_s + \Xi\xi_N, \quad \Xi\xi := -\int_0^{\infty} P_\star \left[X^s\right]^{-1} P_T Q\left(s\right) P_N \xi \mathrm{d}s. \tag{3.14}$$

Since $P_T\eta\left(t;\xi\right) \equiv \eta\left(t;\xi\right)$, then $\eta\left(\cdot;\xi\right)$ satisfies both (3.13) and (3.10).

Hence, for arbitrary ξ_s, ξ_N, we have found $\xi = \xi_s + \Xi\xi_N + \xi_N$ such that

$$X^t\xi = P_T X^t\xi + P_N X^t \xi_N = \eta\left(t;\xi\right) + P_N X^t \xi_N$$

and thus,

$$\left\|X^t\xi\right\| \leq \left(c_3 + c_0\right) \mathrm{e}^{-\lambda t} \left\|\xi\right\| \qquad \forall t \geq 0.$$

Now it is naturally to define the projection

$$\Pi := P_s P_T + \Xi + P_N,$$

and the corresponding sub-bundle

$$W := \Pi\left(T\mathcal{M} \oplus N\mathcal{M}\right).$$

The uniform convergence of integral (3.14) ensures that the splitting $W \oplus V^*$ is continuous. One can easily verify that Π has the projection property $\Pi^2 = \Pi$. Besides, $P_N W = P_N N\mathcal{M} = N\mathcal{M}$.

It remains to verify that the splitting $W \oplus V^\star$ is X^t-invariant. Note that if $\xi \notin W$, than on account of (3.4) the Lyapunov exponent of $\left\|X^t\xi\right\|$ exceeds $-\lambda$. Since,

$$\left\|X^t X^\tau \xi\right\| = \left\|X^{t+\tau}\xi\right\| \leq \left(c_3 + c_0\right) \mathrm{e}^{-\lambda(t+\tau)} \left\|\xi\right\|$$

for any $\xi \in W$, $\tau \in \mathbb{R}$ and $t \geq -\tau$, then the Lyapunov exponent of $\left\|X^t X^\tau \xi\right\|$ does not exceed $-\lambda$. Hence, $X^\tau\xi \in W$ for all $\tau \in \mathbb{R}$, provided that $\xi \in W$. Thus $X^t W \subseteq W$, and since X^t is non-degenerate, then $X^t W = W$. As a consequence, $\Pi X^t \xi = X^t \Pi \xi$ for any $\xi \in W$, but since both W and V^* are X^t-invariant, than the above equality holds true for any $\xi \in T\mathcal{M} \oplus N\mathcal{M}$. This yields that $\mathrm{Id} - \Pi$ commutes with X^t as well:

$$\left(\mathrm{Id} - \Pi\right) X^t\xi = X^t\xi - \Pi X^t \xi = X^t\left(\xi - \Pi\xi\right) = X^t\left(\mathrm{Id} - \Pi\right)\xi \qquad \forall \xi \in T\mathcal{M} \oplus N\mathcal{M}. \qquad \square$$

Corollary 3.2. *There is a constant $K > 0$ such that the following inequalities hold true:*

$$\left\|X^t \Pi \left[X^\tau\right]^{-1}\right\| \leq K\mathrm{e}^{-\lambda(t-\tau)}, \qquad 0 \leq \tau \leq t,$$
$$\left\|X^t\left(\mathrm{Id} - \Pi\right)\left[X^\tau\right]^{-1}\right\| \leq K\mathrm{e}^{-\sigma(t-\tau)}, \qquad 0 \leq t < \tau.$$

4 Existence of local exponentially stable set for a given orbit

After introducing the new variable y by

$$x = \chi^t(z) + y,$$

system (2.1) takes the form

$$\dot{y} = v'\left(\chi^t(z)\right) y + w(t,z,y) \tag{4.1}$$

where
$$w(t,z,y) := v\left(\chi^t(z)+y\right) - v\left(\chi^t(z)\right) - v'\left(\chi^t(z)\right)y,$$
and $z \in \mathcal{M}$ is considered as a parameter. From now on throughout this section, we do not show explicitly the variable z among arguments of mappings whenever it does not cause a confusion.

In order to apply the Lyapunov–Perron method of integral equations, introduce the Green function
$$G(t,\tau) := \begin{cases} X^t\Pi\left[X^\tau\right]^{-1}, & 0 \le \tau \le t, \\ X^t\left(\Pi\text{- Id}\right)\left[X^\tau\right]^{-1}, & 0 \le t < \tau \end{cases}$$
and use the following standard statement.

Proposition 4.1. *A mapping $y(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}^n$ with upper Lyapunov exponent not exceeding $-\lambda$ is a solution of (4.1) if and only if there is $\zeta \in W \cap \pi^{-1}(z)$ such that $y(\cdot) = y(\cdot,\zeta)$ satisfies the integral equation*
$$y(t,\zeta) = X^t\zeta + \int_0^\infty G(t,\tau)w(\tau,y(\tau,\zeta))\mathrm{d}\tau =: \mathcal{G}[y](t,\zeta), \tag{4.2}$$
as well as the condition $\Pi y(0,\zeta) = \zeta$.

Proof. Note that Corollary 3.2 together with inequality $\lambda > \sigma$ yields
$$\begin{aligned}\int_0^\infty \mathrm{e}^{-2\lambda\tau}\left\|G(t,\tau)\right\|\mathrm{d}\tau &\le K\mathrm{e}^{-\lambda t}\left[\int_0^t \mathrm{e}^{-\lambda\tau}\mathrm{d}\tau + \mathrm{e}^{(\lambda-\sigma)t}\int_t^\infty \mathrm{e}^{(\sigma-\lambda)\tau}\mathrm{e}^{-\lambda\tau}\mathrm{d}\tau\right] \\ &\le K\mathrm{e}^{-\lambda t}\int_0^\infty \mathrm{e}^{-\lambda\tau}\mathrm{d}\tau \le \frac{K}{\lambda}\mathrm{e}^{-\lambda t},\end{aligned} \tag{4.3}$$
and since v is C^2-vector field, then $\|w(t,z,y)\| = O\big(\|y\|^2\big)$ as $\|y\| \to 0$. If now $y(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}^n$ is a solution of (4.1) with upper Lyapunov exponent not exceeding $-\lambda$, then by means of direct calculations it is not hard to verify that
$$\tilde{y}(t) := \int_0^\infty G(t,\tau)w(\tau,y(\tau))\mathrm{d}\tau = O\big(\mathrm{e}^{-\lambda t}\big), \quad t \to \infty,$$
is a solution of the linear non-homogeneous system
$$\dot{y} = v'\left(\chi^t(z)\right)y + w(t,z,y(t)).$$
The last one has the solution $t \mapsto y(t) = O\left(\mathrm{e}^{-\lambda t}\right)$, $t \to \infty$, as well. Hence, there exists $\zeta \in W \cap \pi^{-1}(z)$ such that $y(t) - \tilde{y}(t) = X^t\zeta$. From $\Pi\tilde{y}(0) = 0$ it follows that $\Pi y(0) = \zeta$.

Vice versa, by means of direct calculations one can easily verify that any solution $t \mapsto y(t,\zeta) = O\left(\mathrm{e}^{-\lambda t}\right), t \to \infty$, of (4.2) is a solution of (4.1) such that $\Pi y(0,\zeta) = \zeta$. □

By means of the mapping Id $+\Xi$ (see (3.14)), we define an isomorphic image of $N\mathcal{M}_r$ as
$$U_r := (\mathrm{Id}+\Xi)\left(N\mathcal{M}_r\right) \equiv \bigcup_{\zeta\in N\mathcal{M}_r}\left\{\zeta + \Xi\zeta\right\}.$$
Note that $P_N U_r = N\mathcal{M}_r$, and if we introduce the set
$$W_r := \left\{\zeta \in W : \|\zeta\| < r\right\},$$

then $U_r = \{\zeta \in W_r : P_s P_T \zeta = 0\}$.

Let $\mathrm{C}\left(\mathbb{R}_+ \times W_r \mapsto \mathbb{R}^n; \|\cdot\|_\lambda\right)$ stands for a Banach space of mappings endowed with the norm

$$\|\cdot\|_\lambda := \sup_{(t,\zeta)\in\mathbb{R}_+\times W_r} \mathrm{e}^{\lambda t} \|\cdot\|.$$

For a constant $C > 0$, define the closed subset

$$\mathcal{Y}_{r,C} := \left\{ y(\cdot,\cdot) \in \mathrm{C}\left(\mathbb{R}_+ \times W_r \mapsto \mathbb{R}^n; \|\cdot\|_\lambda\right) : \left\|y(t,\zeta) - X^t\zeta\right\| \le C\mathrm{e}^{-\lambda t} \|\zeta\|^2 \right\}.$$

Proposition 4.2. *There exist positive numbers r and C such that:*

(i) *equation (4.1) has a unique solution $y_*(\cdot,\cdot) \in \mathcal{Y}_{r,C}$;*

(ii) *the mapping $y_*(\cdot,\cdot)$ has a continuous derivative along every fiber $W(z) := W \cap \pi^{-1}(z)$, $z \in \mathcal{M}$.*

Proof. One can prove assertion (i) in a standard way by means of the Banach contraction principle. For the sake of completeness, we present here some essential details.

Firstly, impose conditions on r, C ensuring inclusion $\mathcal{G}[\mathcal{Y}_{r,C}] \subset \mathcal{Y}_{r,C}$. Since v is C^2-vector field, then there is a constant $C_w > 0$ such that

$$\|w(t,y,z)\| \le \frac{C_w}{2} \|y\|^2, \qquad \left\|w'_y(t,y,z)\right\| \le C_w \|y\|, \qquad \left\|w''_{yy}(t,y,z)\right\| \le C_w \tag{4.4}$$

for all $(t,z) \in \mathbb{R} \times \mathcal{M}$, $\|y\| \le 1$. Now, on account of (4.3), for any $y(\cdot,\cdot) \in \mathcal{Y}_{r,C}$, we obtain

$$\Pi\mathcal{G}[y](0,\zeta) = \Pi\zeta = \zeta,$$

$$\left\|\mathcal{G}[y](t,\zeta) - X^t\zeta\right\| \le \frac{KC_w}{2\lambda}(c + Cr)^2 \mathrm{e}^{-\lambda t} \|\zeta\|^2 \le C\mathrm{e}^{-\lambda t} \|\zeta\|^2$$

provided that

$$cr + Cr^2 < 1, \qquad \frac{KC_w}{2\lambda}(c + Cr)^2 \le C.$$

If we set $C := 2KC_w c^2/\lambda$ then it is sufficient to require that r is small enough to satisfy the inequalities

$$2cr < 1, \qquad Cr \le c. \tag{4.5}$$

Now let us find conditions under which $\mathcal{G}[\cdot]$ is a contraction mapping in $\mathcal{Y}_{r,C}$. Since

$$\begin{aligned} \|w(t,y_1) - w(t,y_2)\| &\le \left\| \int_0^1 \left[v'\left(\chi^t(z) + \theta y_1 + (1-\theta)y_2\right) - v'\left(\chi^t(z)\right) \right] \mathrm{d}\theta \right\| \|y_1 - y_2\| \\ &\le \frac{C_w}{2}\left(\|y_1\| + \|y_2\|\right)\|y_1 - y_2\| \qquad \forall y_1, y_2 : \|y_1\|, \|y_2\| \le 1, \end{aligned}$$

then for every $y_1(\cdot,\cdot), y_2(\cdot,\cdot) \in \mathcal{Y}_{r,C}$ we obtain

$$\|\mathcal{G}[y_1](t,\zeta) - \mathcal{G}[y_2](t,\zeta)\|_\lambda \le \frac{KC_w}{\lambda}\left(cr + Cr^2\right)\|y_1(\cdot,\cdot) - y_2(\cdot,\cdot)\|_\lambda. \tag{4.6}$$

The inequality

$$\frac{KC_w}{\lambda}\left(cr + Cr^2\right) \le \frac{1}{2}, \tag{4.7}$$

ensures that $\mathcal{G}[\cdot]$ is a contraction in $\mathcal{Y}_{r,C}$ and then, by the Banach contraction principle, equation (4.2) has a unique solution $y_*(\cdot,\cdot) \in \mathcal{Y}_{r,C}$. Taking into account (4.5) and definition of C, to satisfy (4.7) it is sufficient to replace the second inequality in (4.5) with $2Cr \leq c$. This completes the proof of assertion (i).

To prove (ii), firstly observe that every point $z_0 \in \mathcal{M}$ has a neighborhood $\mathcal{N}(z_0) \subset \mathcal{M}$ such that $\pi^{-1}(\mathcal{N}(z_0)) \cap W_r$ is homeomorphic to $\mathcal{N}(z_0) \times \mathcal{B}_r^k(0)$ where $k = \dim W$ and $\mathcal{B}_r^k(0) \subset \mathbb{R}^k$ is a ball of radius r centered at the origin. So, we regard $y_*(\cdot,\cdot)$ as a mapping with domain $\mathcal{N}(z_0) \times \mathcal{B}_r^k(0)$. Now for $\rho \in (0,r)$, $\delta \in (0, r-\rho)$ and unit vector $e \in \mathbb{R}^k$, consider a family of mappings $\left\{u_s(\cdot,\cdot;e) : \mathbb{R}_+ \times \mathcal{N}(z_0) \times \mathcal{B}_\rho^k(0) \mapsto \mathbb{R}^n\right\}_{s\in[-\delta,\delta]\setminus\{0\}}$ defined by

$$u_s(t,\zeta;e) := \frac{1}{s}\left[y_*(t,\zeta+se) - y_*(t,\zeta)\right]$$

(recall that we agreed not to show explicitly the dependence on z). We aim to establish the existence of

$$\partial_e y_*(t,\zeta) := \lim_{s\to 0} u_s(t,\zeta;e)$$

and show that $\partial_e y_*(\cdot,\zeta)$ is a solution of the linear integral equation

$$u(t,\zeta;e) = X^t e + \int_0^\infty G(t,\tau)\, w_y'(\tau, y_*(\tau,\zeta))\, u(\tau,\zeta;e)\,\mathrm{d}\tau. \tag{4.8}$$

Similarly to the previous reasoning, introduce the Banach space

$$\mathfrak{B} := \mathrm{C}\left(\mathbb{R}_+ \times \mathcal{N}(z_0) \times \mathcal{B}_\rho^k(0) \mapsto \mathbb{R}^n; \|\cdot\|_\lambda\right)$$

endowed with the norm

$$\|\cdot\|_\lambda := \sup\left\{\mathrm{e}^{\lambda t}\|\cdot\| : (t,z,\zeta) \in \mathbb{R}_+ \times \mathcal{N}(z_0) \times \mathcal{B}_\rho^k(0)\right\}.$$

On account of (4.4), (4.3) and (4.7), one can easily obtain the estimate

$$\int_0^\infty \left\|G(t,\tau)\, w_y'(\tau, y_*(\tau,\zeta))\right\| \mathrm{e}^{-\lambda\tau}\mathrm{d}\tau \leq \mathrm{e}^{-\lambda t}\frac{KC_w}{\lambda}\left(c\|\zeta\| + C\|\zeta\|^2\right) \leq \frac{1}{2}\mathrm{e}^{-\lambda t} \tag{4.9}$$

which allows us to apply the Banach contraction principle and prove that (4.8) has a unique solution $u_*(\cdot,\cdot;e) \in \mathfrak{B}$ satisfying

$$\|u_*(\cdot,\cdot;e)\|_\lambda \leq 2c.$$

Besides, by means of (4.6) and (4.7) we obtain

$$\|u_s(\cdot,\cdot;e)\|_\lambda \leq c + \frac{1}{2}\|u_s(\cdot,\cdot;e)\|_\lambda \quad \Longrightarrow \quad \|u_s(\cdot,\cdot;e)\|_\lambda \leq 2c.$$

Next, we have

$$\begin{aligned}\|u_s(t,\zeta;e) - u_*(t,\zeta;e)\| \leq &\int_0^\infty \left\|G(t,\tau)\, w_y'(\tau, y_*(\tau,\zeta))\right\| \|u_s(\tau,\zeta;e) - u_0(\tau,\zeta;e)\|\,\mathrm{d}\tau \\ &+ \int_0^\infty \|G(t,\tau)\, H(\tau,\zeta,s;e)\|\, \|u_s(\tau,\zeta;e)\|\,\mathrm{d}\tau\end{aligned}$$

where

$$H(\tau,\zeta,s;e) := \int_0^1 \left[w_y'(\theta y_*(\tau,\zeta+se) + (1-\theta)y_*(\tau,\zeta)) - w_y'(\tau, y_*(\tau,\zeta))\right]\mathrm{d}\theta.$$

Since

$$\|H(\tau,\zeta,s;e)\| \le \frac{C_w}{2}\|y_*(\tau,\zeta+se)-y_*(\tau,\zeta)\|$$

and (4.9) yields

$$\begin{aligned}&\sup_{t\ge 0}\mathrm{e}^{\lambda t}\int_0^\infty \left\|G(t,\tau)\,w_y'(\tau,y_*(\tau,\zeta))\right\|\|u_s(\tau,\zeta;e)-u_0(\tau,\zeta;e)\|\,\mathrm{d}\tau\\ &\qquad\le \frac{1}{2}\sup_{t\ge 0}\mathrm{e}^{\lambda t}\|u_s(\tau,\zeta;e)-u_0(\tau,\zeta;e)\|,\end{aligned}$$

then, on account of (4.3), we obtain

$$\begin{aligned}&\lim_{s\to 0}\sup_{t\ge 0}\mathrm{e}^{\lambda t}\|u_s(t,\zeta;e)-u_0(t,\zeta;e)\|\\ &\quad\le cC_w\lim_{s\to 0}\sup_{t\ge 0}\left\|\mathrm{e}^{\lambda t}\int_0^\infty \mathrm{e}^{-2\lambda\tau}\|G(t,\tau)\|\left[\mathrm{e}^{\lambda\tau}\|y_*(\tau,\zeta+se)-y_*(\tau,\zeta)\|\right]\mathrm{d}\tau\right\|\\ &\quad\le cC_wK\lim_{s\to 0}\int_0^\infty \mathrm{e}^{-\lambda\tau}\left[\mathrm{e}^{\lambda\tau}\|y_*(\tau,\zeta+se)-y_*(\tau,\zeta)\|\right]\mathrm{d}\tau\\ &\quad\le cC_wK\lim_{T\to\infty}\lim_{s\to 0}\left[\int_0^T\|y_*(\tau,\zeta+se)-y_*(\tau,\zeta)\|\,\mathrm{d}\tau+\frac{4e^{-\lambda T}}{\lambda}\|y_*(\cdot,\cdot)\|_\lambda\right]=0.\end{aligned}$$

This completes the proof of assertion (ii). □

Corollary 4.3. *For all $(t,\zeta)\in\mathbb{R}_+\times W_r$ and every unite vector $e\in W_r\cap\pi^{-1}(z)$, where $z:=\pi(\zeta)$, the following inequalities hold:*

$$\|\partial_e y_*(t,\zeta)\|\le 2c\mathrm{e}^{-\lambda t},\qquad \left\|\partial_e y_*(t,\zeta)-X^te\right\|\le \mathrm{e}^{-\lambda t}\frac{2cKC_w}{\lambda}\|\zeta\|.$$

Proposition 4.4. *Let C, C_w and r be the constants specified according to Proposition 4.2. If $y(\cdot):\mathbb{R}_+\mapsto\mathbb{R}^n$ is a solution of (4.1) such that $\sup_{t\in\mathbb{R}_+}\mathrm{e}^{\lambda t}\|y(t)\|\le\min\{\lambda/(KC_w),1\}$ and $\zeta:=\Pi y(0)\in W_r$, then*

$$\left\|y(t)-X^t\zeta\right\|\le C\mathrm{e}^{-\lambda t}\|\zeta\|^2\qquad\forall t\ge 0,$$

and thus, $y(t)\equiv y_*(t,\zeta)$.

Proof. By Proposition 4.1 $y(\cdot)$ satisfies integral equation (4.4) with $\zeta=\Pi y(0)$. Then on account of (4.3) and (4.4) we have

$$\sup_{t\in\mathbb{R}_+}\mathrm{e}^{\lambda t}\|y(t)\|\le c\|\zeta\|+\frac{KC_w}{2\lambda}\min\{\lambda/(KC_w),1\}\sup_{t\in\mathbb{R}_+}\mathrm{e}^{\lambda t}\|y(t)\|,$$

and thus

$$\sup_{t\in\mathbb{R}_+}\mathrm{e}^{\lambda t}\|y(t)\|\le 2c\|\zeta\|\le 2cr<1.$$

This inequality, in its turn, implies

$$\left\|y(t)-X^t\zeta\right\|\le\frac{2KC_wc^2}{\lambda}\mathrm{e}^{-\lambda t}\|\zeta\|^2=C\|\zeta\|^2\qquad\forall t\ge 0.$$

To end the proof it remains only to refer to assertion (i) from Proposition 4.2 which ensures the uniqueness of $y_*(\cdot,\cdot)$. □

Define the sets

$$W_r(z) := W_r \cap \pi^{-1}(z), \qquad \mathcal{W}_r(z) := z + y_*(0, W_r(z))$$
$$U_r(z) := U_r \cap \pi^{-1}(z), \qquad \mathcal{U}_r(z) := z + y_*(0, U_r(z)).$$

Corollary 4.3 yields the following

Proposition 4.5. *There is a sufficiently small positive r such that the sets $\mathcal{W}_r(z)$ and $\mathcal{U}_r(z)$ are differentiable manifolds diffeomorphic to $W_r(z)$ and $P_N U_r(z)$ respectively. Besides, $T_z\mathcal{W}_r(z) = W_r(z)$, $T_z\mathcal{U}_r(z) = U_r(z)$.*

Let $z_0 \in \mathcal{M}$ be a given point and $\mathcal{O}(z_0) := \bigcup_{\tau\in\mathbb{R}} \{\chi^\tau(z_0)\}$ stands for its orbit.

Definition 4.6. We say that the set $\mathcal{W}_r(\mathcal{O}(z_0)) = \bigcup_{z\in\mathcal{O}(z_0)} \mathcal{W}_r(z)$ is a local λ-stable set of the orbit $\mathcal{O}(z_0)$.

Theorem 4.7. *Let system (2.1) satisfy conditions* **H1**, **H2**. *Then there exist positive constants r, C, T and $\rho \in (0, r]$ such that for every $z_0 \in \mathcal{M}$, the set $\mathcal{W}_r(\mathcal{O}(z_0))$ has the following properties:*

(a) for every $x \in \mathcal{W}_r(\mathcal{O}(z_0))$, there exist $z \in \mathcal{O}(z_0)$ and $\zeta \in W_r(z)$ such that

$$\left\|\chi^t(x) - \chi^t(z) - X^t\zeta\right\| \le C\mathrm{e}^{-\lambda t}\|\zeta\|^2 \qquad \forall t \ge 0,$$

and thus, the motion $t \mapsto \chi^t(x)$ has an asymptotic phase;

(b) there hold the inclusions

$$\chi^t(\mathcal{W}_\rho(z)) \subset \mathcal{W}_r(\chi^t(z)) \qquad \forall t \ge 0, \quad \text{and} \quad \chi^t(\mathcal{W}_r(z)) \subset \mathcal{W}_r(\chi^t(z)) \qquad \forall t \ge T;$$

(c) if in addition the vector field v has no singular points on $\mathcal{M}$, then for every $z \in \mathcal{O}(z_0)$, there is a sufficiently small arc $\mathcal{O}_\delta(z) := \bigcup_{|\tau|<\delta}\{\chi^\tau(z)\}$, $0 < \delta \ll 1$, such that that $\mathcal{W}_r(z_1) \cap \mathcal{W}_r(z_2) = \varnothing$ for any $z_1, z_2 \in \mathcal{O}_\delta(z)$, and the set $\mathcal{W}_r(\mathcal{O}(z_0))$ is an immersed into $\mathbb{R}^n$ topological manifold.

Proof. Let r and C be specified via Proposition 4.2, and let $x \in \mathcal{W}_r(z)$ for some $z \in \mathcal{O}(z_0)$. Then there is $\zeta \in W_r(z)$ such that $x = z + y_*(0,\zeta)$ is an initial value for the solution $t \mapsto \chi^t(z) + y_*(t,\zeta)$ of system (2.1). Hence, $\chi^t(x) \equiv \chi^t(z) + y_*(t,\zeta)$, and now (a) is a direct consequence of Proposition 4.2.

Now we proceed to (b). Let $\rho \in (0, r]$ and $x \in \mathcal{W}_\rho(z)$. Then there is $\zeta \in W_\rho(z)$ such that $x = z + y_*(0,\zeta)$, $\Pi y_*(0,\zeta) = \zeta$ and

$$\chi^s(x) = \chi^s(z) + y_*(s,\zeta) = \chi^s(z) + \Pi y_*(s,\zeta) + (\mathrm{Id} - \Pi)\, y_*(s,\zeta).$$

Put $\zeta^s := \Pi y_*(s,\zeta)$. By the definition of Π, we have $\zeta^s \in W(\chi^s(z))$, and by means of estimates from the proof of Proposition 4.2 we obtain,

$$\|\zeta^s\| = \left\| X^s\zeta + \int_0^s X^s\Pi\,[X^\tau]^{-1} w(\tau, y_*(\tau,\zeta))\,\mathrm{d}\tau \right\| \le \mathrm{e}^{-\lambda s}\left(c\rho + C\rho^2\right).$$

Hence, if $\rho \in (0,\rho_0)$, where ρ_0 is small enough to satisfy $c\rho_0 + C\rho_0^2 \le r$, then $\zeta^s \in W_r(\chi^s(z))$ for all $s \ge 0$. And if $\rho = r$, then $\zeta^s \in W_r(\chi^s(z))$ for all $s \ge T$, provided that T is large enough to satisfy $\mathrm{e}^{-\lambda T}(cr + Cr^2) \le r$. Besides, property (a) implies

$$\begin{aligned}\left\|\chi^t \circ \chi^s(x) - \chi^t \circ \chi^s(z)\right\| &\le \left\|\chi^{t+s}(x) - \chi^{t+s}(z) - X^{t+s}\zeta\right\| + \left\|X^{t+s}\zeta\right\| \\ &\le \mathrm{e}^{-\lambda(t+s)}\left(C\|\zeta\|^2 + c\|\zeta\|\right).\end{aligned}$$

Hence, if $\rho \in (0, \rho_0)$, whereρ_0 is small enough to satisfy additional condition $c\rho_0 + C\rho_0^2 \leq \lambda / KC_w$, then

$$\left\| \chi^t \circ \chi^s(x) - \chi^t \circ \chi^s(z) \right\| \leq \mathrm{e}^{-\lambda t} \frac{\lambda}{KC_\omega} \qquad \forall t \geq 0, \ \forall s \geq 0;$$

and if $\rho = r$ then

$$\left\| \chi^t \circ \chi^s(x) - \chi^t \circ \chi^s(z) \right\| \leq \mathrm{e}^{-\lambda t} \frac{\lambda}{KC_\omega} \qquad \forall t \geq 0, \ \forall s \geq T,$$

provided that T is large enough to ensure $\mathrm{e}^{-\lambda T}\left(cr + Cr^2\right) \leq \lambda / \left(KC_\omega\right)$. Now the mapping $t \mapsto \chi^t \circ \chi^s(x) - \chi^t \circ \chi^s(z)$, as a solution of (4.1), satisfies conditions of Proposition 4.4 where z and ζ should be replaced with $\chi^s(z)$ and ζ^s respectively. Hence,

$$\chi^t \circ \chi^s(x) - \chi^t \circ \chi^s(z) = y_*(t, \zeta^s),$$

and thus, $\chi^s(x) = \chi^s(z) + y_*(0, \zeta^s)$. As a consequence,

$$\chi^s(x) \in \begin{cases} \mathcal{W}_r(\chi^s(z)) & \forall s \geq 0 \text{ if } x \in \mathcal{W}_\rho(z) \text{ and } \rho \in (0, \rho_0); \\ \mathcal{W}_r(\chi^s(z)) & \forall s \geq T \text{ if } x \in \mathcal{W}_r(z). \end{cases}$$

Finally, let us prove (c) by reasoning ad absurdum. Suppose that for every $z \in \mathcal{O}(z_0)$ there is no $\delta > 0$ such that $\mathcal{W}_r(z_1) \cap \mathcal{W}_r(z_2) = \varnothing$ for any pair of different points $z_1, z_2 \in \mathcal{O}_\delta(z)$. Then there exist sequences $\{t_{1,k}\}_{k\in\mathbb{N}}$, $\{t_{2,k}\}_{k\in\mathbb{N}}$ such that $t_{i,k} \to 0$, $k \to \infty$, $i \in \{1,2\}$, $t_{1,k} > t_{2,k}$, as well as the sequence

$$\left\{x_k \in \mathcal{W}_r\left(\chi^{t_{1,k}}(z)\right) \cap \mathcal{W}_r\left(\chi^{t_{2,k}}(z)\right)\right\}_{k\in\mathbb{N}}.$$

Now, for any $k \in \mathbb{N}$, we obtain

$$\left\|\chi^{t+t_{1,k}}(z) - \chi^{t+t_{2,k}}(z)\right\| \leq \left\|\chi^t(x_k) - \chi^t \circ \chi^{t_{1,k}}(z)\right\| + \left\|\chi^t(x_k) - \chi^t \circ \chi^{t_{2,k}}(z)\right\| \to 0, \qquad t \to \infty,$$

and thus,

$$\lim_{t\to\infty} \left\| \chi^{T_k} \circ \chi^t(z) - \chi^t(z) \right\| = 0 \tag{4.10}$$

where $T_k = t_{1,k} - t_{2,k} \neq 0$. Since $\mathcal{M}$ is compact, then ω-limit set of $\mathcal{O}(z)$ contains at least one point, e.g. $z_* \in \mathcal{M}$, and (4.10) implies that z_* is T_k -periodic for all $k \in \mathbb{N}$. But, as is easily seen, from $T_k \to 0$ it follows that $v(z_*) = 0$, and we arrive at contradiction.

Now we see that the continuous mapping $\mathcal{O}(z_0) \ni z \mapsto \mathcal{W}_r(z)$ is locally one-to-one. Since each $\mathcal{W}_r(z)$ is diffeomorphic to an open d-dimensional ball of Euclidean space (Proposition (4.5)), and $\mathcal{W}_r(\mathcal{O}(z_0))$ is given by the equation

$$x = \chi^t(z_0) + y_*(0, \zeta), \qquad \zeta \in W_r\left(\chi^t(z_0)\right), \ t \in \mathbb{R},$$

then $\mathcal{W}_r(\mathcal{O}(z_0))$ is an immersed $(d+1)$-dimensional topological manifold. □

Remark 4.8. On contrary to [16], Theorem 4.7 do not guarantee that $\mathcal{W}_r(z_1) \cap \mathcal{W}_r(z_2) = \varnothing$ for any pair of different points $z_1, z_2 \in \mathcal{M}$.

Corollary 4.9. *If there exist $z_1, z_2 \in \mathcal{O}(z_0)$ such that $\mathcal{W}_r(z_1) \cap \mathcal{W}_r(z_2) \neq \varnothing$, then ω-limit set of $\mathcal{O}(z_0)$ contains at least one closed orbit.*

5 Existence of asymptotic phase

Now we are in position to prove the following theorem on the existence of asymptotic phase.

Theorem 5.1. *Let system (2.1) satisfy conditions* **H1**, **H2**. *Then there exists $\varepsilon > 0$ such that a motion $t \mapsto \chi^t(x)$ has the asymptotic phase, provided that $\mathcal{O}(x) \cap N\mathcal{M}_\varepsilon \neq \varnothing$.*

Proof. Let x_0 be a point in the tubular neighborhood $N\mathcal{M}_\varepsilon \subset N\mathcal{M}_r$, where r is specified in Theorem 4.7. Then $x_0 = z_0 + \xi_0$, where $z_0 \in \mathcal{M}$, $\xi_0 \in N_{z_0}\mathcal{M}$, $\|\xi_0\| < \varepsilon$. We have to show that if $\varepsilon \in (0, r)$ is sufficiently small, then there exists $z(x_0) \in \mathcal{M}$ such that $x_0 \in \mathcal{U}_r(z(x_0))$ (see Proposition 4.5 concerning $\mathcal{U}_r(z)$). Since $\mathcal{U}_r(z(x_0)) \subset \mathcal{W}_r(z(x_0))$, then by Theorem 4.7 the above inclusion implies

$$\|\chi^t(x_0) - \chi^t(z(x_0))\| \to 0, \qquad t \to \infty,$$

meaning that the motion $t \mapsto \chi^t(x_0)$ has the asymptotic phase.

Let us prove the existence of $z(x_0)$. Note that if r is sufficiently small, then there are local coordinates in $N\mathcal{M}_r$

$$(q_1, \ldots, q_m, p_1, \ldots, p_{n-m}) = (q, p), \qquad m := \dim \mathcal{M},$$

with the following properties: (i) the coordinates of z_0 are $(0, 0)$; (ii) the manifold $\mathcal{M}$ is given by a local equation $x = z(q)$, where $z(\cdot)$ is a C^1-mapping defined in a neighborhood of $q = 0$; (iii) the columns of the matrix $\mathbf{T}(0)$, where $\mathbf{T}(q) := \left[\frac{\partial z_i(q)}{\partial q_j}\right]_{i=1,j=1}^{n,\ m}$, are pairwise orthogonal unit vectors; (iv) if (q, p) are local coordinates of a point $x \in N\mathcal{M}_r$, then

$$x = z(q) + \mathbf{N}(q)p,$$

in particular $x_0 = z(0) + \mathbf{N}(0)p_0$, where $\mathbf{N}(q)$ is $n \times (n - m)$-matrix whose columns are unit vectors pairwise mutually orthogonal, and orthogonal to $\mathcal{M}$ at $z(q)$ as well, thus $\mathbf{N}^\top(q)\mathbf{T}(q) = 0$*; (v) both mappings $q \mapsto \mathbf{N}(q)$ and $q \mapsto \mathbf{T}(q)$ are continuous in a neighborhood of 0.

Having analyzed the mapping $y_*(0, \cdot)$, one can make a conclusion that the manifold $\mathcal{U}_r(z)$ is given by the equation

$$x = z(q) + \mathbf{N}(q)\, p + \mathbf{T}(q)\left[\mathbf{L}(q) + \mathbf{M}(q, p)\right] p,$$

where $\mathbf{L}(q)$ and $\mathbf{M}(q, p)$ are $m \times (n - m)$-matrices with continuous elements, and $\|\mathbf{M}(q, p)\| \to 0$ as $\|p\| \to \mathbf{0}$. Now for a given p_0 such that $\|p_0\| < \varepsilon \ll 1$, we have to solve the equation

$$z(q) + \mathbf{N}(q)\, p + \mathbf{T}(q)\left[\mathbf{L}(q) + \mathbf{M}(q, p)\right] p = z(0) + \mathbf{N}(0)p_0.$$

Since $z(q) - z(0) = \left[\mathbf{T}(0) + \mathbf{T}_1(q)\right] \mathbf{q}$, where $\|\mathbf{T}_1(q)\| = o(1)$, $\|q\| \to 0$, then the above equation can be represented in the form

$$\mathbf{T}(0)\left[q + \mathbf{L}(0)p\right] + \mathbf{N}(0)p = \mathbf{F}(q, p) + \mathbf{N}(0)p_0$$

where

$$\mathbf{F}(q, p) := \left[\mathbf{T}(0)\,\mathbf{L}(0) - \mathbf{T}(q)\,\mathbf{L}(q) + \mathbf{N}(0) - \mathbf{N}(q) - \mathbf{M}(q, p)\right] p - \mathbf{T}_1(q)q.$$

*Here we use the notation $\mathbf{N}(q)$ instead of $\mathbf{N}(z(q))$ where $\mathbf{N}(z)$ is the matrix defined in Section 2.

Note, that $\|\mathbf{F}(q,p)\| = o(\|q\|+\|p\|)$, $\|q\|+\|p\| \to 0$. After the change of variables $q = u - \mathbf{L}(0)p$, on account that the matrix $[\mathbf{T}(0);\mathbf{N}(0)]$ is orthogonal, we arrive at the equation

$$\begin{pmatrix} u \\ p \end{pmatrix} = \mathbf{H}(u,p) + \begin{pmatrix} 0 \\ p_0 \end{pmatrix}, \tag{5.1}$$

where $\mathbf{H}(u,p) = [\mathbf{T}(0);\mathbf{N}(0)]^\top \mathbf{F}(q,p)\big|_{q=u-\mathbf{L}(0)p}$. It is obvious that $\|\mathbf{H}(u,p)\| = o(\|u\|+\|p\|)$, $\|u\|+\|p\| \to 0$. Now we are in position to apply the Brouwer fixed point theorem. Namely, let $D_\varepsilon := \left\{ (u,p)^\top : \|u\|^2 + \|p\|^2 \le \varepsilon^2 \right\}$. If $0 < \varepsilon \ll 1$ and $\|p_0\| \le \varepsilon$, then $\max_{(u,p)\in D_{2\varepsilon}} \|\mathbf{H}(u,p)\| \le \varepsilon$. Hence,

$$D_{2\varepsilon} \ni \begin{pmatrix} u \\ p \end{pmatrix} \mapsto \mathbf{H}(u,p) + \begin{pmatrix} 0 \\ p_0 \end{pmatrix} \in D_{2\varepsilon},$$

and by the Brouwer fixed point theorem equation (5.1) has at least one solution. □

6 Global λ-stable sets of orbits on $\mathcal{M}$

In this section we analyze the geometrical structure of sets formed by motions approaching a given orbit with exponential rate and having asymptotic phases.

Definition 6.1. For a given $z \in \mathcal{M}$ the set

$$\mathcal{W}(z) := \left\{ x \in \mathcal{D} : \|\chi^t(x) - \chi^t(z)\| = O\left(e^{-\lambda t}\right),\ t \to 0 \right\}$$

is said to be (a global) λ-stable set of the point z. For a given $z_0 \in \mathcal{M}$, the set $\mathcal{W}(\mathcal{O}(z_0)) := \bigcup_{z\in\mathcal{O}(z_0)} \mathcal{W}(z)$ is said to be (a global) λ-stable set of the orbit $\mathcal{O}(z_0)$.

Theorem 6.2. *Let system (2.1) satisfy conditions* **H1**, **H2**, *and let r and T be specified according to Theorem 4.7. Then for every $z_0 \in \mathcal{M}$ the λ-stable set for the orbit $\mathcal{O}(z_0)$ has the following properties:*

- *(a) the λ-stable set of any $z \in \mathcal{O}(z_0)$ is an immersed into $\mathbb{R}^n$ differentiable manifold admitting the representation $\mathcal{W}(z) = \bigcup_{k\in\mathbb{Z}_+} \chi^{-kT}\left(\mathcal{W}_r\left(\chi^{kT}(z)\right)\right)$;*
- *(b) the foliation of $\mathcal{W}(\mathcal{O}(z_0))$ by the family of manifolds $\{\mathcal{W}(z)\}_{z\in\mathcal{O}(z_0)}$ is χ^t-invariant, meaning that*
$$\chi^t(\mathcal{W}(z)) = \mathcal{W}\left(\chi^t(z)\right) \qquad \forall t \in \mathbb{R};$$
- *(c) if the vector field v does not have singular points on $\mathcal{M}$, then for every $z \in \mathcal{O}(z_0)$ there is a sufficiently small arc $\mathcal{O}_\delta(z)$, $0 < \delta \ll 1$, such that $\mathcal{W}(z_1) \cap \mathcal{W}(z_2) = \varnothing$ for any $z_1, z_2 \in \mathcal{O}_\delta(z)$, and $\mathcal{W}(\mathcal{O}(z_0))$ is an immersed into $\mathbb{R}^n$ topological manifold.*

Proof. Let $z \in \mathcal{O}(z_0)$ and $x \in \mathcal{W}(z)$. By the definition of $\mathcal{W}(z)$,

$$R = R(x,z) := \sup_{t\ge 0} e^{\lambda t} \|\chi^t(x) - \chi^t(z)\| < \infty.$$

For a $k \in \mathbb{Z}_+$, define

$$x_k := \chi^{kT}(x), \qquad z_k := \chi^{kT}(z), \qquad \zeta_k := \Pi(x_k - z_k), \qquad y_k(t) := \chi^t(x_k) - \chi^t(z_k).$$

Then

$$\|\zeta_k\| \le \sup_{z\in\mathcal{O}(z_0)} \|\Pi\|\, \mathrm{e}^{-kT} R \le \max_{z\in\mathcal{M}} \|\Pi\|\, \mathrm{e}^{-kT} R,$$
$$\|y_k(t)\| = \left\|\chi^{t+kT}(x) - \chi^{t+kT}\right\| \le R\mathrm{e}^{-\lambda(t+kT)} \qquad \forall t \ge 0,$$

and by Proposition 4.1 we have

$$\zeta_k \in W_r(z_k), \qquad y_k(t) = y_*(t, \zeta_k),$$

provided that k is sufficiently large. Hence,

$$x_k = z_k + y_*(0, \zeta_k) \in \mathcal{W}_r(z_k) \qquad \Longrightarrow \qquad x \in \chi^{-kT}\mathcal{W}_r\left(\chi^{kT}(z)\right),$$

and the last inclusion implies the required representation for $\mathcal{W}(z)$.

By Theorem 4.7, for any $z \in \mathcal{O}(z_0)$, the set $\mathcal{W}_r(z)$ is a differentiable manifold and $\mathcal{W}_r(z) \subset \chi^{-T}\mathcal{W}_r(\chi^T(z))$. Hence,

$$\chi^{-kT}\mathcal{W}_r\left(\chi^{kT}(z)\right) \subset \chi^{-kT} \circ \chi^{-T}\mathcal{W}_r\left(\chi^T \circ \chi^{kT}(z)\right) = \chi^{-(k+1)T}\mathcal{W}_r\left(\chi^{(k+1)T}(z)\right),$$

and thus,

$$\mathcal{W}_r(z) \subset \cdots \chi^{-kT}\mathcal{W}_r\left(\chi^{kT}(z)\right) \subset \chi^{-(k+1)T}\mathcal{W}_r\left(\chi^{(k+1)T}(z)\right) \subset \cdots = \mathcal{W}(z).$$

Now it is obvious that $\mathcal{W}(z)$ is a differentiable manifold. The proof of (a) is complete.

We proceed to assertion (b). Let $x \in \mathcal{W}(z)$ and $t \in \mathbb{R}$. Then there are $i, k \in \mathbb{Z}_+$ such that $x \in \chi^{-kT}\mathcal{W}_r(\chi^{kT}(z))$ and $iT + t \ge T$. Now we obtain

$$\begin{aligned}\chi^t(x) \in \chi^{t-kT}\left(\mathcal{W}_r\left(\chi^{kT}(z)\right)\right) &= \chi^{-(k+i)T} \circ \chi^{iT+t}\left(\mathcal{W}_r\left(\chi^{kT}(z)\right)\right) \\ &\subset \chi^{-(k+i)T}\left(\mathcal{W}_r\left(\chi^{(k+i)T} \circ \chi^t(z)\right)\right) \subset \mathcal{W}\left(\chi^t(z)\right).\end{aligned}$$

Hence, $\chi^t(\mathcal{W}(z)) \subset \mathcal{W}\left(\chi^t(z)\right)$ for all $t \in \mathbb{R}$. But then $\mathcal{W}\left(\chi^{-t}(z)\right) \subset \chi^{-t}(\mathcal{W}(z))$ for all $-t \in \mathbb{R}$. This completes the proof of (b).

The proof of assertion (c) is the same as in Theorem 4.7. □

Corollary 6.3. *A λ-stable set of $\mathcal{O}(z_0)$ is generated by λ-stable set of z_0, i.e.*

$$\mathcal{W}(\mathcal{O}(z_0)) = \bigcup_{t\in\mathbb{R}} \chi^t(\mathcal{W}(z_0)).$$

7 Asymptotic phase for motions attracting by semi-invariant domains

Let us consider a more general case where the system under consideration satisfies conditions like **H1**, **H2** not on the whole invariant manifold, but on some forward χ^t-semi-invariant domain $\mathcal{M}^+ \subset \mathcal{M}$. Namely,

H1$^+$ The tangent bundle $T\mathcal{M}^+$ splits into a continuous Whitney sum $T\mathcal{M}^+ = V^s \oplus V^\star$ of forward X^t-semi-invariant vector sub-bundles $V^s = \coprod_{z\in\mathcal{M}^+} V^s_z$, $V^\star = \coprod_{z\in\mathcal{M}^+} V^\star_z$, and there exist constants $c_0 \geq 1$, $\nu > 0$, $\sigma \in [0,\nu)$ such that

$$\left\|X^t\xi\right\| \leq c_0 \mathrm{e}^{-\nu t} \left\|\xi\right\| \qquad \forall t \geq 0,\ \forall \xi \in V^s, \tag{7.1}$$

$$\left\|X^t\xi\right\| \geq c_0^{-1} \mathrm{e}^{-\sigma t} \left\|\xi\right\| \qquad \forall t \geq 0,\ \forall \xi \in V^\star. \tag{7.2}$$

H2$^+$ The natural projections

$$P_s : T\mathcal{M}^+ \mapsto V^s, \qquad P_\star : T\mathcal{M}^+ \mapsto V^\star$$

are uniformly bounded.

H3$^+$ There exists $\gamma > \sigma$ such that

$$\left\|P_N X^t P_N \xi\right\| \leq c_0 \mathrm{e}^{-\gamma t} \left\|\xi\right\| \qquad \forall t \geq 0,\ \forall \xi \in T\mathcal{M}^+.$$

It turns out that conditions **H1$^+$**,**H2$^+$** imply a counterpart of inequalities (3.8), namely, there exists a constant c_1^+ such that

$$\left\| X^t P_s \left[X^\tau\right]^{-1} P_T\big|_{T\chi^\tau(\mathcal{M}^+)} \right\| \leq c_1^+ \mathrm{e}^{-\nu(t-\tau)}, \qquad 0 \leq \tau \leq t,$$

$$\left\| X^t P_\star \left[X^\tau\right]^{-1} P_T\big|_{T\chi^\tau(\mathcal{M}^+)} \right\| \leq c_1^+ \mathrm{e}^{-\sigma(t-\tau)}, \qquad 0 \leq t < \tau.$$

E.g., derive the last inequality. Let $0 \leq t < \tau$. For any $\zeta \in V^\star\big|_{\chi^\tau(\mathcal{M}^+)}$ define $\xi = X^{-t}\zeta$. Then (3.2) implies $\left\|X^{-t}\zeta\right\| \leq c_0 \mathrm{e}^{\sigma t} \left\|\zeta\right\|$. Hence,

$$\left\|X^{t-\tau}\zeta\right\| \leq c_0 \mathrm{e}^{-\sigma(t-\tau)} \left\|\zeta\right\| \qquad \forall \zeta \in V^\star\big|_{\chi^{\tau-t}(\mathcal{M}^+)},\ 0 \leq t \leq \tau.$$

But $V^\star\big|_{\chi^\tau(\mathcal{M}^+)} \subset V^\star\big|_{\chi^{\tau-t}(\mathcal{M}^+)}$, and thus, for any $\eta \in T\chi^\tau\left(\mathcal{M}^+\right)$, we obtain

$$\begin{aligned}\left\| X^t P_\star [X^\tau]^{-1} \eta \right\| &= \left\|X^t X^{-\tau} P_\star \eta\right\| = \left\|X^{t-\tau} P_\star \eta\right\| \leq c_0 \mathrm{e}^{-\sigma(t-\tau)} \left\|P_* \eta\right\| \\ &\leq c_1^+ \mathrm{e}^{-\sigma(t-\tau)} \left\|\eta\right\|, \qquad 0 \leq t \leq \tau.\end{aligned}$$

Now in our case, one can perform all steps analogous to those of Sections 3, 4. We first observe that the projections P_s and $P_\star$ are uniformly bounded in $\mathcal{M}^+$ and satisfy counterparts of inequalities (3.8) with constant c_1^+ instead of c_1. Everywhere in what follows the mapping $[X^\tau(z)]^{-1}$ will act on $T_{\chi^\tau(z)}\mathcal{M}^+$ with $\tau \geq 0$, $z \in \mathcal{M}^+$. In view of this fact and since

$$P_T Q(\tau)\xi \in T_{\chi^\tau(z)}\mathcal{M}^+ \qquad \forall z \in \mathcal{M}^+,\ \xi \in T_z\mathcal{M}^+,\ \tau \geq 0,$$

then, for all $t \geq 0$, $\eta(t;\xi)$ is correctly defined via (3.12) and satisfies the inequality

$$\left\|\eta(t;\xi)\right\| \leq c_3 \mathrm{e}^{-\lambda t} \left\|\xi\right\|$$

with an appropriately redefined constant $c_3 > 0$. Then, in the same way as in the proof of Proposition 3.1, we define the mapping Ξ via (3.14), the projection Π and the corresponding sub-bundle

$$W^+ := \Pi\left(T\mathcal{M}^+ \oplus N\mathcal{M}^+\right).$$

Note that Π is uniformly bounded in $\mathcal{M}^+$. To prove that W^+ is forward semi-invariant, it is sufficient to take into account that the Lyapunov exponent of $\|X^t X^\tau\|$ does not exceed $-\lambda$ for any $\xi \in W^+$ and $t, \tau \in \mathbb{R}_+$ (but, in general case, not for all $\tau \in \mathbb{R}$ and $t \geq -\tau$ as in Proposition 3.1). Now, as in Corollary 3.2, we obtain the estimates for $\|X^t \Pi [X^\tau]^{-1}\|$ and $\|X^t (\mathrm{Id} - \Pi) [X^\tau]^{-1}\|$ with appropriately redefined constant K.

Next, in Section 4, up to Proposition 4.2 we need to replace $\mathcal{M}, W, \mathcal{M}_r$ with $\mathcal{M}^+, W^+, \mathcal{M}_r^+$ respectively. As a consequence, W_r, U_r will be replaced by W_r^+, U_r^+. The proofs of counterparts to Propositions 4.1–4.4 need no changes, except that starting from Proposition 4.2 the constants C and r should be found via the relevant inequalities, e.g. (4.5), (4.7), involving redefined constant K. As a result, for any $z \in \mathcal{M}^+$, we are able to define the sets

$$\mathcal{W}_r^+(z) := z + y_* \left(0, W_r^+(z)\right), \qquad \mathcal{U}_r^+(z) := z + y_* \left(0, U_r^+(z)\right),$$

where

$$W_r^+(z) := W_r^+ \cap \pi^{-1}(z), \qquad U_r^+(z) := U_r^+ \cap \pi^{-1}(z).$$

Since $\mathcal{M}^+$ is forward semi-invariant, then for any $z_0 \in \mathcal{M}^+$, it is natural to define the phase curve

$$\mathcal{O}^+(z_0) := \mathcal{O}(z_0) \cap \mathcal{M}^+$$

which includes all points of orbit $\mathcal{O}(z_0)$ containing in $\mathcal{M}^+$. And since $\mathcal{M}^+$ is open, then there exists $\epsilon(z_0) > 0$ such that

$$\bigcup_{-\epsilon(z_0) < t < \infty} \{\chi^t(z_0)\} \subset \mathcal{O}^+(z_0) \subset \mathcal{M}^+.$$

Finally, we define the local λ-stable set of the phase curve $\mathcal{O}^+(z_0)$ as

$$\mathcal{W}_r\left(\mathcal{O}^+(z_0)\right) := \bigcup_{z \in \mathcal{O}^+(z_0)} \mathcal{W}_r^+(z).$$

Now the assertions (a), (b), (c) of Theorem 4.7 as well as their proofs remain correct for every $z_0 \in \mathcal{M}^+$ after we replace $\mathcal{M}, \mathcal{O}(z_0), W, \mathcal{W}$ with $\mathcal{M}^+, \mathcal{O}^+(z_0), W^+, \mathcal{W}^+$ respectively. As a consequence, we obtain the following counterpart of Theorem 5.1.

Theorem 7.1. *Let system (2.1) satisfy conditions* **H1**$^+$ – **H3**$^+$ *in a forward χ^t-semi-invariant domain $\mathcal{M}^+ \subset \mathcal{M}$. Then there exists $\varepsilon > 0$ such that a motion $t \mapsto \chi^t(x)$ has an asymptotic phase, provided that $\mathcal{O}(x) \cap N\mathcal{M}_\varepsilon^+ \neq \varnothing$, where $N\mathcal{M}_\varepsilon^+$ is a portion of the tubular neighborhood $N\mathcal{M}_\varepsilon$ over $\mathcal{M}^+$.*

Now let us apply Theorem 7.1 to each forward semi-invariant domain $\chi^{-k}(\mathcal{M}^+)$. Then we obtain a sequence of positive numbers $\{\varepsilon_k\}$ and the corresponding sequence of sets $\{N\mathcal{M}_{\varepsilon_k}^+\}$. Next define χ^t-invariant domains in $\mathcal{M}$ and $\mathbb{R}^n$ respectively

$$\mathcal{M}' := \bigcup_{k \in \mathbb{N}} \chi^{-k}(\mathcal{M}^+), \qquad \mathcal{D}' = \bigcup_{t \geq 0} \chi^{-t}\left(\bigcup_{k \in \mathbb{N}} N\mathcal{M}_{\varepsilon_k}^+\right).$$

Finally we arrive at the following result

Theorem 7.2. *Let system (2.1) satisfy conditions* **H1**$^+$–**H3**$^+$ *in a forward χ^t-semi-invariant domain $\mathcal{M}^+ \subset \mathcal{M}$. Then for any $x \in \mathcal{D}'$ the motion $t \mapsto \chi^t(x)$ has an asymptotic phase in $\mathcal{M}'$.*

8 A system on cotangent bundle of a compact homogeneous space $\mathrm{SL}(2;\mathbb{R})/\Gamma$

Consider a (right) homogeneous space $\mathcal{Q} =: \mathfrak{G}/\Gamma := \{q = G\Gamma : G \in \mathfrak{G}\}$ where $\mathfrak{G} := \mathrm{SL}(2;\mathbb{R})$ and Γ is a discrete subgroup of $\mathfrak{G}$ such that $\mathcal{Q}$ is compact. As is well known, homogeneous spaces of such a kind are naturally associated with compact Riemannian surfaces of constant negative curvature, and the geodesic flows on such surfaces are classical examples of Anosov dynamical systems [1,2,5,30]. We aims to apply the results of previous sections to a specific system defined on cotangent bundle $T^*\mathcal{Q}$. To obtain such a system, we first construct an appropriate right-invariant system on cotangent bundle $T^*\mathfrak{G}$ and then factorize it by the right action of the lattice Γ.

Recall that the group $\mathfrak{G}$ generates a Poissonian action on $T^*\mathfrak{G}$ (see [3]). Namely, let Λ be the Liouville 1-form ("$p\mathrm{d}q$"-form) on $T^*\mathfrak{G}$. The exact 2-form $\omega^2 := \mathrm{d}\Lambda$ defines a standard symplectic structure on $T^*\mathfrak{G}$. For any $A \in \mathfrak{g} := \mathrm{sl}(2,\mathbb{R})$, denote by $AG := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{e}^{At}G$ the right-invariant vector field generating the left action of one-parameter subgroup $\{\mathrm{e}^{At}\}$ on $\mathfrak{G}$. There is a natural lift of this action to $T^*\mathfrak{G}$ as the flow of Hamiltonian system with right invariant Hamiltonian function

$$h_a(x) = \Lambda\left(A\pi\left(x\right)\right), \qquad x \in T^*\mathfrak{G},$$

where $\pi : T^*\mathfrak{G} \mapsto \mathfrak{G}$ is the natural projection. For any $A, B \in \mathfrak{g}$, the Poisson bracket of Hamiltonians $h_A(\cdot)$, $h_B(\cdot)$ satisfies

$$\{h_A, h_B\}(x) := \omega\left(A\pi(x), A\pi(x)\right) = h_{[A,B]}(x), \qquad x \in T^*\mathfrak{G},$$

meaning that $\mathfrak{G}$-action on $T^*\mathfrak{G}$ is Poissonian. Let $m(\cdot) : T^*\mathfrak{G} \mapsto \mathfrak{g}^*$ be the corresponding momentum map, thus $m(x)A := h_A(x)$. Choose a standard base in $\mathfrak{g}$ represented, respectively, by the matrices

$$A_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and define the corresponding components of co-vector $m(x)$ by setting

$$m_k(x) := m(x)A_k, \qquad k \in \{1,2,3\}.$$

Since

$$[A_1, A_2] = -2A_3, \qquad [A_1, A_3] = 2A_2, \qquad [A_2, A_3] = 2A_1,$$

then

$$\{m_1, m_2\} = -2m_3, \qquad \{m_1, m_3\} = 2m_2, \qquad \{m_2, m_3\} = 2m_1 \tag{8.1}$$

The diffeomorphism

$$T^*\mathfrak{G} \ni x \mapsto (m(x), \pi(x)) \in \mathfrak{g}^* \times \mathfrak{G}$$

induces a Poissonian structure on $\mathfrak{g}^* \times \mathfrak{G}$ such that the brackets $\{m_i, m_j\}$ satisfy (8.1), the bracket of any pair of functions $f_i(\cdot), f_j(\cdot) : \mathfrak{G} \mapsto \mathbb{R}$ equals zero, and

$$\{G, m_k\} = A_k G, \tag{8.2}$$

meaning that $\mathfrak{G}$-component of Hamiltonian vector field with Hamiltonian function m_k is the right-invariant vector field $A_k G$.

Now introduce a Hamiltonian of the form

$$H(m) := \frac{1}{2}\sum_{k=1}^{3}\lambda_k m_k^2$$

where $\lambda_1, \lambda_2, \lambda_3$, are given positive numbers. The corresponding Hamiltonian system on $\mathfrak{g}^* \times \mathfrak{G}$ reads

$$\dot{m} = \{m, H(m)\}\,, \tag{8.3}$$

$$\dot{G} = \sum_{k=1}^{3}\lambda_k m_k A_k G. \tag{8.4}$$

Here

$$\{m, H(m)\} := \begin{pmatrix}\{m_1, H(m)\}\\ \{m_2, H(m)\}\\ \{m_3, H(m)\}\end{pmatrix} \equiv \begin{pmatrix}2\,(\lambda_3 - \lambda_2)\, m_2 m_3\\ 2\,(\lambda_1 + \lambda_3)\, m_1 m_3\\ -2\,(\lambda_1 + \lambda_2)\, m_1 m_2\end{pmatrix}.$$

Note that $H(m)$ in a standard way defines a right-invariant metrics $\langle\cdot,\cdot\rangle$ on $\mathfrak{G}$, and thus, one can consider system (8.3)–(8.4) as the Hamiltonian form of Lagrangian system for geodesics on $\mathfrak{G}$.

It is easily seen that, except $H(m)$, system (8.3) has an additional first integral (the Casimir function for the Poisson bracket on $\mathfrak{g}^*$)

$$J(m) = m_1^2 - m_2^2 - m_3^2.$$

If we consider sub-system (8.3) in $\mathfrak{g}^*$, then for any constants c_1 and c_2 satisfying

$$\min\{c_1 - \lambda_1 c_2, c_1 + \lambda_2 c_2, c_1 + \lambda_3 c_2\} > 0$$

the set $H^{-1}(c_1) \cap J^{-1}(c_2)$ is a union of two closed phase curves. Let $\mathcal{C}$ be one of such curves, and c_1^0, c_2^0 be the corresponding values of the constants. There is a tubular neighborhood $\mathcal{W}$ of $\mathcal{C}$ that is diffeomorphic to a direct product $D \times \mathbb{S}^1$ where $D \subset \mathbb{R}^2$ is a disc centered at the origin and $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. The set $\mathcal{W}$ is foliated by closed phase curves of system (8.3). By means of an appropriate diffeomorphism $\mu(\cdot) : \mathbb{S}^1 \times D \mapsto \mathcal{W}$ one can introduce an action-angular-type coordinates $(y_1, y_2, \theta|\bmod 1)$ in $D \times \mathbb{S}^1$ in such a way that the following relations are satisfied

$$\begin{aligned} y_1 &= H(m) - c_1^0, & y_2 &= J(m) - c_2^0,\\ H \circ \mu(\theta, y) &= y_1 + c_1^0, & J \circ \mu(\theta, y) &= y_2 + c_2^0,\\ \{\theta, y_1\} &= \omega(c^0 + y), & \{\theta, y_2\} &= 0. \end{aligned}$$

Here $\omega(c) > 0$ is a frequency of periodic motion over the closed phase curve given by the equation $m = \mu(\theta, c)$, and thus, being a component of $H^{-1}(c_1) \cap J^{-1}(c_2)$.

Now, instead of (8.3), consider a system

$$\dot{m} = \{m, H(m)\} + F(m) \tag{8.5}$$

with a perturbation term $F(m)$ under the impact of which the cycle $\mathcal{C}$ becomes asymptotically stable. For example, if we set

$$\begin{aligned} F(m) :=& -\varepsilon\left[(H(m) - c_1^0)\,\|\nabla J(m)\|^2 - (J(m) - c_2^0)\,\langle\nabla H(m), \nabla J(m)\rangle\right]\nabla H(m)\\ &- \varepsilon\left[(J(m) - c_2^0)\,\|\nabla H(m)\|^2 - (H(m) - c_1^0)\,\langle\nabla H(m), \nabla J(m)\rangle\right]\nabla J(m) \end{aligned} \tag{8.6}$$

where ε is a (small) positive parameter, then the derivative of the Lyapunov function

$$U(m) := \left(H(m) - c_1^0\right)^2 + \left(J(m) - c_2^0\right)^2$$

by virtue of system (8.5) is

$$\dot{U}(m) = -\varepsilon U(m)\left[\|\nabla H(m)\|^2 \|\nabla J(m)\|^2 - \langle \nabla H(m), \nabla J(m)\rangle^2\right].$$

Hence, both $U(\cdot)$ and $\dot{U}(\cdot)$ vanish along $\mathcal{C}$. Furthermore, since

$$\min_{m\in\mathcal{C}}\left\{\|\nabla H(m)\|^2 \|\nabla J(m)\|^2 - \langle \nabla H(m), \nabla J(m)\rangle^2\right\} > 0,$$

then there is $\varkappa > 0$ such that the inequality

$$\dot{U}(m) \leq -\varepsilon\varkappa U(m) \tag{8.7}$$

is satisfied in $\mathcal{W}$, provided that this tubular neighborhood is sufficiently small. This inequality ensures the exponential stability of $\mathcal{C}$ as a limit cycle of perturbed system (8.5). In fact, note that in the coordinates (y, θ) system (8.5) can be presented in the form

$$\dot{y} = P(\theta)y + O(\|y\|^2), \qquad \dot{\theta} = \omega\left(c^0\right) + \left\langle \nabla\omega\left(c^0\right) + b(\theta), y\right\rangle + O(\|y\|^2) \tag{8.8}$$

with a 1-periodic (2×2)-matrix $P(\cdot)$ and 2-vector $b(\cdot)$, and on account of (8.7) the derivative of the function $U \circ \mu\left(\theta, c^0 + y\right) = \|y\|^2$ by virtue of this system does not exceed $-\varepsilon\varkappa\|y\|^2$. Furthermore, the derivative of $\|y\|^2$ by virtue of linearized system

$$\dot{y} = P(\theta)y, \qquad \dot{\theta} = \omega\left(c^0\right) \tag{8.9}$$

is $2\langle P(\theta)y, y\rangle$, and thus, does not exceed $-\varepsilon\varkappa\|y\|^2/2$, provided $\mathcal{W}$ is small enough. (Note that that the last system generates the normal co-cycle associated with the flow on $\mathcal{C}$.) The obtained inequalities imply that y- components of solutions starting in $\mathcal{W}$ of both systems (8.8) and (8.9) vanish with exponential rate as $t \to \infty$.

To find stable limit cycles of system (8.5) in the case where F is a small vector field of general kind one can apply the well developed perturbation theory of periodic solutions (see e.g. [18]).

Now, after we have established that $\mathcal{C}$ is exponentially stable limit cycle of system (8.5), we proceed to analyze the structure of the flow generated by system (8.5) – (8.4) (or, what is the same, of system (8.3) – (8.4)) on its invariant manifold $\mathcal{C} \times \mathfrak{G}$. Since the motion of a point $m_0 = \mu(\theta) := \mu\left(c^0, \theta\right)$ on $\mathcal{C}$ is given by $t \mapsto \mu\left(\omega t + \theta\right)$, $\omega := \omega\left(c^0\right)$, we arrive at the linear system with $\frac{1}{\omega}$-periodic coefficients

$$\dot{G} = A(\omega t + \theta)G \tag{8.10}$$

where

$$A(\theta) := \left[\sum_{k=1}^{3} \lambda_k \mu_k(\theta) A_k\right] \tag{8.11}$$

Let $\mathcal{G}^t(\theta)$ stands for a fundamental matrix of (8.10) such that $\mathcal{G}^0(\theta) = \mathrm{Id}$. Thus, the motion of arbitrary point $(\mu(\theta), G) \in \mathcal{C} \times \mathfrak{G}$ in virtue of system (8.3)–(8.4) is governed by the mapping

$$t \mapsto \left(\mu(\omega t + \theta), \mathcal{G}^t(\theta)G\right).$$

Then for any $a \in \mathbb{R}$ and $B \in \mathrm{sl}(2;\mathbb{R})$, the motion of tangent vector $(a\mu'(\theta), BG)$ under the action of the corresponding tangent co-cycle is governed by the mapping

$$t \mapsto \left(a\mu'(\omega t+\theta), a\left[\mathcal{G}^t(\theta)\right]'_\theta G + \mathcal{G}^t(\theta)BG\right). \tag{8.12}$$

The co-cycle property of $\mathcal{G}^t(\theta)$ yields

$$\mathcal{G}^{t+s}(\theta) = \mathcal{G}^t(\omega s+\theta)\mathcal{G}^s(\theta) \quad \Longrightarrow \quad \mathcal{G}^t(\theta) = \mathcal{G}^{t+\theta/\omega}(0)\left[\mathcal{G}^{\theta/\omega}(0)\right]^{-1}, \tag{8.13}$$

$$\frac{\partial}{\partial s}\Big|_{s=0}\mathcal{G}^{t+s}(\theta) = \frac{\partial}{\partial s}\Big|_{s=0}\mathcal{G}^t(\omega s+\theta)\mathcal{G}^s(\theta) \quad \Longrightarrow \quad \frac{\partial}{\partial t}\mathcal{G}^t(\theta) = \omega\left[\mathcal{G}^t(\theta)\right]'_\theta + \mathcal{G}^t(\theta)A(\theta). \tag{8.14}$$

Now we see that the tangent bundle of $\mathcal{C}\times\mathfrak{G}$ splits into two invariant sub-bundles: the first one is spanned by the vector field of the flow $(\omega\mu'(\theta), A(\theta)G)$ (on account of (8.12), (8.14) this fact is also the consequence of (8.12) and (8.14) for $a=\omega$, $B=A(\theta)$), and the second one is naturally identified with the tangent bundle $T\mathfrak{G}$ by the correspondence $(\mu(\theta), G)\mapsto(0, T_G\mathfrak{G})$. Thus, it remains to analyze properties of the tangent co-cycle action on $T_G\mathfrak{G}$.

In what follows, we will focus on the hyperbolic case where the monodromy matrix $M := \mathcal{G}^{1/\omega}(0)$ has real eigenvalues (the Floquet multipliers) $\rho_1=\rho$ and $\rho_2=\rho^{-1}$, $|\rho|>1$. Numerical experiments show that this case actually takes place for an appropriate range of parameters $\lambda_1,\lambda_2,\lambda_3$ and c_0, c_1. E.g., in particular case where $\lambda_1 = 3/2$; $\lambda_2 = 3$; $\lambda_3 = 3/2$, $m_1(0)\equiv 4/5$, $m_2(0)=1$, $m_3(0)=0$, and thus, $c_1^0 = 3.96$, $c_2^0 = -0.36$, we obtain

$$M \approx \begin{pmatrix} -6.84081991830724 & 2.57720475804614 \\ -2.57720426780894 & 0.82475266189202 \end{pmatrix}, \qquad \rho \approx -5.84498051556855.$$

By the Floquet theorem, there exists a mapping $\Phi(\cdot):\mathbb{R}\mapsto \mathrm{SL}(2;\mathbb{R})$ such that

$$\Phi(\theta+1) = \operatorname{sign}\rho\,\Phi(\theta), \qquad \mathcal{G}^t(0) = \Phi(\omega t)\mathrm{e}^{Lt}$$

where $L := \omega\ln(\operatorname{sign}\rho M)\in \mathrm{sl}(2;\mathbb{R})$. Now (8.13) implies

$$\begin{aligned}\mathcal{G}^t(\theta)BG &= \Phi(\omega t+\theta)\mathrm{e}^{Lt}\Phi^{-1}(\theta)BG \\ &= \Phi(\omega t+\theta)\left[\mathrm{e}^{Lt}\Phi^{-1}(\theta)B\Phi(\theta)\,\mathrm{e}^{-Lt}\right]\mathrm{e}^{Lt}\Phi^{-1}(\theta)G.\end{aligned}$$

We see that the properties of the tangent co-cycle action on $T_G\mathfrak{G}$ are completely determined by the adjoint action of the one-parameter sub-group $\{\mathrm{e}^{Lt}\}$ on $\mathfrak{g}$, and thus, by the spectrum of the corresponding operator $\mathrm{ad}_L:\mathfrak{g}\mapsto\mathfrak{g}$, $\mathrm{ad}_L A := LA - AL$. It is not hard to calculate

$$\sigma(\mathrm{ad}_L) = \left\{0, 2\sqrt{-\det L}, -2\sqrt{-\det L}\right\} = \left\{0, \ln\rho^2, -\ln\rho^2\right\}.$$

Now consider a system on $T^*\mathcal{Q}$ obtained by factorization of system (8.5)–(8.4)

$$\dot m = \{m, H(m)\} + F(m), \qquad \dot q = Q(m,q), \tag{8.15}$$

where $q\in\mathcal{Q}$, and $Q(m,q) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\exp\left(\sum_{k=1}^3\lambda_k m_k A_k\right)q$. The above reasoning implies that this system has 4-D compact exponentially stable invariant manifold $\mathcal{M}=\mathcal{C}\times\mathcal{Q}$. The tangent bundle $T\mathcal{M}$ admits invariant splitting into a Whitney sum $V^s\oplus V^c\oplus V^u$ of three sub-bundles: 1-D stable V^s, 1-D unstable V^u, and 2-D center V^c (every co-cycle orbit on V^c is bounded). To show that any motion starting close to $\mathcal{M}$ has an asymptotic phase we are going to apply Theorem 5.1. It should be noted that the mentioned theorem concerns systems situated in

Euclidean spaces. Hence, we have to embed system (8.15) into an auxiliary system possessing the same exponentially stable invariant manifold $\mathcal{M}$ and defined in a domain of Euclidean space.

Since $\mathcal{Q}$ is parallelizable, then it is a π-manifold [24], and thus, $\mathcal{Q}$ can be embedded in a Euclidean space $\mathbb{R}^d$ of a sufficiently high dimension d with trivial normal bundle $N\mathcal{Q} \sim \mathcal{Q} \times \mathbb{R}^{d-3}$. Hence, for sufficiently small $\delta > 0$, there exists a diffeomorphism $\psi(\cdot)$ of $\mathcal{Q} \times \mathcal{B}_\delta^{d-3}(0)$** onto a tubular neighborhood of $\mathcal{Q}$ in $\mathbb{R}^d$ such that $\psi\big(\{q\} \times \mathcal{B}_\delta^{d-3}(0)\big) \subset N_q\mathcal{Q}$ for any $q \in \mathcal{Q}$. Finally, to obtain the required auxiliary system we embed the system

$$\dot{m} = \{m, H(m)\} + F(m), \qquad \dot{q} = Q(m,q), \qquad y = -y$$

into $\mathbb{R}^3 \times \mathbb{R}^d$ by means of the diffeomorphism $\mathrm{id}(\cdot) \times \psi(\cdot) : \mathbb{R}^3 \times \mathcal{Q} \times \mathcal{B}_\delta^{d-3}(0) \mapsto \mathbb{R}^3 \times \mathbb{R}^d$.

9 Conclusion

In order to simplify our exposition we restrict ourselves to the case where the invariant manifold is situated in Euclidean space. Actually, this is not a serious restriction. If we deal with a system defined on a manifold $\mathfrak{M}$ and $\mathcal{M}$ is an attracting invariant submanifold, than we can apply the same trick as in Section 8. Namely, we have to embed the manifold $\mathfrak{M}$ into Euclidean space of sufficiently high dimension d and to extend the initial system to an auxiliary d-dimensional system such that its domain is a neighborhood of $\mathfrak{M}$ in $\mathbb{R}^d$ and its motions are attracted by $\mathcal{M}$.

It would be interesting to consider the case where the attracting invariant manifold admits a partition into subsets with different types of partial hyperbolicity. Just this case can happen when, for an appropriate perturbation $F(\cdot)$, the attracting manifold of system (8.5) is some level set of the Hamiltonian. We expect that in such a situation, to tackle the problem on the existence of asymptotic phase, the results of Section 7 might be useful.

Acknowledgements

We thank the anonymous referee for the helpful comments, and constructive remarks on our manuscript.

This research was partially supported by the Ministry of Education and Science of Ukraine (grant No. 0119U100334).

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**Recall the notation $\mathcal{B}_\delta^{d-3}(0) := \{y \in \mathbb{R}^{d-3} : \|y\| < \delta\}$.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 37, 1–23; https://doi.org/10.14232/ejqtde.2021.1.37 www.math.u-szeged.hu/ejqtde/

Sign-changing solutions for fourth-order elliptic equations of Kirchhoff type with critical exponent

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Received 27 January 2021, appeared 17 April 2021
Communicated by Roberto Livrea

Abstract. In this paper, we study the existence of ground state sign-changing solutions for the following fourth-order elliptic equations of Kirchhoff type with critical exponent. More precisely, we consider

$$\begin{cases} \Delta^2 u - \left(1 + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) + |u|^{2^{**}-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator, $N = \{5,6,7\}$, $2^{**} = 2N/(N-4)$ is the Sobolev critical exponent and $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary and b, λ are some positive parameters. By using constraint variational method, topological degree theory and the quantitative deformation lemma, we prove the existence of ground state sign-changing solutions with precisely two nodal domains.

Keywords: Kirchhoff type problem, fourth-order elliptic equation, critical growth, sign-changing solution.

2020 Mathematics Subject Classification: 35A15, 35J60, 47G20.

1 Introduction and main results

In this paper, we are interested in the existence of least energy nodal solutions for the following Kirchhoff-type fourth-order Laplacian equations with critical growth:

$$\begin{cases} \Delta^2 u - \left(1 + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) + |u|^{2^{**}-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ^2 is the biharmonic operator, $N = \{5,6,7\}$, $2^{**} = 2N/(N-4)$ is the Sobolev critical exponent, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, and b, λ are some positive parameters.

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Now we introduce the assumptions on the function f that will in full force throughout the paper. More precisely, we suppose that $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following conditions:

(f_1) $\lim_{t\to 0} \frac{f(x,t)}{|t|^3} = 0$;

(f_2) There exist $\theta \in (4, 2^{**})$ and $C > 0$ such that $|f(x,t)| \leq C(1 + |t|^{\theta-1})$ for all $t \in \mathbb{R}$;

(f_3) $\frac{f(x,t)}{|t|^3}$ is a strictly increasing function of $t \in \mathbb{R} \setminus \{0\}$.

A simple example of function satisfying the above assumptions (f_1)–(f_3) is $f(t) = t|t|^{\theta-2}$ for any $t \in \mathbb{R}$, where $\theta \in (4, 2^{**})$.

Our motivation for studying problem (1.1) is two-fold. On the one hand, there is a vast literature concerning the existence and multiplicity of solutions for the following Dirichlet problem of Kirchhoff type

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Problem (1.2) is a generalization of a model introduced by Kirchhoff. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical d'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. The problem (1.2) is related to the stationary analogue of problem (1.3). Problem (1.2) received much attention only after Lions [17] proposed an abstract framework to the problem. For example, some important and interesting results can be found in [5,9,10,12–14,16,25,26,39]. We note that the results dealing with the problem (1.2) with critical nonlinearity are relatively scarce. The main difficulty in the study of these problems is due to the lack of compactness caused by the presence of the critical Sobolev exponent.

Recently, many researchers devoted themselves to the following fourth-order elliptic equations of Kirchhoff type

$$\begin{cases} \Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

In fact, this is related to the following stationary analogue of the Kirchhoff-type equation:

$$u_{tt} + \Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), \quad x \in \Omega, \tag{1.5}$$

where $a, b > 0$. In [2,4], Eq. (1.5) was used to describe some phenomena appearing in different physical, engineering and other sciences for dimension $N \in \{1, 2\}$, as a good approximation for describing nonlinear vibrations of beams or plates. Different approaches have been taken to deal with this problem under various hypotheses on the nonlinearity. For example, Ma in [21] considered the existence and multiplicity of positive solutions for the fourth-order equation by using the fixed point theorems in cones of ordered Banach spaces. By variational methods, Wang and An in [34] studied the following fourth-order equation of Kirchhoff type

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

and obtained the existence and multiplicity of solutions, see [19, 20, 34] for more results. For $M(t) = \lambda(a + bt)$, Wang *et al.* in [35] proved the existence of solutions for problem (1.6) as λ small, by employing the mountain pass theorem and the truncation method. In [30], Song and Shi obtained the existence and multiplicity of solutions for problem (1.6) critical exponent in bounded domains by using the concentration-compactness principle and variational method. In [41], by variational methods together with the concentration-compactness principle, Zhao *et al.* investigated the existence and multiplicity of solutions for problem (1.6) with critical nonlinearity. In [15], by using the same method as in [41], Liang and Zhang obtained the existence and multiplicity of solutions for perturbed biharmonic equation of Kirchhoff type with critical nonlinearity in the whole space.

On the other hand, many authors paid attention to finding sign-changing solutions for problem (1.2) or similar Kirchhoff-type equations, and consequently some interesting results have been obtained recently. For example, Zhang and Perera in [40] and Mao and Zhang in [23] used the method of invariant sets of descent flow to obtain the existence of a sign-changing solution of problem (1.2). In [7], Figueiredo and Nascimento studied the following Kirchhoff equation of type:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.7}$$

where Ω is a bounded domain in $\mathbb{R}^3$, M is a general C^1 class function, and f is a superlinear C^1 class function with subcritical growth. By using the minimization argument and a quantitative deformation lemma, the existence of a sign-changing solution for this Kirchhoff equation was obtained. In unbounded domains, Figueiredo and Santos Júnior in [8] studied a class of nonlocal Schrödinger–Kirchhoff problems involving only continuous functions. Using a minimization argument and a quantitative deformation lemma, they got a least energy sign-changing solution to Schrödinger–Kirchhoff problems. Moreover, the authors obtained that the problem has infinitely many nontrivial solutions when it presents symmetry.

It is worth mentioning that combining constraint variational methods and quantitative deformation lemma, Shuai in [29] proved that problem (1.2) has one least energy sign-changing solution u_b and the energy of u_b strictly larger than the ground state energy. Moreover, the author investigated the asymptotic behavior of u_b as the parameter $b \searrow 0$. Later, under some more weak assumptions on f (especially, Nehari type monotonicity condition been removed), with the aid of some new analytical skills and Non-Nehari manifold method, Tang and Cheng in [32] improved and generalized some results obtained in [29]. In [6], Deng *et al.* studied the following Kirchhoff-type problem:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(x,u), \qquad x \in \mathbb{R}^3. \tag{1.8}$$

The authors obtained the existence of radial sign-changing solutions with prescribed numbers of nodal domains for Kirchhoff problem (1.8), by using a Nehari manifold and gluing solution pieces together, when $V(x) = V(|x|)$, $f(x,u) = f(|x|,u)$ and satisfies some assumtions.

Precisely, they proved the existence of a sign-changing solution, which changes signs exactly k times for any $k \in \mathbb{N}$. Moreover, they investigated the energy property and the asymptotic behavior of the sign-changing solution. By using a combination of the invariant set method and the Ljusternik-Schnirelman type minimax method, Sun *et al.* in [31] obtained infinitely many sign-changing solutions for Kirchhoff problem (1.8) when $f(x,u) = f(u)$ and f is odd in u. It is worth noticing that, in [31], the nonlinear term may not be 4-superlinear at infinity; In particular, it encloses the power-type nonlinearity $|u|^{p-2}u$ with $p \in (2,4]$. In [33], the authors obtained the existence of least energy sign-changing solutions of Kirchhoff-type equation with critical growth by using the constraint variational method and the quantitative deformation lemma. For more results on sign-changing solutions for Kirchhoff-type equations, we refer the reader to [6,11,18,22,36] and the references therein.

However, concerning the existence of sign-changing solutions for fourth-order elliptic equations of Kirchhoff type with critical exponent, to the best of our knowledge, so far there has been no paper in the literature where existence of sign-changing solutions to problem (1.1). Hence, a natural question is whether or not there exists sign-changing solutions of problem (1.1)? The goal of the present paper is to give an affirmative answer.

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth open domain, $E = H^2(\Omega) \cap H_0^1(\Omega)$ be the Hilbert space equipped with the inner product

$$\langle u, v\rangle_E = \int_\Omega (\Delta u \Delta v + \nabla u \nabla v) dx$$

and the deduced norm

$$\|u\|_E^2 = \int_\Omega (|\Delta u|^2 + |\nabla u|^2) dx.$$

It is well know that $\|u\|_E$ is equivalent to

$$\|u\| := \left(\int_\Omega |\Delta u|^2 dx\right)^{\frac{1}{2}}.$$

And there exists $\tau > 0$ such that

$$\|u\| \le \|u\|_E \le \tau \|u\|.$$

For the weak solution, we mean the one satisfies the following definition.

Definition 1.1. We say that $u \in E$ is a (weak) solution of problem (1.1) if

$$\begin{aligned}\int_\Omega (\Delta u \cdot \Delta v + \nabla u \nabla v)\, dx + b\left(\int_\Omega |\nabla u|^2 dx\right) \int_\Omega \nabla u \cdot \nabla v dx \\ = \int_\Omega \left(|u|^{2^{**}-2} uv + \lambda f(x,u) v\right) \mathrm{d}x \quad (1.9)\end{aligned}$$

for any $v \in E$.

The corresponding energy functional $I_b^\lambda : E \to \mathbb{R}$ to problem (1.1) is defined by

$$\begin{aligned} I_b^\lambda(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 + |\nabla u|^2)\, dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 \\ - \lambda \int_\Omega F(x,u) dx - \frac{1}{2^{**}} \int_\Omega |u|^{2^{**}} dx. \end{aligned} \quad (1.10)$$

It is easy to see that I_b^λ belongs to $C^1(E,\mathbb{R})$ and the critical points of I_b^λ are the solutions of (1.1). Furthermore, if we write $u^+(x) = \max\{u(x),0\}$ and $u^-(x) = \min\{u(x),0\}$ for $u \in E$, then every solution $u \in E$ of problem (1.1) with the property that $u^\pm \neq 0$ is a sign-changing solution of problem (1.1).

Our goal in this paper is then to seek for the least energy sign-changing solutions of problem (1.1). As well known, there are some very interesting studies, which studied the existence and multiplicity of sign-changing solutions for the following problem:

$$-\Delta u + V(x)u = f(x,u), \quad x \in \Omega, \tag{1.11}$$

where Ω is an open subset of $\mathbb{R}^N$. However, these methods of seeking sign-changing solutions heavily rely on the following decompositions:

$$J(u) = J(u^+) + J(u^-), \tag{1.12}$$

$$\langle J'(u),u^+\rangle = \langle J'(u^+),u^+\rangle, \langle J'(u),u^-\rangle = \langle J'(u^-),u^-\rangle, \tag{1.13}$$

where J is the energy functional of (1.11) given by

$$J(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + V(x)u^2)dx - \int_\Omega F(x,u)dx.$$

However, if $b > 0$, the energy functional I_b^λ does not possess the same decompositions as (1.12) and (1.13). In fact, a straightforward computation yields that

$$I_b^\lambda(u) > I_b^\lambda(u^+) + I_b^\lambda(u^-),$$

$$\langle (I_b^\lambda)'(u),u^+\rangle > \langle (I_b^\lambda)'(u^+),u^+\rangle \quad \text{and} \quad \langle (I_b^\lambda)'(u),u^-\rangle > \langle (I_b^\lambda)'(u^-),u^-\rangle$$

for $u^\pm \neq 0$. Therefore, the classical methods to obtain sign-changing solutions for the local problem (1.11) do not seem applicable to problem (1.1). In this paper, we follow the approach in [3] by defining the following constrained set

$$\mathcal{M}_b^\lambda = \Big\{u \in E, u^\pm \neq 0 \text{ and } \langle (I_b^\lambda)'(u),u^+\rangle = \langle (I_b^\lambda)'(u),u^-\rangle = 0\Big\} \tag{1.14}$$

and considering a minimization problem of I_b^λ on $\mathcal{M}_b^\lambda$. Indeed, by using the parametric method and implicit theorem, Shuai in [29] proved $\mathcal{M}_b^\lambda \neq \emptyset$ in the absence of the nonlocal term. However, the nonlocal term in problem (1.1), consisting of the biharmonic operator and the nonlocal term will cause some difficulties. Roughly speaking, compared to the general Kirchhoff type problem (1.2), decompositions (1.12) and (1.13) corresponding to I_b^λ are much more complicated. This results in some technical difficulties during the proof of the nonempty of $\mathcal{M}_b^\lambda$. Moreover, we find that the parametric method and implicit theorem are not applicable for problem (1.1) due to the complexity of the nonlocal term there. Therefore, our proof is based on a different approach which is inspired by [1], namely, we make use of a modified Miranda's theorem (cf. [24]). In addition, we are also able to prove that the minimizer of the constrained problem is also a sign-changing solution via the quantitative deformation lemma and degree theory.

Now we can present our first main result.

Theorem 1.2. *Assume that* (f_1)*–*(f_3) *hold. Then, there exists* $\lambda^* > 0$ *such that for all* $\lambda \geq \lambda^*$*, problem* (1.1) *has a least energy sign-changing solution* u_b.

Another goal of this paper is to establish the so-called energy doubling property (cf. [37]), i.e., the energy of any sign-changing solution of problem (1.1) is strictly larger than twice the ground state energy. For the semilinear equation problem (1.13), the conclusion is trivial. Indeed, if we denote the Nehari manifold associated to problem (1.13) by

$$\mathcal{N} = \{u \in E \setminus \{0\} \mid \langle J'(u), u\rangle = 0\}$$

and define

$$c = \inf_{u\in\mathcal{N}} J(u) \tag{1.15}$$

then it is easy to verify that $u^{\pm} \in \mathcal{N}$ for any sign-changing solution $u \in E$ for problem (1.13). Moreover, if the nonlinearity $f(x,t)$ satisfies some conditions (see [3]) which is analogous to (f_1)–(f_3), we can deduce that

$$J(w) = J(w^+) + J(w^-) \geq 2c. \tag{1.16}$$

We point out that the minimizer of (1.14) is indeed a ground state solution of problem (1.11) and $c > 0$ is the least energy of all weak solutions of problem (1.11). Therefore, by (1.15), it follows that the energy of any sign-changing solution of problem (1.11) is larger than twice the least energy. When $b > 0$, a similar result was obtained by Shuai [29] in a bounded domain Ω. We are also interested in that whether property (1.15) is still true for problem (1.1). To answer this question, we have the following result:

Theorem 1.3. *Assume that (f_1)–(f_3) hold. Then, there exists $\lambda^{**} > 0$ such that for all $\lambda \geq \lambda^{**}$, the $c^* := \inf_{u\in\mathcal{N}_b^\lambda} I_b^\lambda(u) > 0$ is achieved and $I_b^\lambda(u) > 2c^*$, where $\mathcal{N}_b^\lambda = \{u \in E \setminus \{0\} \mid \langle (I_b^\lambda)'(u), u\rangle = 0\}$ and u is the least energy sign-changing solution obtained in Theorem 1.2. In particular, $c^* > 0$ is achieved either by a positive or a negative function.*

The plan of this paper is as follows: Section 2 covers the proof of the achievement of least energy for the constraint problem (1.1), Section 3 is devoted to the proof of our main theorems.

Throughout this paper, we use standard notations. For simplicity, we use " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence in the related function space respectively. Various positive constants are denoted by C and C_i. We use ":=" to denote definitions and $B_r(x) := \{y \in \mathbb{R}^N \mid |x - y| < r\}$. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified.

2 Some technical lemmas

Now, fixed $u \in E$ with $u^{\pm} \neq 0$, we define function $\psi_u : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and mapping $T_u : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ by

$$\psi_u(\alpha,\beta) = I_b^\lambda(\alpha u^+ + \beta u^-) \tag{2.1}$$

and

$$T_u(\alpha,\beta) = \left(\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+\rangle, \langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^-\rangle\right). \tag{2.2}$$

Lemma 2.1. *Assume that (f_1)–(f_3) hold, if $u \in E$ with $u^{\pm} \neq 0$, then there is the unique maximum point pair (α_u, β_u) of the function ψ such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_b^\lambda$.*

Proof. Our proof will be divided into three steps.

Step 1. For any $u \in E$ with $u^{\pm} \neq 0$, in the following, we will prove the existence of α_u and β_u.

From (f_1) and (f_2), for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon|t| + C_\varepsilon|t|^{\theta-1} \quad \text{for all } t \in \mathbb{R}. \tag{2.3}$$

Then, by the Sobolev embedding theorem, we have that

$$\begin{aligned}
&\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+\rangle \\
&\quad \geq \alpha^2\|u^+\|^2 + b\alpha^4\|u^+\|^4 + b\alpha^2\beta^2\|u^+\|^2\|u^-\|^2 \\
&\qquad - \lambda\alpha^2\varepsilon\int_\Omega |u^+|^2 dx - \lambda C_\varepsilon \alpha^\theta \int_\Omega |u^+|^\theta dx - \alpha^{2^{**}}\int_\Omega |u|^{2^{**}} dx \\
&\quad \geq \alpha^2\|u^+\|^2 + b\alpha^4\|u^+\|^4 - \lambda\alpha^2\varepsilon C_1\|u^+\|^2 - \lambda C_\varepsilon\alpha^\theta C_2\|u^+\|^\theta - C_3\alpha^{2^{**}}\|u^+\|^{2^{**}} \\
&\quad = (1-\lambda\varepsilon C_1)\,\alpha^2\|u^+\|^2 + b\alpha^4\|u^+\|^4 - \lambda C_\varepsilon\alpha^\theta C_2\|u^+\|^\theta - C_3\alpha^{2^{**}}\|u^+\|^{2^{**}}.
\end{aligned}$$

Choose $\varepsilon > 0$ such that $(1-\lambda\varepsilon C_1) > 0$. Since $2^{**}, \theta > 4$, we have that $\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+\rangle > 0$ for α small enough and all $\beta \geq 0$.

Similarly, we obtain that $\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^-\rangle > 0$ for β small enough and all $\alpha \geq 0$.

Therefore, there exists $\delta_1 > 0$ such that

$$\langle (I_b^\lambda)'(\delta_1 u^+ + \beta u^-), \delta_1 u^+\rangle > 0, \quad \langle (I_b^\lambda)'(\alpha u^+ + \delta_1 u^-), \delta_1 u^-\rangle > 0 \tag{2.4}$$

for all $\alpha, \beta \geq 0$.

On the other hand, by (f_2) and (f_3), we have that

$$f(x,t)t > 0, \quad t \neq 0; \quad F(x,t) \geq 0, \quad t \in \mathbb{R}. \tag{2.5}$$

In fact, by (f_2) and (f_3), we obtain that $f(x,t) > 0 (< 0)$ for $t > 0 (< 0)$ and almost every $x \in \Omega$. Moreover, by (f_2) and continuity of f, it follows that $f(x,0) = 0$ for almost every $x \in \Omega$. Therefore, $F(x,t) \geq 0$ for $t \geq 0$ and almost every $x \in \Omega$.

If $t < 0$, by (f_3), we have

$$F(x,t) = \int_0^t \frac{f(x,s)}{s^3}s^3 ds \geq \frac{f(x,t)}{t^3}\int_0^t s^3 ds = \frac{1}{4}f(x,t)t > 0, \quad \text{a.e. } x \in \Omega,$$

since $t \leq s < 0$ and $f(x,t) < 0$ for a.e. $x \in \Omega$.

From the above arguments, we conclude that (2.5) holds.

Therefore, choose $\alpha = \delta_2^* > \delta_1$, if $\beta \in [\delta_1, \delta_2^*]$ and δ_2^* is large enough, it follows that

$$\begin{aligned}
&\langle (I_b^\lambda)'(\delta_2^* u^+ + \beta u^-), \delta_2^* u^+\rangle \\
&\quad \leq \tau(\delta_2^*)^2\|u^+\|^2 + b(\delta_2^*)^4\|u^+\|^4 + b(\delta_2^*)^4\|u^+\|^2\|u^-\|^2 - (\delta_2^*)^{2^{**}}\int_\Omega |u^+|^{2^{**}}dx \leq 0.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
&\langle (I_b^\lambda)'(\alpha u^+ + \delta_2^* u^-), \delta_2^* u^-\rangle \\
&\quad \leq \tau(\delta_2^*)^2\|u^-\|^2 + b(\delta_2^*)^4\|u^+\|^4 + b(\delta_2^*)^4\|u^+\|^2\|u^-\|^2 - (\delta_2^*)^{2^{**}}\int_\Omega |u^-|^{2^{**}}dx \leq 0.
\end{aligned}$$

Let $\delta_2 > \delta_2^*$ be large enough, we obtain that

$$\langle (I_b^\lambda)'(\delta_2^* u^+ + \beta u^-), \delta_2^* u^+\rangle < 0 \quad \text{and} \quad \langle (I_b^\lambda)'(\alpha u^+ + \delta_2^* u^-), \delta_2^* u^-\rangle < 0 \tag{2.6}$$

for all $\alpha, \beta \in [\delta_1, \delta_2]$.

Combining (2.4) and (2.6) with Miranda's theorem [24], there exists $(\alpha_u, \beta_u) \in (0, +\infty) \times (0, +\infty)$ such that $T_u(\alpha, \beta) = (0, 0)$, i.e., $\alpha u^+ + \beta u^- \in \mathcal{M}_b^\lambda$.

Step 2. In this step, we prove the uniqueness of the pair (α_u, β_u).

• Case $u \in \mathcal{M}_b^\lambda$.

If $u \in \mathcal{M}_b^\lambda$, we have that

$$\|u^+\|_E^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 = \lambda \int_\Omega f(x, u^+)u^+ dx + \int_\Omega |u^+|^{2^{**}} dx \tag{2.7}$$

and

$$\|u^-\|_E^2 + b\|u^-\|^4 + b\|u^+\|^2\|u^-\|^2 = \lambda \int_\Omega f(x, u^-)u^- dx + \int_\Omega |u^-|^{2^{**}} dx. \tag{2.8}$$

We show that $(\alpha_u, \beta_u) = (1, 1)$ is the unique pair of numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_b^\lambda$.

Let (α_0, β_0) be a pair of numbers such that $\alpha_0 u^+ + \beta_0 u^- \in \mathcal{M}_b^\lambda$ with $0 < \alpha_0 \leq \beta_0$. Hence, one has that

$$\alpha_0^2\|u^+\|_E^2 + b\alpha_0^4\|u^+\|^4 + b\alpha_0^2\beta_0^2\|u^+\|^2\|u^-\|^2 = \lambda \int_\Omega f(x, \alpha_0 u^+)\alpha_0 u^+ dx + \alpha_0^{2^{**}} \int_\Omega |u^+|^{2^{**}} dx \tag{2.9}$$

and

$$\begin{aligned}\beta_0^2\|u^-\|_E^2 &+ b\beta_0^4\|u^-\|^4 + b\alpha_0^2\beta_0^2\|u^+\|^2\|u^-\|^2 \\ &= \lambda \int_\Omega f(x, \beta_0 u^-)\beta_0 u^- dx + \beta_0^{2^{**}} \int_\Omega |u^-|^{2^{**}} dx.\end{aligned} \tag{2.10}$$

According to $0 < \alpha_0 \leq \beta_0$ and (2.10), we have that

$$\frac{\|u^-\|_E^2}{\beta_0^2} + b\|u^-\|^4 + b\|u^+\|^2\|u^-\|^2 \geq \lambda \int_\Omega \frac{f(x, \beta_0 u^-)}{(\beta_0 u^-)^3}(u^-)^4 dx + \beta_0^{2^{**}-4} \int_\Omega |u^-|^{2^{**}} dx. \tag{2.11}$$

If $\beta_0 > 1$, by (2.8) and (2.11), one has that

$$\left(\frac{1}{\beta_0^2} - 1\right)\|u^-\|_E^2 \geq \lambda \int_\Omega \left[\frac{f(x, \beta_0 u^-)}{(\beta_0 u^-)^3} - \frac{f(x, u^-)}{(u^-)^3}\right](u^-)^4 dx + (\beta_0^{2^{**}-4} - 1) \int_\Omega |u^-|^{2^{**}} dx.$$

Thus, for any $\beta_0 > 1$, the left side of the above inequality is negative, the right-hand side above is greater than zero by condition (f_3), which is a contradiction. Therefore, we conclude that $0 < \alpha_0 \leq \beta_0 \leq 1$.

Similarly, by (2.9) and $0 < \alpha_0 \leq \beta_0$, we have that

$$\left(\frac{1}{\alpha_0^2} - 1\right)\|u^+\|_E^2 \leq \lambda \int_\Omega \left[\frac{f(x, \alpha_0 u^+)}{(\alpha_0 u^+)^3} - \frac{f(x, u^-)}{(u^+)^3}\right](u^+)^4 dx + (\alpha_0^{2^{**}-4} - 1) \int_\Omega |u^+|^{2^{**}} dx.$$

According to condition (f_3), we obtain that $\alpha_0 \geq 1$.

Consequently, $\alpha_0 = \beta_0 = 1$.

• Case $u \notin \mathcal{M}_b^\lambda$.

Suppose that there exist (α_1, β_1), (α_2, β_2) such that

$$\omega_1 = \alpha_1 u^+ + \beta_1 u^- \in \mathcal{M}_b^\lambda \quad \text{and} \quad \omega_2 = \alpha_2 u^+ + \beta_2 u^- \in \mathcal{M}_b^\lambda.$$

Hence

$$\omega_2 = \left(\frac{\alpha_2}{\alpha_1}\right)\alpha_1 u^+ + \left(\frac{\beta_2}{\beta_1}\right)\beta_1 u^- = \left(\frac{\alpha_2}{\alpha_1}\right)\omega^+ + \left(\frac{\beta_2}{\beta_1}\right)\omega^- \in \mathcal{M}_b^\lambda.$$

By $\omega_1 \in \mathcal{M}_b^\lambda$, one has that

$$\frac{\alpha_2}{\alpha_1} = \frac{\beta_2}{\beta_1} = 1.$$

Hence, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$.

Step 3. In this step, we will prove that (α_u, β_u) is the unique maximum point of ψ_u on $[0,\infty) \times [0,\infty)$.

In fact, by (2.3), we have that

$$\begin{aligned}
\psi_u(\alpha,\beta) &= I_b^\lambda(\alpha u^+ + \beta u^-) \\
&= \frac{1}{2}\|\alpha u^+ + \beta u^-\|_E^2 + \frac{b}{4}\|\alpha u^+ + \beta u^-\|^4 \\
&\quad -\lambda \int_\Omega F(x, \alpha u^+ + \beta u^-)dx - \frac{1}{2^{**}}\int_\Omega |\alpha u^+ + \beta u^-|^{2^{**}}dx \\
&= \frac{\alpha^2}{2}\|u^+\|_E^2 + \frac{\beta^2}{2}\|u^-\|_E^2 + \frac{b\alpha^4}{4}\|u^+\|^4 + \frac{b\beta^4}{4}\|u^-\|^4 + \frac{b\alpha^2\beta^2}{2}\|u^+\|^2\|u^-\|^2 \\
&\quad -\lambda\int_\Omega F(x,\alpha u^+)dx - \lambda\int_\Omega F(x,\beta u^-)dx - \frac{\alpha^{2^{**}}}{2^{**}}\int_\Omega |u^+|^{2^{**}}dx - \frac{\beta^{2^{**}}}{2^{**}}\int_\Omega |u^-|^{2^{**}}dx \\
&\le \frac{\tau\alpha^2}{2}\|u^+\|^2 + \frac{\tau\beta^2}{2}\|u^-\|^2 + \frac{b\alpha^4}{4}\|u^+\|^4 + \frac{b\beta^4}{4}\|u^-\|^4 + \frac{b\alpha^2\beta^2}{2}\|u^+\|^2\|u^-\|^2 \\
&\quad -\frac{\alpha^{2^{**}}}{2^{**}}\int_\Omega |u^+|^{2^{**}}dx - \frac{\beta^{2^{**}}}{2^{**}}\int_\Omega |u^-|^{2^{**}}dx,
\end{aligned}$$

which implies that $\lim_{|(\alpha,\beta)|\to\infty}\psi(\alpha,\beta) = -\infty$ thanks to $2^{**} > 4$.

Hence, (α_u, β_u) is the unique critical point of ψ_u in $[0,\infty)\times[0,\infty)$. So it is sufficient to check that a maximum point cannot be achieved on the boundary of $[0,\infty)\times[0,\infty)$. By contradiction, we suppose that $(0,\beta_0)$ is a maximum point of ψ_u with $\beta_0 \ge 0$. Then, we have that

$$\begin{aligned}
\psi_u(\alpha,\beta_0) &= \frac{\alpha^2}{2}\|u^+\|_E^2 + \frac{b\alpha^4}{4}\|u^+\|^4 - \lambda\int_\Omega F(x,\alpha u^+)dx - \frac{\alpha^{2^{**}}}{2^{**}}\int_\Omega |u^+|^{2^{**}}dx \\
&\quad + \frac{\beta_0^2}{2}\|u^-\|_E^2 + \frac{b\beta_0^4}{4}\|u^-\|^4 - \lambda\int_\Omega F(x,\beta_0 u^-)dx - \frac{\beta^{2^{**}}}{2^{**}}\int_\Omega |u^-|^{2^{**}}dx \\
&\quad + \frac{b\alpha^2\beta_0^2}{2}\|u^+\|^2\|u^-\|^2.
\end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned}
(\psi_u)'_\alpha(\alpha,\beta_0) &= \alpha\|u^+\|_E^2 + b\alpha^3\|u^+\|^4 + b\alpha\beta_0^2\|u^+\|^2\|u^-\|^2 \\
&\quad -\lambda\int_\Omega f(x,\alpha u^+)u^+dx - \alpha^{2^{**}-1}\int_\Omega |u^+|^{2^{**}}dx \\
&\ge \alpha\|u^+\|^2 + b\alpha^3\|u^+\|^4 + b\alpha\beta_0^2\|u^+\|^2\|u^-\|^2 \\
&\quad -\lambda\int_\Omega f(x,\alpha u^+)u^+dx - \alpha^{2^{**}-1}\int_\Omega |u^+|^{2^{**}}dx \\
&> 0,
\end{aligned}$$

if α is small enough. That is, ψ_u is an increasing function with respect to α if α is small enough. This yields the contradiction. Similarly, ψ_u can not achieve its global maximum on $(\alpha, 0)$ with $\alpha \ge 0$. □

Lemma 2.2. *Assume that* (f_1)–(f_3) *hold, if* $u \in E$ *with* $u^{\pm} \neq 0$ *such that* $\langle (I_b^\lambda)'(u), u^{\pm} \rangle \leq 0$. *Then, the unique maximum point of* ψ_u *on* $[0,\infty) \times [0,\infty)$ *satisfies* $0 < \alpha_u, \beta_u \leq 1$.

Proof. In fact, if $\alpha_u \geq \beta_u > 0$. On the one hand, by $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_b^\lambda$, we have

$$\begin{aligned} &\alpha_u^2 \|u^+\|_E^2 + b\alpha_u^4 \|u^+\|^4 + b\alpha_u^4 \|u^+\|^2 \|u^-\|^2 \\ &\quad \geq \alpha_u^2 \|u^+\|_E^2 + b\alpha_u^4 \|u^+\|^4 + b\alpha_u^2 \beta_u^2 \|u^+\|^2 \|u^-\|^2 \\ &\quad = \lambda \int_\Omega f(x, \alpha_u u^+) \alpha_u u^+ dx + \alpha^{2^{**}} \int_\Omega |u^+|^{2^{**}} dx. \end{aligned} \tag{2.12}$$

On the other hand, by $\langle (I_b^\lambda)'(u), u^+ \rangle \leq 0$, we have

$$\|u^+\|_E^2 + b\|u^+\|^4 + b\|u^+\|^2 \|u^-\|^2 \leq \lambda \int_\Omega f(x, u^+) u^+ dx + \int_\Omega |u^+|^{2^{**}} dx. \tag{2.13}$$

So, according to (2.12) and (2.13), we have that

$$\left(\frac{1}{\alpha_u^2} - 1 \right) \|u^+\|_E^2 \geq \lambda \int_\Omega \left[\frac{f(x, \alpha_u u^+)}{(\alpha_u u^+)^3} - \frac{f(x, u^+)}{(u^+)^3} \right] (u^+)^4 dx + (\alpha_u^{2^{**}-2} - 1) \int_\Omega |u^+|^{2^{**}} dx.$$

Thanks to condition (f_3), we conclude that $\alpha_u \leq 1$. Thus, we have that $0 < \alpha_u, \beta_u \leq 1$. □

Lemma 2.3. *Let* $c_b^\lambda = \inf_{u \in \mathcal{M}_b^\lambda} I_b^\lambda(u)$, *then we have that* $\lim_{\lambda \to \infty} c_b^\lambda = 0$.

Proof. For any $u \in \mathcal{M}_b^\lambda$, we have

$$\|u^\pm\|_E^2 + b\|u^\pm\|^4 + b\|u^+\|^2 \|u^-\|^2 = \lambda \int_\Omega f(x, u^\pm) u^\pm dx + \int_\Omega |u^\pm|^{2^{**}} dx.$$

Then, by (2.3) and Sobolev inequalities, we have that

$$\|u^\pm\|^2 \leq \lambda \int_\Omega f(x, u^\pm) u^\pm dx + \int_\Omega |u^\pm|^{2^{**}} dx \leq \lambda \varepsilon C_1 \|u^\pm\|^2 + \lambda C_\varepsilon C_2 \|u^\pm\|^\theta + C_3 \|u^\pm\|^{2^{**}}.$$

Thus, we get

$$(1 - \lambda \varepsilon C_1) \|u^\pm\|^2 \leq \lambda C_\varepsilon C_2 \|u^\pm\|^\theta + C_3 \|u^\pm\|^{2^{**}}.$$

Choosing ε small enough such that $1 - \lambda \varepsilon C_1 > 0$, since $2^{**} > 4$, there exists $\rho > 0$ such that

$$\|u^\pm\| \geq \rho \quad \text{for all} \ \ u \in \mathcal{M}_b^\lambda. \tag{2.14}$$

On the other hand, for any $u \in \mathcal{M}_b^\lambda$, it is obvious that $\langle (I_b^\lambda)'(u), u \rangle = 0$. Thanks to (f_2) and (f_3), we obtain that

$$\Theta(x, t) := f(x, t)t - 4F(x, t) \geq 0 \tag{2.15}$$

and is increasing when $t > 0$ and decreasing when $t < 0$ for almost every $x \in \Omega$. Then, we have

$$I_b^\lambda(u) = I_b^\lambda(u) - \frac{1}{4} \langle (I_b^\lambda)'(u), u \rangle \geq \frac{1}{4} \|u\|^2.$$

From above discussions, we have that $I_b^\lambda(u) > 0$ for all $u \in \mathcal{M}_b^\lambda$. Therefore, I_b^λ is bounded below on $\mathcal{M}_b^\lambda$, that is $c_b^\lambda = \inf_{u \in \mathcal{M}_b^\lambda} I_b^\lambda(u)$ is well defined.

Let $u \in E$ with $u^{\pm} \neq 0$ be fixed. By Lemma 2.1, for each $\lambda > 0$, there exist $\alpha_\lambda, \beta_\lambda > 0$ such that $\alpha_\lambda u^+ + \beta_\lambda u^- \in \mathcal{M}_b^\lambda$. By using Lemma 2.1 again, we have that

$$\begin{aligned}
0 \leq c_b^\lambda &= \inf_{u \in \mathcal{M}_b^\lambda} I_b^\lambda(u) \leq I_b^\lambda(\alpha_\lambda u^+ + \beta_\lambda u^-) \\
&\leq \frac{1}{2}\|\alpha_\lambda u^+ + \beta_\lambda u^-\|_E^2 + \frac{b}{4}\|\alpha_\lambda u^+ + \beta_\lambda u^-\|^4 \\
&\leq \alpha_\lambda^2 \|u^+\|_E^2 + \beta_\lambda^2 \|u^-\|_E^2 + 2b\alpha_\lambda^4 \|u^+\|^4 + 2b\beta_\lambda^4 \|u^-\|^4.
\end{aligned}$$

To the end, we just prove that $\alpha_\lambda \to 0$ and $\beta_\lambda \to 0$ as $\lambda \to \infty$.

Let

$$\mathcal{T}_u = \{(\alpha_\lambda, \beta_\lambda) \in [0,\infty) \times [0,\infty) : T_u(\alpha_\lambda, \beta_\lambda) = (0,0), \lambda > 0\},$$

where T_u is defined as (2.2). By (2.3), we have that

$$\begin{aligned}
&\alpha_\lambda^{2^{**}} \int_\Omega |u^+|^{2^{**}} dx + \beta_\lambda^{2^{**}} \int_\Omega |u^-|^{2^{**}} dx \\
&\quad \leq \alpha_\lambda^{2^{**}} \int_\Omega |u^+|^{2^{**}} dx + \beta_\lambda^{2^{**}} \int_\Omega |u^-|^{2^{**}} dx \\
&\qquad + \lambda \int_\Omega f(x, \alpha_\lambda u^+)\alpha_\lambda u^+ dx + \lambda \int_\Omega f(x, \beta_\lambda u^-)\beta_\lambda u^- dx \\
&\quad = \|\alpha_\lambda u^+ + \beta_\lambda u^-\|_E^2 + b\|\alpha_\lambda u^+ + \beta_\lambda u^-\|^4 \\
&\quad \leq 2\tau^2 \alpha_\lambda^2 \|u^+\|^2 + 2\tau^2 \beta_\lambda^2 \|u^-\|^2 + 4b\alpha_\lambda^4 \|u^+\|^4 + 4b\beta_\lambda^4 \|u^-\|^4.
\end{aligned}$$

Hence, $\mathcal{T}_u$ is bounded. Let $\{\lambda_n\} \subset (0,\infty)$ be such that $\lambda_n \to \infty$ as $n \to \infty$. Then, there exist α_0 and β_0 such that $(\alpha_{\lambda_n}, \beta_{\lambda_n}) \to (\alpha_0, \beta_0)$ as $n \to \infty$.

Now, we claim $\alpha_0 = \beta_0 = 0$. Suppose, by contradiction, that $\alpha_0 > 0$ or $\beta_0 > 0$. By $\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^- \in \mathcal{M}_b^{\lambda_n}$, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|_E^2 + b\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|^4 \\
&\quad = \lambda_n \int_\Omega f(x, \alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-) dx + \int_\Omega |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^{2^{**}} dx. \qquad (2.16)
\end{aligned}$$

Thanks to $\alpha_{\lambda_n} u^+ \to \alpha_0 u^+$ and $\beta_{\lambda_n} u^- \to \beta_0 u^+$ in E, (2.3) and (2.4), we have that

$$\int_\Omega f(x, \alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-) dx \to \int_\Omega f(x, \alpha_0 u^+ + \beta_0 u^-)(\alpha_0 u^+ + \beta_0 u^-) dx > 0$$

as $n \to \infty$.

It follows from $\lambda_n \to \infty$ as $n \to \infty$ and $\{\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\}$ is bounded in E, which contradicts equality (2.16). Hence, $\alpha_0 = \beta_0 = 0$.

Hence, we conclude that $\lim_{\lambda \to \infty} c_b^\lambda = 0$. □

Lemma 2.4. *There exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, the infimum c_b^λ is achieved.*

Proof. By the definition of c_b^λ, there exists a sequence $\{u_n\} \subset \mathcal{M}_b^\lambda$ such that

$$\lim_{n \to \infty} I_b^\lambda(u_n) = c_b^\lambda.$$

Obviously, $\{u_n\}$ is bounded in E. Then, up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in E$ such that $u_n \rightharpoonup u$. Since the embedding $E \hookrightarrow L^t(\Omega)$ is compact for all $t \in (2, 2^{**})$ (see [27]), we have

$$u_n \to u \quad \text{in } L^t(\Omega), \quad u_n \to u \quad \text{a.e. } x \in \Omega.$$

Hence

$$\begin{aligned} u_n^{\pm} &\rightharpoonup u^{\pm} \quad \text{in } E, \\ u_n^{\pm} &\to u^{\pm} \quad \text{in } L^t(\Omega), \\ u_n^{\pm} &\to u^{\pm} \quad \text{a.e. } x \in \Omega. \end{aligned}$$

By Lemma 2.1, we have

$$I_b^{\lambda}(\alpha u_n^+ + \beta u_n^-) \leq I_b^{\lambda}(u_n)$$

for all $\alpha, \beta \geq 0$.

Then, by the Brézis–Lieb lemma and Fatou's lemma, we have that

$$\begin{aligned} \liminf_{n\to\infty} I_b^{\lambda}(\alpha u_n^+ + \beta u_n^-) \geq{}& \frac{\alpha^2}{2} \lim_{n\to\infty} (\|u_n^+ - u^+\|_E^2 + \|u^+\|_E^2) \\ &+ \frac{\beta^2}{2} \lim_{n\to\infty} (\|u_n^- - u^-\|_E^2 + \|u^-\|_E^2) \\ &+ \frac{b\alpha^4}{4} \left[\lim_{n\to\infty} (\|u_n^+ - u^+\|^2 + \|u^+\|^2)\right]^2 \\ &+ \frac{b\beta^4}{4} \left[\lim_{n\to\infty} (\|u_n^- - u^-\|^2 + \|u^-\|^2)\right]^2 \\ &- \frac{\alpha^{2^{**}}}{2^{**}} \left[\lim_{n\to\infty} \int_\Omega |u_n^+ - u^+|^{2^{**}} dx + \lim_{n\to\infty} \int_\Omega |u^+|^{2^{**}} dx\right] \\ &- \frac{\beta^{2^{**}}}{2^{**}} \lim_{n\to\infty} \left[\int_\Omega |u_n^- - u^-|^{2^{**}} dx + \int_\Omega |u^-|^{2^{**}} dx\right] \\ &- \lambda \int_\Omega F(x, \alpha u^+) dx - \lambda \int_\Omega F(x, \beta u^-) dx + \frac{b\alpha^2\beta^2}{2} \liminf_{n\to\infty} (\|u_n^+\|^2 \|u_n^-\|^2) \\ \geq{}& I_b^{\lambda}(\alpha u^+ + \beta u^-) + \frac{\alpha^2}{2} \lim_{n\to\infty} \|u_n^+ - u^+\|_E^2 + \frac{\beta^2}{2} \lim_{n\to\infty} \|u_n^- - u^-\|_E^2 \\ &+ \frac{b\alpha^4}{2} \lim_{n\to\infty} \|u_n^+ - u^+\|^2 \|u^+\|^2 + \frac{b\beta^4}{2} \lim_{n\to\infty} \|u_n^- - u^-\|^2 \|u^-\|^2 \\ &+ \frac{b\alpha^4}{4} (\lim_{n\to\infty} \|u_n^+ - u^+\|^2)^2 + \frac{bt^4}{4} (\lim_{n\to\infty} \|u_n^- - u^-\|^2)^2 \\ &- \frac{\alpha^{2^{**}}}{2^{**}} \int_\Omega |u_n^+ - u^+|^{2^{**}} dx - \frac{\beta^{2^{**}}}{2^{**}} \int_\Omega |u_n^- - u^-|^{2^{**}} dx \\ \geq{}& I_b^{\lambda}(\alpha u^+ + \beta u^-) + \frac{\alpha^2}{2} A_1 + \frac{b\alpha^4}{2} A_1 \|u^+\|^2 + \frac{b\alpha^4}{4} A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}} B_1 \\ &+ \frac{\beta^2}{2} A_2 + \frac{b\beta^4}{2} A_2 \|u^-\|^2 + \frac{b\beta^4}{4} A_2^2 - \frac{\beta^{2^{**}}}{2^{**}} B_2, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \lim_{n\to\infty} \|u_n^+ - u^+\|^2, \quad A_2 = \lim_{n\to\infty} \|u_n^- - u^-\|^2, \\ B_1 &= \lim_{n\to\infty} |u_n^+ - u^+|_{2^{**}}^{2^{**}}, \quad B_2 = \lim_{n\to\infty} |u_n^- - u^-|_{2^{**}}^{2^{**}}. \end{aligned}$$

That is, one has that

$$\begin{aligned} I_b^{\lambda}(\alpha u^+ + \beta u^-) + \frac{\alpha^2}{2} A_1 + \frac{b\alpha^4}{2} A_1 \|u^+\|^2 + \frac{b\alpha^4}{4} A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}} B_1 & \\ + \frac{\beta^2}{2} A_2 + \frac{b\beta^4}{2} A_2 \|u^-\|^2 + \frac{b\beta^4}{4} A_2^2 - \frac{\beta^{2^{**}}}{2^{**}} B_2 \leq c_b^{\lambda} & \quad (2.17) \end{aligned}$$

for all $\alpha \geq 0$ and all $\beta \geq 0$.

Now, we claim that $u^{\pm} \neq 0$.

In fact, since the situation $u^{-} \neq 0$ is analogous, we just prove $u^{+} \neq 0$. By contradiction, we suppose $u^{+} = 0$. Hence, let $\beta = 0$ in (2.17) and we have that

$$\frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1 \leq c_b^{\lambda} \tag{2.18}$$

for all $\alpha \geq 0$.

Case 1: $B_1 = 0$.

If $A_1 = 0$, that is, $u_n^{+} \to u^{+}$ in E. In view of Lemma (2.14), we obtain $\|u^{+}\| > 0$, which contradicts our supposition. If $A_1 > 0$, by (2.18), we have that

$$\frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 \leq c_b^{\lambda}$$

for all $\alpha \geq 0$, which is absurd by Lemma 2.3. Anyway, we have a contradiction.

Case 2: $B_1 > 0$.

One one hand, by Lemma 2.3, there exists $\lambda^* > 0$ such that

$$c_b^{\lambda} < \frac{2}{N}S^{-2/N} \quad \text{for all } \lambda \geq \lambda^*, \tag{2.19}$$

where $S := \inf\left\{\int_{\Omega}|\Delta u|^2 dx : \int_{\Omega}|u|^{2^{**}}dx = 1\right\}$.

On the other hand, since $B_1 > 0$, we obtain $A_1 > 0$. Hence, in view of (2.18), we have that

$$\frac{2}{N}S^{-2/N} \leq \frac{2}{N}\left[\frac{A_1^{\frac{2^{**}}{2}}}{B_1}\right]^{\frac{2}{2^{**}-2}} \leq \max_{\alpha\geq 0}\left\{\frac{\alpha^2}{2}A_1 - \frac{\alpha^{2^{**}}}{2^{**}}B_1\right\} \leq \max_{\alpha\geq 0}\left\{\frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1\right\} \leq c_b^{\lambda},$$

which is a contradiction. That is, we deduce that $u^{\pm} \neq 0$.

Next we prove $B_1 = B_2 = 0$.

Since the situation $B_2 = 0$ is analogous, we only prove $B_1 = 0$. By contradiction, we suppose that $B_1 > 0$.

Case 1: $B_2 > 0$.

According to $B_1, B_2 > 0$ and Sobolev embedding, we obtain that $A_1, A_2 > 0$. Let

$$\varphi(\alpha) = \frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1 \quad \text{for all } \ \alpha \geq 0.$$

It is easy to see that $\varphi(\alpha) > 0$ for $\alpha > 0$ small enough and $\varphi(\alpha) < 0$ for $\alpha < 0$ large enough. Hence, by continuous of $\varphi(\alpha)$, there exists $\hat{\alpha} > 0$ such that

$$\frac{\hat{\alpha}^2}{2}A_1 + \frac{b\hat{\alpha}^4}{4}A_1^2 - \frac{\hat{\alpha}^{2^{**}}}{2^{**}}B_1 = \max_{\alpha\geq 0}\left\{\frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1\right\}.$$

Similarly, there exists $\hat{\beta} > 0$ such that

$$\frac{\hat{\beta}^2}{2}A_2 + \frac{b\hat{\beta}^4}{4}A_2^2 - \frac{\hat{\beta}^{2^{**}}}{2^{**}}B_2 = \max_{\alpha\geq 0}\left\{\frac{\beta^2}{2}A_2 + \frac{b\beta^4}{4}A_2^2 - \frac{\beta^{2^{**}}}{2^{**}}B_2\right\}.$$

Since $[0,\hat{\alpha}]\times[0,\hat{\beta}]$ is compact and ψ is continuous, there exists $(\alpha_u,\beta_u)\in[0,\hat{\alpha}]\times[0,\hat{\beta}]$ such that

$$\psi(\alpha_u,\beta_u)=\max_{(\alpha,\beta)\in[0,\hat{\alpha}]\times[0,\hat{\beta}]}\psi(\alpha,\beta).$$

Now, we prove that $(\alpha_u,\beta_u)\in(0,\hat{\alpha})\times(0,\hat{\beta})$.

Note that, if β is small enough, we have that

$$\psi(\alpha,0)=I_b^\lambda(\alpha u^+)<I_b^\lambda(\alpha u^+)+I_b^\lambda(\beta u^-)\le I_b^\lambda(\alpha u^+ +\beta u^-)=\psi(\alpha,\beta)$$

for all $\alpha\in[0,\hat{\alpha}]$.

Hence, there exists $\beta_0\in[0,\hat{\beta}]$ such that

$$\psi(\alpha,0)\le\psi(\alpha,\beta_0)\quad\text{for all } \alpha\in[0,\hat{\alpha}].$$

That is, any point of $(\alpha,0)$ with $0\le\alpha\le\hat{\alpha}$ is not the maximizer of ψ. Hence, $(\alpha_u,\beta_u)\notin[0,\hat{\alpha}]\times\{0\}$. Similarly, we obtain $(\alpha_u,\beta_u)\notin\{0\}\times[0,\hat{\alpha}]$.

On the other hand, it is easy to see that

$$\frac{\alpha^2}{2}A_1+\frac{b\alpha^4}{2}A_1\|u^+\|^2+\frac{b\alpha^4}{4}A_1^2-\frac{\alpha^{2^{**}}}{2^{**}}B_1>0 \tag{2.20}$$

and

$$\frac{\beta^2}{2}A_2+\frac{b\beta^4}{2}A_2\|u^-\|^2+\frac{b\beta^4}{4}A_2^2-\frac{\beta^{2^{**}}}{2^{**}}B_2>0 \tag{2.21}$$

for $\alpha\in(0,\hat{\alpha}]$, $\beta\in(0,\hat{\beta}]$.

Then, we have that

$$\begin{aligned}\frac{2}{N}S^{-2/N}\le{}&\frac{\hat{\alpha}^2}{2}A_1+\frac{b\hat{\alpha}^4}{4}A_1^2-\frac{\hat{\alpha}^{2^{**}}}{2^{**}}B_1+\frac{b\hat{\alpha}^4}{2}A_1\|u^+\|^2\\&+\frac{\beta^2}{2}A_2+\frac{b\beta^4}{2}A_2\|u^-\|^2+\frac{b\beta^4}{4}A_2^2-\frac{\beta^{2^{**}}}{2^{**}}B_2\end{aligned}$$

and

$$\begin{aligned}\frac{2}{N}S^{-2/N}\le{}&\frac{\tilde{\beta}^2}{2}A_2+\frac{b\tilde{\beta}^4}{4}A_2^2-\frac{\tilde{\beta}^{2^{**}}}{2^{**}}B_2+\frac{b\tilde{\beta}^4}{2}A_2\|u^-\|^2\\&+\frac{\alpha^2}{2}A_1+\frac{b\alpha^4}{2}A_1\|u^+\|^2+\frac{b\alpha^4}{4}A_1^2-\frac{\alpha^{2^{**}}}{2^{**}}B_1\end{aligned}$$

for all $\alpha\in[0,\tilde{\alpha}]$ and all $\beta\in[0,\tilde{\beta}]$.

Therefore, according to (2.17), we conclude that

$$\psi(\alpha,\hat{\beta})\le 0,\quad \psi(\hat{\alpha},\beta)\le 0$$

for all $\alpha\in[0,\hat{\alpha}]$ and all $\beta\in[0,\hat{\beta}]$.

Hence,$(\alpha_u,\beta_u)\notin\{\tilde{\alpha}\}\times[0,\tilde{\beta}]$ and $(\alpha_u,\beta_u)\notin[0,\tilde{\alpha}]\times\{\tilde{\beta}\}$.

Finally, we get that $(\alpha_u,\beta_u)\in(0,\hat{\alpha})\times(0,\hat{\beta})$. Hence, it follows that (α_u,β_u) is a critical point of ψ.

Hence, $\alpha_u u^+ +\beta_u u^-\in\mathcal{M}_b^\lambda$. From (2.17), (2.20), and (2.21), we have that

$$\begin{aligned}c_b^\lambda&\ge I_b^\lambda(\alpha_u u^+ +\beta_u u^-)+\frac{\alpha_u^2}{2}A_1+\frac{b\alpha_u^4}{2}A_1\|u^+\|^2+\frac{b\alpha_u^4}{4}A_1^2-\frac{\alpha^{2^{**}}}{2^{**}}B_1\\&\quad+\frac{\beta_u^2}{2}A_2+\frac{b\beta_u^4}{2}A_2\|u^-\|^2+\frac{b\beta_u^4}{4}A_2^2-\frac{\beta^{2^{**}}}{2^{**}}B_2\\&>I_b^\lambda(\alpha_u u^+ +\beta_u u^-)\ge c_b^\lambda,\end{aligned}$$

which is a contradiction.

Case 2: $B_2 = 0$.

In this case, we can maximize in $[0,\hat{\alpha}] \times [0,\infty)$. Indeed, it is possible to show that there exist $\beta_0 \in [0,\infty)$ such that

$$I_b^\lambda(\alpha_u u^+ + \beta_u u^-) \leq 0 \quad \text{for all } (\alpha,\beta) \in [0,\hat{\alpha}] \times [\beta_0,\infty).$$

Hence, there is $(\alpha_u, \beta_u) \in [0,\hat{\alpha}] \times [0,\infty)$ such that

$$\psi(\alpha_u,\beta_u) = \max_{(\alpha,\beta)\in[0,\hat{\alpha}]\times[0,\infty)} \psi(\alpha,\beta).$$

In the following, we prove that $(\alpha_u,\beta_u) \in (0,\hat{\alpha}) \times (0,\infty)$.

It is noted that $\psi(\alpha,0) < \psi(\alpha,\beta)$ for $\alpha \in [0,\hat{\alpha}]$ and β small enough, so we have $(\alpha_u,\beta_u) \notin [0,\hat{\alpha}] \times \{0\}$.

Meanwhile, $\psi(0,\beta) < \psi(\alpha,\beta)$ for $\beta \in [0,\infty)$ and α small enough, then we have $(\alpha_u,\beta_u) \notin \{0\} \times [0,\infty)$.

On the other hand, it is obvious that

$$\frac{2}{N}S^{-2/N} \leq \frac{\hat{\alpha}^2}{2}A_1 + \frac{b\hat{\alpha}^4}{4}A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1 + \frac{b\hat{\alpha}^4}{2}A_2\|u^+\|^2 + \frac{\beta^2}{2}A_2 + \frac{b\beta^4}{2}A_2\|u^-\|^2 + \frac{b\beta^4}{4}A_2^2$$

for all $\beta \in [0,\infty)$.

Hence, we have that $\psi(\hat{\alpha},\beta) \leq 0$ for all $\beta \in [0,\infty)$. Thus, $(\alpha_u,\beta_u) \notin \{\hat{\alpha}\} \times [0,\infty)$. Hence, $(\alpha_u,\beta_u) \in (0,\hat{\alpha}) \times (0,\infty)$. That is, (α_u,β_u) is an inner maximizer of ψ in $[0,\hat{\alpha}) \times [0,\infty)$. Hence, $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_b^\lambda$.

Hence, in view of (2.20), we have that

$$\begin{aligned}
c_b^\lambda &\geq I_b^\lambda(\alpha_u u^+ + \beta_u u^-) + \frac{\alpha_u^2}{2}A_1 + \frac{b\alpha_u^4}{2}A_1\|u^+\|^2 + \frac{b\alpha_u^4}{4}A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1 \\
&\quad + \frac{\beta_u^2}{2}A_2 + \frac{b\beta_u^4}{2}A_2\|u^-\|^2 + \frac{b\beta_u^4}{4}A_2^2 \\
&> I_b^\lambda(\alpha_u u^+ + \beta_u u^-) \geq c_b^\lambda,
\end{aligned}$$

which is a contradiction.

Therefore, from the above arguments, we have that $B_1 = B_2 = 0$.

Finally, we prove c_b^λ is achieved.

Since $u^\pm \neq 0$, by Lemma 2.1, there exist $\alpha_u, \beta_u > 0$ such that

$$\bar{u} := \alpha_u u^+ + \beta_u u^- \in \mathcal{M}_b^\lambda.$$

Furthermore, it is easy to see that

$$\langle (I_b^\lambda)'(u), u^\pm \rangle \leq 0.$$

By Lemma 2.2, we obtain $0 < \alpha_u, \beta_u < 1$.

Since $u_n \in \mathcal{M}_b^\lambda$, according to Lemma 2.3, we get

$$I_b^\lambda(\alpha_u u_n^+ + \beta_u u_n^-) \leq I_b^\lambda(u_n^+ + u_n^-) = I_b^\lambda(u_n).$$

Thanks to (f_3), $B_1 = B_2 = 0$ and that the norm in E is lower semicontinuous, we have that

$$\begin{aligned}
c_b^\lambda &\le I_b^\lambda(\bar{u}) - \frac{1}{4}\langle (I_b^\lambda)'(\bar{u}), \bar{u}\rangle \\
&\le \frac{1}{4}\|\bar{u}\|_E^2 + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega |\bar{u}|^{2^{**}} dx + \frac{\lambda}{4}\int_\Omega [f(x,\bar{u})\bar{u} - 4F(x,\bar{u})]dx \\
&= \frac{1}{4}(\|\alpha_u u^+\|_E^2 + \|\beta_u u^-\|_E^2) + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\left[\int_\Omega |\alpha_u u^+|^{2^{**}} dx + \int_\Omega |\beta_u u^-|^{2^{**}} dx\right] \\
&\quad + \frac{\lambda}{4}\int_\Omega [f(x,\alpha_u u^+)(\alpha_u u^+) - 4F(x,\alpha_u u^+)]dx + \frac{\lambda}{4}\int_\Omega [f(x,\beta_u u^-)(\beta_u u^-) - 4F(x,\beta_u u^-)]dx \\
&\le \frac{1}{4}\|u\|_E^2 + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega |u|^{2^{**}} dx + \frac{\lambda}{4}\int_\Omega [f(x,u)u - 4F(x,u)]dx \\
&\le \liminf_{n\to\infty}\left[I_b^\lambda(u_n) - \frac{1}{4}\langle (I_b^\lambda)'(u_n), u_n\rangle\right] \le c_b^\lambda.
\end{aligned}$$

Therefore, $\alpha_u = \beta_u = 1$, and c_b^λ is achieved by $u_b := u^+ + u^- \in \mathcal{M}_b^\lambda$. □

3 Proof of Theorems 1.2–1.3

In this section, we prove our main results. First, we prove Theorem 1.2. In fact, by means of Lemma 2.4, we just prove that the minimizer u_b for c_b^λ is indeed a sign-changing solution of problem (1.1).

Proof of Theorem 1.2. Since $u_b \in \mathcal{M}_b^\lambda$, we have $\langle (I_b^\lambda)'(u_b), u_b^+\rangle = \langle (I_b^\lambda)'(u_b), u_b^-\rangle = 0$. By Lemma 2.4, for $(\alpha,\beta) \in (\mathbb{R}_+ \times \mathbb{R}_+)\backslash(1,1)$, we have

$$I_b^\lambda(\alpha u_b^+ + \beta u_b^-) < I_b^\lambda(u_b^+ + u_b^-) = c_b^\lambda. \tag{3.1}$$

If $(I_b^\lambda)'(u_b) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that

$$\|(I_b^\lambda)'(v)\| \ge \theta \quad \text{for all } \|v - u_b\| \ge 3\delta.$$

Choose $\tau \in (0, \min\{1/2, \frac{\delta}{\sqrt{2}\|u_b\|}\})$. Let

$$D := (1-\tau, 1+\tau) \times (1-\tau, 1+\tau)$$

and

$$g(\alpha,\beta) = \alpha u_b^+ + \beta u_b^- \quad \text{for all } (\alpha,\beta) \in D.$$

In view of (3.1), it is easy to see that

$$\bar{c}_\lambda := \max_{\partial\Omega} I_b^\lambda \circ g < c_{b,\lambda}. \tag{3.2}$$

Let $\varepsilon := \min\{(c_b^\lambda - \bar{c}_\lambda)/2, \theta\delta/8\}$ and $S_\delta := B(u_b,\delta)$, according to Lemma 2.3 in [38], there exists a deformation $\eta \in C([0,1] \times D, D)$ such that

(a) $\eta(1,v) = v$ if $v \notin (I_b^\lambda)^{-1}([c_b^\lambda - 2\varepsilon, c_b^\lambda + 2\varepsilon] \cap S_{2\delta})$,

(b) $\eta(1, (I_b^\lambda)^{c_b^\lambda+\varepsilon} \cap S_\delta) \subset (I_b^\lambda)^{c_{b,\lambda}-\varepsilon}$,

(c) $I_b^\lambda(\eta(1,v)) \le I_b^\lambda)(v)$ for all $v \in E$.

First, we need to prove that

$$\max_{(\alpha,\beta)\in\bar{D}} I_b^\lambda(\eta(1,g(\alpha,\beta))) < c_b^\lambda. \tag{3.3}$$

In fact, it follows from Lemma 2.1 that $I_b^\lambda(g(\alpha,\beta)) \le c_b^\lambda < c_b^\lambda + \varepsilon$. That is,

$$g(\alpha,\beta) \in (I_b^\lambda)^{c_b^\lambda+\varepsilon}.$$

On the other hand, we have

$$\begin{aligned}\|g(\alpha,\beta)-u_b\|^2 &= \|(\alpha-1)u_b^+ + (\beta-1)u_b^-\| \\ &\le 2((\alpha-1)^2\|u_b^+\|^2 + (\beta-1)^2\|u_b^-\|^2) \\ &\le 2\tau\|u_b\|^2 < \delta^2,\end{aligned}$$

which shows that $g(\alpha,\beta)\in S_\delta$ for all $(\alpha,\beta)\in\bar{D}$.

Therefore, by (b), we have $I_b^\lambda(\eta(1,g(s,t))) < c_b^\lambda - \varepsilon$. Hence, (3.3) holds.

In the following, we prove that $\eta(1,g(D))\cap\mathcal{M}_b^\lambda \neq \emptyset$, which contradicts the definition of c_b^λ.

Let $h(\alpha,\beta) := \eta(1,g(\alpha,\beta))$ and

$$\begin{aligned}\Psi_0(\alpha,\beta) :=& \ (\langle (I_b^\lambda)'(g(\alpha,\beta)),u_b^+\rangle, \langle (I_b^\lambda)'(g(\alpha,\beta)),u_b^-\rangle) \\ =& \ (\langle (I_b^\lambda)'(\alpha u_b^+ + \beta u_b^-),u_b^+\rangle, \langle (I_b^\lambda)'(\alpha u_b^+ + \beta u_b^-),u_b^-\rangle) \\ =&: (\varphi_u^1(\alpha,\beta), \varphi_u^2(\alpha,\beta))\end{aligned}$$

and

$$\Psi_1(\alpha,\beta) := \left(\frac{1}{\alpha}\langle (I_b^\lambda)'(h(\alpha,\beta)),\,(g(\alpha,\beta))^+\rangle, \frac{1}{\beta}\langle (I_b^\lambda)'(h(\alpha,\beta)),(h(\alpha,\beta))^-\rangle\right).$$

By the direct calculation, we have

$$\begin{aligned}\left.\frac{\varphi_u^1(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} &= \|u_b^+\|_E^2 + 3b\|u_b^+\|^4 + b\|u_b^+\|^2\|u_b^-\|^2 \\ &\quad - (2^{**}-1)\int_\Omega |u_b^+|^{2^{**}}dx\end{aligned}$$

$$-\lambda\int_\Omega \partial_\alpha f(x,u_b^+)(u_b^+)^2dx, \left.\frac{\varphi_u^1(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} = 2b\|u_b^+\|^2\|u_b^-\|^2, \quad \frac{\varphi_u^2(\alpha,\beta)}{\partial\alpha}\ |_{(1,1)} = 2b\|u_b^+\|^2\|u_b^-\|^2,$$

$$\begin{aligned}\left.\frac{\varphi_u^2(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} &= \|u_b^-\|^2 + 3b\|u_b^-\|^4 + b\|u_b^+\|^2\|u_b^-\|^2 \\ &\quad - (2^{**}-1)\int_\Omega |u_b^-|^{2^{**}}dx - \lambda\int_\Omega \partial_\beta f(x,u_b^-)(u_b^-)^2dx.\end{aligned}$$

Let

$$M = \begin{bmatrix} \left.\frac{\varphi_u^1(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} & \left.\frac{\varphi_u^2(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} \\ \left.\frac{\varphi_u^1(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} & \left.\frac{\varphi_u^2(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} \end{bmatrix}.$$

By (f_3), for $t\neq 0$, we have

$$\partial_t f(x,t)t^2 - 3f(x,t)t > 0$$

for almost every $x \in \Omega$. Then, since $u_b \in \mathcal{M}_{b,\lambda}$, we have

$$
\begin{aligned}
&\det M\\
&= \frac{\varphi_u^1(\alpha,\beta)}{\partial\alpha}\bigg|_{(1,1)} \times \frac{\varphi_u^2(\alpha,\beta)}{\partial\beta}\bigg|_{(1,1)} - \frac{\varphi_u^1(\alpha,\beta)}{\partial\beta}\bigg|_{(1,1)} \times \frac{\varphi_u^2(\alpha,\beta)}{\partial\alpha}\bigg|_{(1,1)}\\
&= \left[2\|u_b^+\|^2+(2^{**}-4)\int_\Omega |u_b^+|^{2^{**}}dx+2b\|u_b^+\|^2\|u_b^-\|^2+\lambda\int_\Omega(\partial_\alpha f(x,u_b^+)(u_b^+)^2-3f(x,u_b^+)(u_b^+)dx\right]\\
&\quad\times\left[2\|u_b^-\|^2+(2^{**}-4)\int_\Omega |u_b^-|^{2^{**}}dx+2b\|u_b^+\|^2\|u_b^-\|^2+\lambda\int_\Omega(\partial_\beta f(x,u_b^-)(u_b^-)^2-3f(x,u_b^-)(u_b^-)dx\right]\\
&\quad-4b^2\|u_b^+\|^4\|u_b^-\|^4\\
&> 0.
\end{aligned}
$$

Since $\Psi_0(\alpha,\beta)$ is a C^1 function and $(1,1)$ is the unique isolated zero point of Ψ_0, by using the degree theory, we deduce that $\deg(\Psi_0, D, 0) = 1$.

Hence, combining (3.3) with (a), we obtain

$$g(\alpha,\beta) = h(\alpha,\beta) \quad \text{on } \partial D.$$

Consequently, we obtain $\deg(\Psi_1, D, 0) = 1$. Therefore, $\Psi_1(\alpha_0,\beta_0) = 0$ for some $(\alpha_0,\beta_0) \in D$ so that

$$\eta(1, g(\alpha_0,\beta_0)) = h(\alpha_0,\beta_0) \in \mathcal{M}_b^\lambda,$$

which is contradicted to (3.3).

From the above discussions, we deduce that u_b is a sign-changing solution for problem (1.1).

Finally, we prove that u has exactly two nodal domains. To this end, we assume by contradiction that

$$u_b = u_1 + u_2 + u_3$$

with

$$u_i \neq 0,\ u_1 \geq 0,\ u_2 \leq 0$$

and

$$\text{suppt}(u_i) \cap \text{suppt}(u_j) = \emptyset \quad \text{for } i \neq j,\ i,j = 1,2,3$$

and

$$\langle (I_b^\lambda)'(u), u_i\rangle = 0 \quad \text{for } i = 1,2,3.$$

Setting $v := u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e., $v^\pm \neq 0$. Then, there exist a unique pair (α_v, β_v) of positive numbers such that

$$\alpha_v u_1 + \beta_v u_2 \in \mathcal{M}_b^\lambda.$$

Hence

$$I_b^\lambda(\alpha_v u_1 + \beta_v u_2) \geq c_b^\lambda.$$

Moreover, using the fact that $\langle (I_b^\lambda)'(u), u_i\rangle = 0$, we obtain $\langle (I_b^\lambda)'(v), v^\pm\rangle < 0$.

From Lemma 2.1 (ii), we have that

$$(\alpha_v, \beta_v) \in (0,1] \times (0,1].$$

On the other hand, we have that

$$
\begin{aligned}
0 = \frac{1}{4}\langle (I_b^\lambda)'(u), u_3\rangle &= \frac{1}{4}\|u_3\|^2 + \frac{b}{4}\|u_1\|^2\|u_3\|^2 + \frac{b}{4}\|u_2\|^2\|u_3\|^2 + \frac{b}{4}\|u_3\|^4 \\
&\quad - \frac{1}{2^{**}}\int_\Omega |u_3|^{2^{**}}dx - \frac{\lambda}{4}\int_\Omega f(x,u_3)u_3 dx \\
&< I_b^\lambda(u_3) + \frac{b}{4}\|u_1\|^2\|u_3\|^2 + \frac{b}{4}\|u_2\|^2\|u_3\|^2.
\end{aligned}
$$

Hence, by (2.15), we can obtain that

$$
\begin{aligned}
c_b^\lambda &\le I_b^\lambda(\alpha_v u_1 + \beta_v u_2) = I_b^\lambda(\alpha_v u_1 + \beta_v u_2) - \frac{1}{4}\langle (I_b^\lambda)'(\alpha_v u_1 + \beta_v u_2), (\alpha_v u_1 + \beta_v u_2)\rangle \\
&= \frac{1}{4}(\|\alpha_v u_1\|_E^2 + \|\beta_v u_2\|_E^2) + \frac{\lambda}{4}\int_\Omega [f(x,\alpha_v u_1)(\alpha_v u_1) - 4F(x,\alpha_v u_1)]dx \\
&\quad + \frac{\lambda}{4}\int_\Omega [f(x,\beta_v u_2)(\beta_v u_2) - 4F(x,\beta_v u_2)]dx \\
&\quad + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega \alpha_v^{2^{**}}|u_1|^{2^{**}}dx + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega \beta_v^{2^{**}}|u_2|^{2^{**}}dx \\
&\le \frac{1}{4}(\|u_1\|_E^2 + \|u_2\|_E^2) + \frac{\lambda}{4}\int_\Omega [f(x,u_1)u_1 - 4F(x,u_1)]dx \\
&\quad + \frac{\lambda}{4}\int_\Omega [f(x,u_2)u_2 - 4F(x,u_2)]dx + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega |u_1|^{2^{**}}dx + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega |u_2|^{2^{**}}dx \\
&= I_b^\lambda(u_1 + u_2) - \frac{1}{4}\langle (I_b^\lambda)'(u_1 + u_2), (u_1 + u_2)\rangle \\
&= I_b^\lambda(u_1 + u_2) + \frac{1}{4}\langle (I_b^\lambda)'(u), u_3\rangle + \frac{b}{4}\|u_1\|^2\|u_3\|^2 + \frac{b}{4}\|u_2\|^2\|u_3\|^2 \\
&< I_b^\lambda(u_1) + I_b^\lambda(u_2) + I_b^\lambda(u_3) + \frac{b}{4}(\|u_2\|^2 + \|u_3\|^2)\|u_1\|^2 \\
&\quad + \frac{b}{4}(\|u_1\|^2 + \|u_3\|^2)\|u_2\|^2 + \frac{b}{4}(\|u_1\|^2 + \|u_2\|^2)\|u_3\|^2 \\
&= I_b^\lambda(u) = c_b^\lambda,
\end{aligned}
$$

which is a contradiction, that is, $u_3 = 0$ and u_b has exactly two nodal domains. □

By Theorem 1.2, we obtain a least energy sign-changing solution u_b of problem (1.1). Next we prove that the energy of u_b is strictly more than twice the ground state energy.

Proof of Theorem 1.3. Similar to the proof of Lemma 2.3, there exists $\lambda_1^* > 0$ such that for all $\lambda \ge \lambda_1^*$, and for each $b > 0$, there exists $v_b \in \mathcal{N}_b^\lambda$ such that $I_b^\lambda(v_b) = c^* > 0$. By standard arguments (see Corollary 2.13 in [9]), the critical points of the functional I_b^λ on $\mathcal{N}_b^\lambda$ are critical points of I_b^λ in E, and we obtain $(I_b^\lambda)'(v_b) = 0$. That is, v_b is a ground state solution of (1.1).

According to Theorem 1.2, we know that the problem (1.1) has a least energy sign-changing solution u_b, which changes sign only once when $\lambda \ge \lambda^*$.

Let

$$\lambda^{**} = \max\{\lambda^*, \lambda_1^*\}.$$

Suppose that $u_b = u_b^+ + u_b^-$. As in the proof of Lemma 2.1, there exist $\alpha_{u_b^+} > 0$ and $\beta_{u_b^-} > 0$ such that

$$\alpha_{u_b^+} u_b^+ \in \mathcal{N}_b^\lambda, \quad \beta_{u_b^-} u_b^- \in \mathcal{N}_{b,\lambda}.$$

Furthermore, Lemma 2.1 implies that $\alpha_{u_b^+}, \beta_{u_b^-} \in (0,1)$.
Therefore, in view of Lemma 2.1, we have that

$$2c^* \leq I_b^\lambda(\alpha_{u_b^+} u_b^+) + I_b^\lambda(\beta_{u_b^-} u_b^-) \leq I_b^\lambda(\alpha_{u_b^+} u_b^+ + \beta_{u_b^-} u_b^-) < I_b^\lambda(u_b^+ + u_b^-) = c_b^\lambda.$$

Hence, it follows that $c^* > 0$ cannot be achieved by a sign-changing function. □

Acknowledgements

S. Liang was supported by the Foundation for China Postdoctoral Science Foundation (No. 2019M662220), Scientific research projects for Department of Education of Jilin Province, China (No. JJKH20210874KJ). B. Zhang was supported by the National Natural Science Foundation of China (No. 11871199), the Heilongjiang Province Postdoctoral Startup Foundation, PR China (No. LBH-Q18109), the Shandong Provincial Natural Science Foundation, PR China (No. ZR2020MA006), and the Cultivation Project of Young and Innovative Talents in Universities of Shandong Province.

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Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 38, 1–20; https://doi.org/10.14232/ejqtde.2021.1.38 www.math.u-szeged.hu/ejqtde/

The asymptotic behavior of solutions to a class of inhomogeneous problems: an Orlicz–Sobolev space approach

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Received 10 January 2021, appeared 19 April 2021
Communicated by Roberto Livrea

Abstract. The asymptotic behavior of the sequence $\{v_n\}$ of nonnegative solutions for a class of inhomogeneous problems settled in Orlicz–Sobolev spaces with prescribed Dirichlet data on the boundary of domain Ω is analysed. We show that $\{v_n\}$ converges uniformly in Ω as $n \to \infty$, to the distance function to the boundary of the domain.

Keywords: weak solution, viscosity solution, nonlinear elliptic equations, asymptotic behavior, Orlicz–Sobolev spaces.

2020 Mathematics Subject Classification: 35D30, 35D40, 35J60, 35J70, 46E30, 46E35.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the family of problems

$$\begin{cases} -\operatorname{div}\left(\frac{\varphi_n(|\nabla v|)}{|\nabla v|}\nabla v\right) = \lambda e^v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where for each positive integer n, the mappings $\varphi_n : \mathbb{R} \to \mathbb{R}$ are odd, increasing homeomorphisms of class C^1 satisfying *Lieberman-type condition*

$$N - 1 < \varphi_n^- - 1 \leq \frac{t\varphi_n'(t)}{\varphi_n(t)} \leq \varphi_n^+ - 1 < \infty, \qquad \forall\, t \geq 0 \tag{1.2}$$

for some constants φ_n^- and φ_n^+ with $1 < \varphi_n^- \leq \varphi_n^+ < \infty$,

$$\varphi_n^- \to \infty \quad \text{as } n \to \infty, \tag{1.3}$$

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and such that

$$\text{there exists a real constant } \beta > 1 \text{ with the property that } \varphi_n^+ \le \beta\varphi_n^-, \ \forall\, n \ge 1 \tag{1.4}$$

and

$$\lim_{n\to\infty} \varphi_n(1)^{1/\varphi_n^-} = 1. \tag{1.5}$$

For some examples of functions satisfying conditions (1.2)–(1.5) the reader is referred to [5, p. 4398]. Here we just point out the fact that in the particular case when $\varphi_n(t) = |t|^{n-2}t$, $n \ge 2$, the differential operator involved in problem (1.1) is the n-Laplacian, which for sufficiently smooth functions v is defined as $\Delta_n v := \operatorname{div}(|\nabla v|^{n-2}\nabla v)$. In this particular case problem (1.1) becomes

$$\begin{cases} -\Delta_n v = \lambda e^v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

which has been extensively studied in the literature (see, e.g. [3,7,12,14,15,18,19,32]). An existence result concerning problem (1.6) for each given $n > N$ and $\lambda > 0$ sufficiently small was proved by Aguilar Crespo & Peral Alonso in [3] by using a fixed-point argument while Mihăilescu *et al.* [32] showed a similar result by using variational techniques. Moreover, in [32] was studied the asymptotic behavoir of solutions as $n \to \infty$. More precisely, it was proved that there exists $\lambda^\star > 0$ (which does not depend on n) such that for each $n > N$ and each $\lambda \in (0, \lambda^\star)$ problem (1.6) possesses a nonnegative solution $u_n \in W_0^{1,n}(\Omega)$ and the sequence of solutions $\{u_n\}$ converges uniformly in $\overline{\Omega}$, as $n \to \infty$, to the unique viscosity solution of the problem

$$\begin{cases} \min\{|\nabla u| - 1, -\Delta_\infty u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.7}$$

which is precisely the distance function to the boundary of the domain $\operatorname{dist}(\cdot, \partial\Omega)$ (see [26, Lemma 6.10]). The result from [32] was extended to the case of equations involving variable exponent growth conditions by Mihăilescu & Fărcăşeanu in [14]. Motivated by these results the goal of this paper is to investigate the asymptotic behaviour of the solutions of the family of problems (1.1), as $n \to \infty$, for $\lambda > 0$ sufficiently small. We will show that the results from [32] and [14] continue to hold true in the case of the family of problems (1.1). In particular, our results generalise the results from [32] and complement the results from [14].

The paper is organized as follows. In Section 2 we give the definitions of the Orlicz and Orlicz–Sobolev spaces which represent the natural functional framework where the problems of type (1.1) should be investigated. Section 3 is devoted to the proof of the existence of weak solutions for problem (1.1) when λ is sufficiently small. Finally, in Section 4 we analyse the asymptotic behavior of the sequence of solutions found in the previous section, as $n \to \infty$, and we prove its uniform convergence to the distance function to the boundary of the domain.

2 Orlicz and Orlicz–Sobolev spaces

In this section we provide a brief overview on the Orlicz and Orlicz–Sobolev spaces and we recall the definitions and some of their main properties. For more details about these spaces the reader can consult the books [2,22,33,34] and papers [4,9,10,20,21].

First, we will introduce the Orlicz spaces. We assume that the function φ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ of class C^1. We define $\Phi : [0,\infty) \to \mathbb{R}$ by

$$\Phi(t) = \int_0^t \varphi(s)\, ds.$$

Note that Φ is a *Young function*, that is Φ vanishes when $t = 0$, Φ is continuous, Φ is convex and $\lim_{t\to\infty} \Phi(t) = \infty$. Moreover, since $\Phi(0) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$, then Φ is called a *N-function* (see [1, 2]). Next, we define the function $\Phi^\star : [0,\infty) \to \mathbb{R}$ given by

$$\Phi^\star(t) = \int_0^t \varphi^{-1}(s)\, ds\,.$$

$\Phi^\star$ is called the *complementary function of* Φ. The functions Φ and $\Phi^\star$ satisfy

$$\Phi^\star(t) = \sup_{s\geq 0} \left(st - \Phi(s)\right) \quad \text{for any } t \geq 0.$$

We note that $\Phi^\star$ is also a N-function, too.

Throughout this paper, we will assume that

$$0 < \varphi^- - 1 \leq \frac{t\varphi'(t)}{\varphi(t)} \leq \varphi^+ - 1 < \infty, \quad \text{for all } t > 0 \tag{2.1}$$

for some positive constants φ^- and φ^+. By [28, Lemma 1.1] (see also [31, Lemma 2.1]) we deduce that

$$1 < \varphi^- \leq \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^+ < \infty, \quad \text{for all } t > 0. \tag{2.2}$$

By relation (2.2) it follows that for each $t > 0$ and $s \in (0,1]$ we have

$$-\ln s^{\varphi^-} = \int_{st}^t \frac{\varphi^-}{\tau}\, d\tau \leq \int_{st}^t \frac{\varphi(\tau)}{\Phi(\tau)}\, d\tau = \ln \Phi(t) - \ln \Phi(st) \leq \int_{st}^t \frac{\varphi^+}{\tau}\, d\tau = -\ln s^{\varphi^+}$$

or

$$s^{\varphi^+}\Phi(t) \leq \Phi(st) \leq s^{\varphi^-}\Phi(t), \qquad \forall\, t > 0,\ s \in (0,1]\,. \tag{2.3}$$

Similarly, for each $t > 0$ and $s > 1$ we have

$$\ln s^{\varphi^-} = \int_t^{st} \frac{\varphi^-}{\tau}\, d\tau \leq \int_t^{st} \frac{\varphi(\tau)}{\Phi(\tau)}\, d\tau = \ln \Phi(st) - \ln \Phi(t) \leq \int_t^{st} \frac{\varphi^+}{\tau}\, d\tau = \ln s^{\varphi^+}$$

or

$$s^{\varphi^-}\Phi(t) \leq \Phi(st) \leq s^{\varphi^+}\Phi(t), \qquad \forall\, t > 0,\ s > 1\,. \tag{2.4}$$

Inequalities (2.3) and (2.4) can be reformulated as follows

$$\min\{s^{\varphi^-}, s^{\varphi^+}\}\Phi(t) \leq \Phi(st) \leq \max\{s^{\varphi^-}, s^{\varphi^+}\}\Phi(t) \quad \text{for any } s,t > 0\,. \tag{2.5}$$

Similarly, by [31, Lemma 2.1] we deduce that

$$\min\{s^{\varphi^- -1}, s^{\varphi^+ -1}\}\varphi(t) \leq \varphi(st) \leq \max\{s^{\varphi^- -1}, s^{\varphi^+ -1}\}\varphi(t), \qquad \forall\, s,t > 0. \tag{2.6}$$

Next, if we let $s = \varphi^{-1}(t)$ then we have

$$\frac{t(\varphi^{-1})'(t)}{\varphi^{-1}(t)} = \frac{\varphi(s)}{\varphi'(s)s}\,.$$

By (2.1) we deduce that

$$\frac{1}{\varphi^+ - 1} \leq \frac{t(\varphi^{-1})'(t)}{\varphi^{-1}(t)} \leq \frac{1}{\varphi^- - 1}, \qquad \forall\, t > 0\,.$$

The above relation implies that

$$1 < \frac{\varphi^+}{\varphi^+ - 1} \leq \frac{t\varphi^{-1}(t)}{\Phi^\star(t)} \leq \frac{\varphi^-}{\varphi^- - 1} < \infty \quad \text{for all } t > 0. \tag{2.7}$$

Examples. We point out some example of functions φ which are odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and φ and the corresponding primitive Φ satisfy condition (2.2) (see [10, Examples 1–3, p. 243]):

1. $\varphi(t) = |t|^{p-2}t$, $\Phi(t) = \frac{|t|^p}{p}$ with $p > 1$ and $\varphi^- = \varphi^+ = p$.
2. $\varphi(t) = \log(1 + |t|^r)|t|^{p-2}t$, $\Phi(t) = \log(1 + |t|^r)\frac{|t|^p}{p} - \frac{r}{p}\int_0^{|t|} \frac{s^{p+r-1}}{1+s^r}\, ds$ with $p, r > 1$ and $\varphi^- = p$, $\varphi^+ = p + r$.
3. $\varphi(t) = \frac{|t|^{p-2}t}{\log(1+|t|)}$ for $t \neq 0$, $\varphi(0) = 0$, $\Phi(t) = \frac{|t|^p}{p\log(1+|t|)} + \frac{1}{p}\int_0^{|t|} \frac{s^p}{(1+s)(\log(1+s))^2}\, ds$ with $p > 2$ and $\varphi^- = p - 1$, $\varphi^+ = p = \liminf_{t\to\infty} \frac{\log\Phi(t)}{\log t}$.

For each bounded domain $\Omega \subset \mathbb{R}^N$, the *Orlicz space* $L^\Phi(\Omega)$ defined by the N-function Φ (see [1,2,9]) is the set of real-valued measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^\Phi(\Omega)} := \sup\left\{ \int_\Omega u(x)v(x)\, dx;\ \int_\Omega \Phi^\star(|v(x)|)\, dx \leq 1 \right\} < \infty.$$

Then, the Orlicz space $L^\Phi(\Omega)$ endowed with the *Orlicz norm* $\|\cdot\|_{L^\Phi(\Omega)}$ is a Banach space and its Orlicz norm $\|\cdot\|_{L^\Phi(\Omega)}$ is equivalent to the so-called *Luxemburg norm* defined by

$$\|u\|_\Phi := \inf\left\{ \mu > 0\,;\ \int_\Omega \Phi\left(\frac{u(x)}{\mu}\right)\, dx \leq 1 \right\}. \tag{2.8}$$

In the case of Orlicz spaces, the following relations hold true (see, e.g. [17, Lemma 2.1]):

$$\|u\|_\Phi^{\varphi^+} \leq \int_\Omega \Phi(|u(x)|)\, dx \leq \|u\|_\Phi^{\varphi^-} \qquad \forall\, u \in L^\Phi(\Omega) \text{ with } \|u\|_\Phi < 1, \tag{2.9}$$

$$\|u\|_\Phi^{\varphi^-} \leq \int_\Omega \Phi(|u(x)|)\, dx \leq \|u\|_\Phi^{\varphi^+} \qquad \forall\, u \in L^\Phi(\Omega) \text{ with } \|u\|_\Phi > 1 \tag{2.10}$$

and

$$\int_\Omega \Phi(|u(x)|)\, dx = 1 \Longleftrightarrow \|u\|_\Phi = 1, \qquad \forall\, u \in L^\Phi(\Omega). \tag{2.11}$$

Next, we recall that for each bounded domain $\Omega \subset \mathbb{R}^N$, the *Orlicz–Sobolev space* $W^{1,\Phi}(\Omega)$ defined by the N-function Φ is the set of all functions u such that u and its distributional derivatives of order 1 lie in Orlicz space $L^\Phi(\Omega)$. More exactly, $W^{1,\Phi}(\Omega)$ is the space given by

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^\Phi(\Omega);\ \frac{\partial u}{\partial x_j} \in L^\Phi(\Omega),\ \ j \in \{1, \ldots, N\} \right\}.$$

It is a Banach space with respect to the following norm

$$\|u\|_{1,\Phi} := \|u\|_{\Phi} + \|\,|\nabla u|\,\|_{\Phi}.$$

By $W_0^{1,\Phi}(\Omega)$ we denoted the closure of all functions of class C^∞ with compact support over Ω with respect to norm of $W^{1,\Phi}(\Omega)$, i.e.

$$W_0^{1,\Phi}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\Phi}}.$$

Note that the norms $\|\cdot\|_{1,\Phi}$ and $\|\cdot\|_{W_0^{1,\Phi}} := \|\,|\nabla\cdot|\,\|_{\Phi}$ are equivalent on the Orlicz–Sobolev space $W_0^{1,\Phi}(\Omega)$ (see [21, Lemma 5.7]).

Under conditions (2.2) and (2.7), Φ and $\Phi^\star$ satisfy the Δ_2-*condition*, i.e.

$$\Phi(2t) \le C\Phi(t), \qquad \forall\, t \ge 0, \tag{2.12}$$

for some constant $C > 0$ (see [2, p. 232]). Therefore, $L^\Phi(\Omega)$, $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are reflexive Banach spaces (see [2, Theorem 8.19] and [2, p. 232]).

Remark 2.1. For each real number $p > 1$ let $\varphi(t) = |t|^{p-2}t$, $t \in \mathbb{R}$. It can be shown that $\varphi^- = \varphi^+ = p$ as mentioned above in Example 1 and the corresponding Orlicz space $L^\Phi(\Omega)$ reduces to the *classical Lebesgue space* $L^p(\Omega)$ while the Orlicz–Sobolev spaces $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ become the *classical Sobolev spaces* $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively. Note also that by [2, Theorem 8.12] the Orlicz space $L^\Phi(\Omega)$ is continuously embedded in the Lebesgue spaces $L^q(\Omega)$ for each $q \in (1, \varphi^-]$.

3 Variational solutions for problem (1.1)

In this section we will show that there exists a certain constant $\lambda^\star > 0$ (independent of n) such that for each $\lambda \in (0, \lambda^\star)$ problem (1.1) possesses a nonnegative weak solution for each integer $n \ge 1$.

We start by introducing the following notations: for each positive integer n we denote by Φ_n a primitive of the function φ_n. More precisely, we define $\Phi_n : [0,\infty) \to \mathbb{R}$ by

$$\Phi_n(t) := \int_0^t \varphi_n(s)\, ds\,.$$

Definition 3.1. We say that v_n is a *weak solution* of problem (1.1) if $v_n \in W_0^{1,\Phi_n}(\Omega)$ and the following relation holds true

$$\int_\Omega \frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|}\nabla v_n \nabla w\, dx = \lambda \int_\Omega e^{v_n} w\, dx, \qquad \forall\, w \in W_0^{1,\Phi_n}(\Omega). \tag{3.1}$$

Note that the integral from the right-hand side of relation (3.1) is well-defined since the Orlicz–Sobolev space $W_0^{1,\Phi_n}(\Omega)$ is continuously embedded in the classical Sobolev space $W_0^{1,\varphi_n^-}(\Omega)$ (see, e.g. [2, Theorem 8.12]) and for $\varphi_n^- > N$ we have $W_0^{1,\varphi_n^-}(\Omega) \subset L^\infty(\Omega)$. Moreover, we recall that Morrey's inequality holds true, i.e. there exists a positive constant C_n such that

$$\|v\|_{L^\infty(\Omega)} \le C_n \|\,|\nabla v|\,\|_{L^{\varphi_n^-}(\Omega)}, \qquad \forall\, v \in W_0^{1,\varphi_n^-}(\Omega)\,. \tag{3.2}$$

By [8, Proposition 3.1] we know that we can choose C_n as follows

$$C_n := \varphi_n^- |B(0,1)|^{-\frac{1}{\varphi_n^-}} N^{-\frac{N(\varphi_n^-+1)}{(\varphi_n^-)^2}} (\varphi_n^- - 1)^{\frac{N(\varphi_n^- -1)}{(\varphi_n^-)^2}} (\varphi_n^- - N)^{\frac{N-(\varphi_n^-)^2}{(\varphi_n^-)^2}} [\lambda_1(\varphi_n^-)]^{\frac{N-\varphi_n^-}{(\varphi_n^-)^2}}, \tag{3.3}$$

where $|B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^N$ and for each real number $p \in (1,\infty)$, $\lambda_1(p)$ denotes the first eigenvalue for the p-Laplace operator with homogeneous Dirichlet boundary conditions, i.e.

$$\lambda_1(p) := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}, \qquad \forall\, p \in (1,\infty).$$

By [8, Proposition 3.1] (see also [13, Theorem 3.2] for a similar result) it is well known that

$$\lim_{n\to\infty} C_n = \| \operatorname{dist}(\cdot, \partial\Omega)\|_{L^\infty(\Omega)}, \tag{3.4}$$

where $\operatorname{dist}(x,\partial\Omega) := \inf_{y\in\partial\Omega} |x-y|$, $\forall\, x \in \Omega$, stands for the distance function to the boundary of Ω.

For each positive integer n and each positive real number λ we introduce the Euler–Lagrange functional associated to problem (1.1) as $J_{n,\lambda} : W_0^{1,\Phi_n}(\Omega) \to \mathbb{R}$ defined by

$$J_{n,\lambda}(v) := \int_\Omega \Phi_n(|\nabla v|)\, dx - \lambda \int_\Omega e^v \, dx, \qquad \forall\, v \in W_0^{1,\Phi_n}(\Omega).$$

Standard arguments can be used in order to show that $J_{n,\lambda} \in C^1(W_0^{1,\Phi_n}(\Omega), \mathbb{R})$ and

$$\langle J_{n,\lambda}'(v), w\rangle = \int_\Omega \frac{\varphi_n(|\nabla v|)}{|\nabla v|} \nabla v \nabla w \, dx - \lambda \int_\Omega e^v w \, dx, \qquad \forall\, v,\ w \in W_0^{1,\Phi_n}(\Omega).$$

Thus, it is clear that v_n is a weak solution of (1.1) if and only if v_n is a critical point of functional $J_{n,\lambda}$.

We point out that we cannot find critical points of $J_{n,\lambda}$ by using the Direct Method in the Calculus of Variations since in the case of our problem $J_{n,\lambda}$ is not coercive. For that reason we propose an analysis of problem (1.1) based on Ekeland's Variational Principle in order to find critical points of $J_{n,\lambda}$.

For each positive integer n we denote

$$\lambda_n^\star := \frac{1}{2|\Omega|} e^{-C_n\left[|\Omega| + \frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-}}, \tag{3.5}$$

where C_n is the constant given by relation (3.3) and $|\Omega|$ stands for the N-dimensional Lebesgue measure of Ω. The starting point of our approach is the following lemma.

Lemma 3.2. *For each positive integer n let $\lambda_n^\star$ be given by relation (3.5). Then for each $\lambda \in (0, \lambda_n^\star)$ we have*

$$J_{n,\lambda}(v) \geq \frac{1}{2}, \qquad \forall\, v \in W_0^{1,\Phi_n}(\Omega) \quad \text{with} \quad \|v\|_{W_0^{1,\Phi_n}} = 1.$$

Proof. Let n be a positive integer arbitrary fixed. By relation (2.5) we get that $\Phi_n(s) \geq \Phi_n(1) s^{\varphi_n^-}$, for all $s > 1$ and thus,

$$s^{\varphi_n^-} \leq 1 + \frac{\Phi_n(s)}{\Phi_n(1)}, \qquad \forall\, s \geq 0.$$

Using this fact we deduce that

$$\int_\Omega |\nabla v|^{\varphi_n^-}\,dx \le |\Omega| + \frac{1}{\Phi_n(1)}\int_\Omega \Phi_n(|\nabla v|)\,dx, \qquad \forall\, v \in W_0^{1,\Phi_n}(\Omega). \tag{3.6}$$

By the above inequality, and since for each $v \in W_0^{1,\Phi_n}(\Omega)$ with $\|v\|_{W_0^{1,\Phi_n}} := \|\,|\nabla v|\,\|_{\Phi_n} = 1$ we have $\int_\Omega \Phi_n(|\nabla v|)\,dx = 1$ (via relation (2.11)), it results

$$\|\,|\nabla v|\,\|_{L^{\varphi_n^-}(\Omega)} \le \left[|\Omega| + \frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-}, \qquad \forall\, v \in W_0^{1,\Phi_n}(\Omega) \quad \text{with } \|v\|_{W_0^{1,\Phi_n}} = 1. \tag{3.7}$$

Next, taking into account that $W_0^{1,\Phi_n}(\Omega)$ is continuously embedded in $W_0^{1,\varphi_n^-}(\Omega)$ and using the fact that $\varphi_n^- > N$ and Morrey's inequality (3.2) we obtain

$$\begin{aligned} J_{n,\lambda}(v) &= \int_\Omega \Phi_n(|\nabla v|)\,dx - \lambda\int_\Omega e^v\,dx \\ &\ge 1 - \lambda|\Omega| e^{\|v\|_{L^\infty(\Omega)}} \\ &\ge 1 - \lambda|\Omega| e^{C_n \|\,|\nabla v|\,\|_{L^{\varphi_n^-}(\Omega)}}, \qquad \forall\, v \in W_0^{1,\Phi_n}(\Omega) \text{ with } \|v\|_{W_0^{1,\Phi_n}} = 1. \end{aligned}$$

Then for each $\lambda \in (0, \lambda_n^\star)$, combining the above estimates with relation (3.7) we get

$$J_{n,\lambda}(v) \ge 1 - \lambda|\Omega| e^{C_n\left[|\Omega| + \frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-}} \ge 1 - \lambda_n^\star|\Omega| e^{C_n\left[|\Omega| + \frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-}} = \frac{1}{2},$$

for all $v \in W_0^{1,\Phi_n}(\Omega)$ with $\|v\|_{W_0^{1,\Phi_n}} = 1$. The proof of the lemma is complete. $\square$

Lemma 3.3. *For each positive integer n let $\lambda_n^\star$ be given by relation (3.5). Define*

$$\lambda^\star := \inf_{n\in\mathbb{N}^*} \lambda_n^\star. \tag{3.8}$$

Then $\lambda^\star > 0$.

Proof. First, we show that there exists a positive constant $K > 0$ such that

$$\left[|\Omega| + \frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-} < K, \qquad \forall\, n \ge 1. \tag{3.9}$$

Indeed, since by (1.5) we have

$$\lim_{n\to\infty} \varphi_n(1)^{1/\varphi_n^-} = 1,$$

it yields that for each positive integer n large enough we get

$$\frac{1}{2} \le \varphi_n(1)^{1/\varphi_n^-},$$

which implies that

$$\frac{1}{\varphi_n(1)} \le 2^{\varphi_n^-}.$$

By (1.2) (via (2.1) and (2.2)) we find that for each positive integer n large enough the following inequalities hold true

$$\frac{1}{\Phi_n(1)} \le \frac{\varphi_n^+}{\varphi_n(1)} \le \varphi_n^+ 2^{\varphi_n^-} \le \beta\varphi_n^- 2^{\varphi_n^-}.$$

Using the above relations we deduce that for each positive integer n large enough we obtain

$$\left[|\Omega| + \frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-} \leq \left[|\Omega| + \beta\varphi_n^- 2^{\varphi_n^-}\right]^{1/\varphi_n^-} \leq \left(\beta\varphi_n^- 2^{\varphi_n^-+1}\right)^{1/\varphi_n^-}.$$

Now, taking into account the fact that $\lim_{n\to\infty}\left(\beta\varphi_n^- 2^{\varphi_n^-+1}\right)^{1/\varphi_n^-} = 2$, the above approximations imply that relation (3.9) holds true.

Next, using (3.9) and the expression of $\lambda_n^\star$ we infer that

$$\lambda_n^\star = \frac{1}{2|\Omega|} e^{-C_n\left[|\Omega|+\frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-}} > \frac{1}{2|\Omega|} e^{-KC_n}, \qquad \forall\, n \geq 1\,.$$

Recalling that $\lim_{n\to\infty} C_n = \|\operatorname{dist}(\cdot,\partial\Omega)\|_{L^\infty(\Omega)}$ (by (3.4)) and taking into account that function $(1,\infty) \ni p \longrightarrow \lambda_1(p)$ is continuous (see, Lindqvist [29] or Huang [23]) we conclude from the above estimates that $\lambda^\star = \inf_{n\in\mathbb{N}^*} \lambda_n^\star > 0$. The proof of Lemma 3.3 is complete. □

The main goal of this section is to prove the existence of weak solutions of problem (1.1) for each positive integer n. This result is the core of the following theorem.

Theorem 3.4. *Let $\lambda^\star > 0$ be given by (3.8). Then for each $\lambda \in (0,\lambda^\star)$ and each $n \in \mathbb{N}^\star$, problem (1.1) has a nonnegative solution $v_n \in \overline{B_1(0)} \subset W_0^{1,\Phi_n}(\Omega)$ identified by $J_{n,\lambda}(v_n) = \inf_{\overline{B_1(0)}} J_{n,\lambda}$, where $B_1(0)$ is the unit ball centered at the origin in the Orlicz–Sobolev space $W_0^{1,\Phi_n}(\Omega)$.*

Proof. We consider $\lambda \in (0,\lambda^\star)$ and $n \in \mathbb{N}^\star$ arbitary fixed. For each $v \in W_0^{1,\Phi_n}(\Omega)$ with $\|v\|_{W_0^{1,\Phi_n}} \leq 1$, in view of relations (2.9) and (2.11), we have

$$\|v\|^{\varphi_n^-}_{W_0^{1,\Phi_n}} \geq \int_\Omega \Phi_n(|\nabla v|)\,dx \geq \|v\|^{\varphi_n^+}_{W_0^{1,\Phi_n}}. \tag{3.10}$$

Thus, taking into account (3.10), Morrey's inequality (3.2) and relation (3.6), for each $v \in \overline{B_1(0)} \subset W_0^{1,\Phi_n}(\Omega)$ we obtain

$$\begin{aligned} J_{n,\lambda}(v) &= \int_\Omega \Phi_n(|\nabla v|)\,dx - \lambda\int_\Omega e^v\,dx \\ &\geq \|v\|^{\varphi_n^+}_{W_0^{1,\Phi_n}} - \lambda|\Omega|e^{\|v\|_{L^\infty(\Omega)}} \\ &\geq -\lambda|\Omega|e^{C_n\|\,|\nabla v|\,\|_{L^{\varphi_n^-}(\Omega)}} \\ &\geq -\lambda|\Omega|e^{C_n\left[|\Omega|+\frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-}}. \end{aligned}$$

Computing $J_{n,\lambda}(0) = -\lambda|\Omega|$ we deduce that

$$J_{n,\lambda}(0) < 0$$

while by Lemma 3.2 we get

$$\inf_{\partial B_1(0)} J_{n,\lambda} \geq \frac{1}{2} > 0,$$

which imply that

$$\gamma_n := \inf_{\overline{B_1(0)}} J_{n,\lambda} \in (-\infty, 0)\,.$$

We consider $\epsilon > 0$ such that

$$\epsilon < \inf_{\partial B_1(0)} J_{n,\lambda} - \inf_{B_1(0)} J_{n,\lambda}. \tag{3.11}$$

Ekeland's variational principle applied to $J_{n,\lambda}$ restricted to $\overline{B_1(0)}$ provides the existence of $v_\epsilon \in \overline{B_1(0)}$ having the properties

$$i)\ J_{n,\lambda}(v_\epsilon) < \inf_{\overline{B_1(0)}} J_{n,\lambda} + \epsilon,$$

$$ii)\ J_{n,\lambda}(v_\epsilon) < J_{n,\lambda}(v) + \epsilon \, \|v - v_\epsilon\|_{W_0^{1,\Phi_n}} \quad \text{for all } \ v \neq v_\epsilon\,.$$

Since $\inf_{\overline{B_1(0)}} J_{n,\lambda} \leq \inf_{B_1(0)} J_{n,\lambda}$ and ϵ is chosen small such that (3.11) holds true, using relation *i)* above we arrive at

$$J_{n,\lambda}(v_\epsilon) < \inf_{\overline{B_1(0)}} J_{n,\lambda} + \epsilon \leq \inf_{B_1(0)} J_{n,\lambda} + \epsilon < \inf_{\partial B_1(0)} J_{n,\lambda}\,,$$

from which we deduce that v_ϵ is not an element on the boundary of the unit ball of space $W_0^{1,\Phi_n}(\Omega)$, $v_\epsilon \notin \partial B_1(0)$, and consequently, v_ϵ is an element in the interior of this ball, that means $v_\epsilon \in B_1(0)$.

Next, we focus on the functional $F_{n,\lambda} : \overline{B_1(0)} \to \mathbb{R}$ defined by $F_{n,\lambda}(v) = J_{n,\lambda}(v) + \epsilon \, \|v - v_\epsilon\|_{W_0^{1,\Phi_n}}$. Obviously, v_ϵ is a minimum point of $F_{n,\lambda}$ (via *ii)*) that infers

$$\frac{F_{n,\lambda}(v_\epsilon + tw) - F_{n,\lambda}(v_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $w \in B_1(0)$. Computing the above relation we find

$$\frac{J_{n,\lambda}(v_\epsilon + tw) - J_{n,\lambda}(v_\epsilon)}{t} + \epsilon \, \|w\|_{W_0^{1,\Phi_n}} \geq 0$$

and then passing to the limit as $t \to 0^+$ it yields that $\langle J'_{n,\lambda}(v_\epsilon), w\rangle + \epsilon \, \|w\|_{W_0^{1,\Phi_n}} \geq 0$ that implies $\|J'_{n,\lambda}(v_\epsilon)\|_{(W_0^{1,\Phi_n}(\Omega))^\star} \leq \epsilon$, where $(W_0^{1,\Phi_n}(\Omega))^\star$ is the dual space of $W_0^{1,\Phi_n}(\Omega)$.

In consideration of that, we draw to the conclusion that there exists a sequence $\{v_m\}_m \subset B_1(0)$ such that

$$\lim_{m\to\infty} J_{n,\lambda}(v_m) = \gamma_n \quad \text{and} \quad \lim_{m\to\infty} J'_{n,\lambda}(v_m) = 0\,. \tag{3.12}$$

The sequence $\{v_m\}_m$ is certainly bounded in $W_0^{1,\Phi_n}(\Omega)$ since $v_m \in B_1(0)$ for all $m \in \mathbb{N}^\star$ and this fact induces the existence of $v_n \in W_0^{1,\Phi_n}(\Omega)$ such that, up to a subsequence, $\{v_m\}_m$ converges weakly to v_n in $W_0^{1,\Phi_n}(\Omega)$ and uniformly in Ω, since $\varphi_n^- > N$, as $m \to \infty$. Furthermore, we infer that

$$\lim_{m\to\infty} \int_\Omega e^{v_m}(v_m - v_n)\, dx = 0$$

and

$$\lim_{m\to\infty} \langle J'_{n,\lambda}(v_m), v_m - v_n\rangle = 0\,,$$

which imply that

$$\lim_{m\to\infty} \int_\Omega \frac{\varphi_n(|\nabla v_m|)}{|\nabla v_m|} \nabla v_m \nabla(v_m - v_n)\, dx = 0. \tag{3.13}$$

Owing to the weak convergence of sequence $\{v_m\}_m$ to v_n in $W_0^{1,\Phi_n}(\Omega)$, as $m \to \infty$, we have that

$$\lim_{m\to\infty} \langle J'_{n,\lambda}(v_n), v_m - v_n\rangle = 0$$

and it follows that

$$\lim_{m\to\infty}\int_\Omega \frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|}\nabla v_n \nabla(v_m - v_n)\,dx = 0. \tag{3.14}$$

Assembling relations (3.13) and (3.14), we conclude that

$$\lim_{m\to\infty}\int_\Omega \left[\frac{\varphi_n(|\nabla v_m|)}{|\nabla v_m|}\nabla v_m - \frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|}\nabla v_n\right]\nabla(v_m - v_n)\,dx = 0. \tag{3.15}$$

By [16, Lemma 3.2] we know that there exists a positive constant k_n such that

$$\left[\frac{\varphi_n(|\xi|)}{|\xi|}\xi - \frac{\varphi_n(|\eta|)}{|\eta|}\eta\right]\cdot(\xi-\eta) \geq k_n \frac{[\Phi_n(|\xi-\eta|)]^{\frac{\varphi_n^-+2}{\varphi_n^-+1}}}{[\Phi_n(|\xi|)+\Phi_n(|\eta|)]^{1/(\varphi_n^-+1)}}, \qquad \forall\, \xi,\eta \in \mathbb{R}^N,\ \xi \neq \eta.$$

In our case, we established that there exist constant $k_n > 0$ so that

$$\int_\Omega \left[\frac{\varphi_n(|\nabla v_m|)}{|\nabla v_m|}\nabla v_m - \frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|}\nabla v_n\right](\nabla v_m - \nabla v_n)\,dx$$
$$\geq k_n \int_\Omega \frac{[\Phi_n(|\nabla v_m - \nabla v_n|)]^{\frac{\varphi_n^-+2}{\varphi_n^-+1}}}{[\Phi_n(|\nabla v_m|)+\Phi_n(|\nabla v_n|)]^{1/(\varphi_n^-+1)}}\,dx.$$

Due to relation (3.15) we deduce that

$$\lim_{m\to\infty}\int_\Omega \Phi_n(|\nabla(v_m-v_n)|)\left[\frac{\Phi_n(|\nabla(v_m-v_n)|)}{\Phi_n(|\nabla v_m|)+\Phi_n(|\nabla v_n|)}\right]^{1/(\varphi_n^-+1)} dx = 0.$$

Since Φ_n is a convex function we obtain by relation (2.5) that

$$\Phi_n(|\nabla(v_m-v_n)|) \leq \frac{\Phi_n(2|\nabla v_m|)+\Phi_n(2|\nabla v_n|)}{2} \leq 2^{\varphi_n^+-1}\left[\Phi_n(|\nabla v_m|)+\Phi_n(|\nabla v_n|)\right].$$

Using assumption (1.4), the last two relations require

$$\lim_{m\to\infty}\int_\Omega \Phi_n(|\nabla(v_m-v_n)|)\,dx = 0\,,$$

and (2.9) generates

$$\lim_{m\to\infty}\|v_m - v_n\|_{W_0^{1,\Phi_n}} = 0\,.$$

That being the case, $\{v_m\}_m$ converges strongly to v_n in $W_0^{1,\Phi_n}(\Omega)$ as $m\to\infty$. Hence, relation (3.12) contribute to

$$J_{n,\lambda}(v_n) = \gamma_n < 0 \quad \text{and} \quad J'_{n,\lambda}(v_n) = 0\,. \tag{3.16}$$

As a result, v_n is the minimizer of $J_{n,\lambda}$ on $B_1(0)$, and also v_n is a critical point of the functional $J_{n,\lambda}$. Of course, v_n is really a weak solution of (1.1). Finally, note that $J_{n,\lambda}(|v|) \leq J_{n,\lambda}(v)$ for any $v \in W_0^{1,\Phi_n}(\Omega)$ and for this reason v_n is a nonnegative function on Ω.

The proof of Theorem 3.4 is complete. □

4 The asymptotic behavior of the sequence of solutions $\{v_n\}_n$ of problem (1.1) given by Theorem 3.4 as $n \to \infty$

The goal of this section is to prove the following result.

Theorem 4.1. *Let $\lambda^\star > 0$ be given by (3.8). For each $\lambda \in (0, \lambda^\star)$ and each $n \in \mathbb{N}^\star$ we denote by v_n the nonnegative weak solution of problem (1.1) given by Theorem 3.4. The sequence $\{v_n\}$ converges uniformly in Ω to $\operatorname{dist}(\cdot, \partial\Omega)$, the distance function to the boundary of Ω.*

In order to prove Theorem 4.1 we first establish the uniform Hölder estimates for the weak solutions of (1.1).

Lemma 4.2. *Let $\lambda^\star > 0$ be given by (3.8). Fix $\lambda \in (0, \lambda^\star)$ and let v_n be the nonnegative solution of problem (1.1) given by Theorem 3.4. Then there is a subsequence $\{v_n\}$ which converges uniformly in Ω, as $n \to \infty$, to a continuous function $v_\infty \in C(\overline{\Omega})$ with $v_\infty \geq 0$ in Ω and $v_\infty = 0$ on $\partial\Omega$.*

Proof. Let $q \geq N$ be an arbitrary real number. By (1.3) we can choose $q < \varphi_n^-$ for sufficiently large positive integer n. Using Hölder's inequality, relation (3.6), recalling that $v_n \in B_1(0) \subset W_0^{1,\Phi_n}(\Omega)$ and taking into account (2.9) we have

$$\begin{aligned}
\left(\int_\Omega |\nabla v_n|^q \, dx\right)^{1/q} &\leq \left(\int_\Omega |\nabla v_n|^{\varphi_n^-} \, dx\right)^{1/\varphi_n^-} |\Omega|^{1/q - 1/\varphi_n^-} \\
&\leq \left[|\Omega| + \frac{1}{\Phi_n(1)} \int_\Omega \Phi_n(|\nabla v_n|) \, dx\right]^{1/\varphi_n^-} |\Omega|^{1/q-1/\varphi_n^-} \\
&\leq \left[|\Omega| + \frac{1}{\Phi_n(1)} \|v_n\|^{\varphi_n^-}_{W_0^{1,\Phi_n}}\right]^{1/\varphi_n^-} |\Omega|^{1/q-1/\varphi_n^-} \\
&\leq \left[|\Omega| + \frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-} |\Omega|^{1/q-1/\varphi_n^-} .
\end{aligned}$$

Thereupon, using (3.9) we find that sequence $\{|\nabla v_n|\}$ is uniformly bounded in $L^q(\Omega)$. It is clear that $q > N$ ensures that the embedding of $W_0^{1,q}(\Omega)$ into $C(\overline{\Omega})$ is compact. Keeping in mind the reflexivity of the Sobolev space $W_0^{1,q}(\Omega)$ we deduce that there exists a subsequence (not relabelled) of $\{v_n\}$ and a function $v_\infty \in C(\overline{\Omega})$ such that $v_n \rightharpoonup v_\infty$ weakly in $W_0^{1,q}(\Omega)$ and $v_n \to v_\infty$ uniformly in Ω as $n \to \infty$. In addition, the facts that $v_n \geq 0$ in Ω and $v_n = 0$ on $\partial\Omega$ for each $\varphi_n^- > N$ hint that $v_\infty \geq 0$ in Ω and $v_\infty = 0$ on $\partial\Omega$. The proof of Lemma 4.2 is complete. □

In Theorem 4.5 below we show that function v_∞ given by Lemma 4.2 is the solution in the viscosity sense (see, Crandall, Ishii & Lions [11]) of a certain limiting problem. Accordingly, we adopt the usual strategy of first proving that continuous weak solutions of problem (1.1) at level n are indeed solutions in the viscosity sense. Before recalling the definition of viscosity solutions for this type of problems, let us note that if we assume for a moment that the solutions v_n of problem (1.1) are sufficiently smooth so that we can perform the differentiation in the PDE

$$-\operatorname{div}\left(\frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \nabla v_n\right) = \lambda e^{v_n}, \quad \text{in } \Omega,$$

we get

$$-\frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \Delta v_n - \frac{|\nabla v_n| \varphi_n'(|\nabla v_n|) - \varphi_n(|\nabla v_n|)}{|\nabla v_n|^3} \Delta_\infty v_n = \lambda e^{v_n}, \quad \text{in } \Omega, \tag{4.1}$$

where Δ stands for the Laplace operator, $\Delta v := \mathrm{Trace}(D^2 v) = \sum_{i=1}^N \frac{\partial^2 v}{\partial x_i^2}$ and Δ_∞ stands for the ∞-Laplace operator,

$$\Delta_\infty v := \langle D^2 v \nabla v, \nabla v\rangle = \sum_{i,j=1}^N \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j},$$

while $D^2 v$ denotes the Hessian matrix of v.

Remark that (4.1) can be reformulated as

$$H_n(v_n, \nabla v_n, D^2 v_n) = 0, \quad \text{in } \Omega$$

with function H_n defined as follows

$$H_n(y,z,S) := -\frac{\varphi_n(|z|)}{|z|}\,\mathrm{Trace}\, S - \frac{|z|\varphi_n'(|z|) - \varphi_n(|z|)}{|z|^3}\,\langle Sz, z\rangle - \lambda e^y,$$

where $y \in \mathbb{R}$, z is a vector in $\mathbb{R}^N$ and S stands for a real symmetric matrix in $\mathbb{M}^{N\times N}$.

Since our main objective in this section is the asymptotic analysis of solutions $\{v_n\}$ as $n \to \infty$, we are now ready to give the definition of viscosity solutions for the homogeneous Dirichlet boundary value problem associated to degenerate elliptic PDE of the type

$$\begin{cases} H_n(v, \nabla v, D^2 v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Definition 4.3.

i) An upper semicontinuous function v is a viscosity subsolution of problem (4.2) if $v \le 0$ on $\partial\Omega$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $v(x_0) = \Psi(x_0)$ and $v(x) < \Psi(x)$ if $x \in B(x_0, r)\setminus\{x_0\}$ for some $r > 0$, we have $H_n(\Psi(x_0), \nabla\Psi(x_0), D^2\Psi(x_0)) \le 0$.

ii) A lower semicontinuous function v is a viscosity supersolution of problem (4.2) if $v \ge 0$ on $\partial\Omega$ and, whenever $x_0 \in \Omega$ and $\Upsilon \in C^2(\Omega)$ are such that $v(x_0) = \Upsilon(x_0)$ and $v(x) > \Upsilon(x)$ if $x \in B(x_0, r)\setminus\{x_0\}$ for some $r > 0$, we have $H_n(\Upsilon(x_0), \nabla\Upsilon(x_0), D^2\Upsilon(x_0)) \ge 0$.

iii) A continuous function v is a viscosity solution of problem (4.2) if it is both viscosity supersolution and viscosity subsolution of problem (4.2).

In the sequel, functions Ψ and Υ stand for test functions touching the graph of v from above and below, respectively.

Our goal now is to prove that any continuous weak solution of (1.1) is also viscosity solution of (1.1) and in order to establish this result we follow the approach by Juutinen, Lindqvist & Manfredi in [27, Lemma 1.8] (see also [35, Lemma 1] for a similar approach but in the framework of inhomogeneous differential operators).

Lemma 4.4. *A continuous weak solution of problem* (1.1) *is also a viscosity solution of* (1.1).

Proof. Firstly, we prove that if v_n is a continuous weak solution of problem (1.1) for a fixed positive integer n, then it is a viscosity subsolution of problem (1.1). We begin by considering $x_n^0 \in \overline{\Omega}$ and a test function $\Psi_n \in C^2(\overline{\Omega})$ such that $v_n(x_n^0) = \Psi_n(x_n^0)$ and $v_n - \Psi_n$ has a strict local maximum at x_n^0, that is $v_n(y) < \Psi_n(y)$ if $y \in B(x_n^0, \rho)\setminus\{x_n^0\}$ for some $\rho > 0$.

Next, we have to show that

$$-\operatorname{div}\left(\frac{\varphi_n(|\nabla\Psi_n(x_n^0)|)}{|\nabla\Psi_n(x_n^0)|}\,\nabla\Psi_n(x_n^0)\right)\leq\lambda e^{\Psi_n(x_n^0)}$$

or

$$-\frac{\varphi_n(|\nabla\Psi_n(x_n^0)|)}{|\nabla\Psi_n(x_n^0)|}\,\Delta\Psi_n(x_n^0)-\frac{|\nabla\Psi_n(x_n^0)|\varphi_n'(|\nabla\Psi_n(x_n^0)|)-\varphi_n(|\nabla\Psi_n(x_n^0)|)}{|\nabla\Psi_n(x_n^0)|^3}\,\Delta_\infty\Psi_n(x_n^0)\leq\lambda e^{\Psi_n(x_n^0)}.$$

Arguing *ad contrarium*, suppose that this is not the case of the above assertion. In other words, we admit that there exists a radius $\rho_n>0$ such that $B(x_n^0,\rho_n)\subset\Omega$ from the Euclidean space $\mathbb{R}^N$ such that

$$-\frac{\varphi_n(|\nabla\Psi_n(y)|)}{|\nabla\Psi_n(y)|}\Delta\Psi_n(y)-\frac{|\nabla\Psi_n(y)|\varphi_n'(|\nabla\Psi_n(y)|)-\varphi_n(|\nabla\Psi_n(y)|)}{|\nabla\Psi_n(y)|^3}\Delta_\infty\Psi_n(y)>\lambda e^{\Psi_n(y)}$$

for all $y\in B(x_n^0,\rho_n)$. For ρ_n small enough, we may presume that $v_n-\Psi_n$ has a strict local maximum at x_n^0, that is $v_n(y)<\Psi_n(y)$ if $y\in B(x_n^0,\rho_n)\setminus\{x_n^0\}$. This fact implies that actually

$$\sup_{\partial B(x_n^0,\rho_n)}(v_n-\Psi_n)<0.$$

Thus, we may consider a perturbation of the test function Ψ_n defined as

$$\overline{w}_n(y):=\Psi_n(y)+\frac{1}{2}\sup_{y\in\partial B(x_n^0,\rho_n)}[v_n-\Psi_n](y)$$

that has the properties

- $\overline{w}_n(x_n^0)<v_n(x_n^0)$;
- $\overline{w}_n>v_n$ on $\partial B(x_n^0,\rho_n)$;
- $-\operatorname{div}\big(\frac{\varphi_n(|\nabla\overline{w}_n|)}{|\nabla\overline{w}_n|}\,\nabla\overline{w}_n\big)>\lambda e^{\Psi_n}$ in $B(x_n^0,\rho_n)$.

Multiplying the above inequality by the positive part of the function $v_n-\overline{w}_n$, i.e. $(v_n-\overline{w}_n)^+$, that vanishes on the boundary of the ball $B(x_n^0,\rho_n)$, and integrating on $B(x_n^0,\rho_n)$, we get

$$\int_{\mathcal{M}_n}\frac{\varphi_n(|\nabla\overline{w}_n(x)|)}{|\nabla\overline{w}_n(x)|}\nabla\overline{w}_n(x)\,\,[\nabla v_n(x)-\nabla\overline{w}_n(x)]\,\,dx>\lambda\int_{\mathcal{M}_n}e^{\Psi_n(x)}\,[v_n(x)-\overline{w}_n(x)]\,\,dx,\quad(4.3)$$

where the set $\mathcal{M}_n:=\{x\in B(x_n^0,\rho_n);\ \overline{w}_n(x)<v_n(x)\}$.

On the other hand, taking the test function in relation (3.1) to be

$$w:\Omega\to\mathbb{R},\ w(x)=\begin{cases}(v_n-\overline{w}_n)^+(x), & \text{if } x\in B(x_n^0,\rho_n),\\ 0, & \text{if } x\in\Omega\setminus B(x_n^0,\rho_n),\end{cases}$$

we obtain

$$\int_{B(x_n^0,\rho_n)}\frac{\varphi_n(|\nabla v_n(x)|)}{|\nabla v_n(x)|}\,\nabla v_n(x)\,\,\nabla(v_n-\overline{w}_n)^+(x)\,dx=\lambda\int_{B(x_n^0,\rho_n)}e^{v_n(x)}(v_n-\overline{w}_n)^+(x)\,dx$$

or

$$\int_{\mathcal{M}_n}\frac{\varphi_n(|\nabla v_n(x)|)}{|\nabla v_n(x)|}\nabla v_n(x)\,\,\nabla(v_n-\overline{w}_n)(x)\,dx=\lambda\int_{\mathcal{M}_n}e^{v_n(x)}(v_n-\overline{w}_n)(x)\,dx$$

since $v_n \leq \overline{w}_n$ in the ball $B(x_n^0, \rho_n)$ outside $\mathcal{M}_n$.

Applying the subtraction of the above equality from inequality (4.3) it produces

$$\int_{\mathcal{M}_n} \left[\frac{\varphi_n(|\nabla \overline{w}_n|)}{|\nabla \overline{w}_n|} \nabla \overline{w}_n - \frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \nabla v_n \right] (\nabla v_n - \nabla \overline{w}_n)\, dx$$
$$> \lambda \int_{\mathcal{M}_n} \left(e^{\Psi_n} - e^{v_n} \right) (v_n - \overline{w}_n)\, dx \geq 0 \quad (4.4)$$

with the aid of the facts that $v_n < \Psi_n$ on $B(x_n^0, \rho_n) \setminus \{x_n^0\}$ and $\overline{w}_n < v_n$ on $\mathcal{M}_n \subset B(x_n^0, \rho_n)$.

Cauchy–Schwarz inequality implies

$$\int_{\mathcal{M}_n} [\varphi_n(|\nabla v_n|) - \varphi_n(|\nabla \overline{w}_n|)] (|\nabla v_n| - |\nabla \overline{w}_n|) dx$$
$$\leq \int_{\mathcal{M}_n} \left[\frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|}\, \nabla v_n - \frac{\varphi_n(|\nabla \overline{w}_n|)}{|\nabla \overline{w}_n|} \nabla \overline{w}_n \right] \nabla (v_n - \overline{w}_n) dx$$

and combined with relation (4.4) leads to

$$\int_{\mathcal{M}_n} [\varphi_n(|\nabla \overline{w}_n|) - \varphi_n(|\nabla v_n|)] (|\nabla \overline{w}_n| - |\nabla v_n|) dx < 0$$

which is a contradiction with the statement that φ_n is an increasing function on $\mathbb{R}$. Actually, it follows that v_n is a viscosity subsolution of problem (1.1).

On the other hand, v_n is a viscosity supersolution of problem (1.1) with similar arguments as above adapted for this case and therefore, these details will be omitted. The proof of Lemma 4.4 is complete. □

By Lemma 4.2 we may select a subsequence $\{v_n\}$ that converges uniformly to v_∞ in Ω as $n \to \infty$. Next, we will focus to identify the limit equation verified by v_∞. The following theorem encloses the main result regarding the asymptotic behavior of the solutions $\{v_n\}$ of problem (1.1).

Theorem 4.5. *Let v_∞ be the function achieved as the uniform limit of a subsequence of $\{v_n\}$ in Lemma 4.2. Then v_∞ is a solution in the viscosity sense of problem*

$$\begin{cases} \min\{-\Delta_\infty v, |\nabla v| - 1\} = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{4.5}$$

Proof. First, we investigate if v_∞ is a viscosity supersolution of (4.5). We consider $y_0 \in \Omega$ and a test function $\Upsilon \in C^2(\Omega)$ such that $v_\infty - \Upsilon$ has a strict local minimum point at y_0. We claim that the uniform convergence of $\{v_n\}$ shown in Lemma 4.2 allows us to extract, up to a subsequence, $\{y_n\} \subset \Omega$ such that y_n converges to y_0 and moreover $v_n - \Upsilon$ achieves a strict local minimum point at y_n. Indeed, since y_0 is a strict minimum point of $v_\infty - \Upsilon$ it follows that $v_\infty(y_0) = \Upsilon(y_0)$ and $v_\infty(y) > \Upsilon(y)$ for every y in a punctured neighborhood of y_0, let's say $B(y_0, r) \setminus \{y_0\}$ with $r > 0$ fixed in such a manner that $B(y_0, 2r) \subset \Omega$. For any positive ρ with $\rho < r$ we get

$$\inf_{B(y_0,r)\setminus B(y_0,\rho)} (v_\infty - \Upsilon) > 0.$$

By the uniform convergence of $\{v_n\}$ to v_∞ in Ω and in particular in $\overline{B(y_0, r)}$, for any positive integer n sufficiently large, the function $v_n - \Upsilon$ attains its zero minimum value in $B(y_0, \rho)$ and thus, the minimum point of $v_n - \Upsilon$ will be represented by $y_n \in B(y_0, \rho)$. Considering a sequence $\rho_k \to 0^+$ as $k \to \infty$, we can construct a subsequence $\{n_k\}$ such that y_{n_k} converges to y_0 as $k \to \infty$. The claim now holds true after an appropriate relabelling of the indices. In other words, taking into account that $v_n, v_\infty \in C(\overline{\Omega})$ for any positive integer n sufficiently large, the uniform convergence of sequence $\{v_n\}$ to v_∞ in Ω implies that since Υ touches v_∞ from below at y_0, then there are points $y_n \to y_0$ such that

$$v_n(y) - \Upsilon(y) > 0 = v_n(y_n) - \Upsilon(y_n) \quad \text{for all } \; y \in B(y_0, \rho) \setminus \{y_0\}$$

for some subsequence (see [6, Theorem 3.1] or [30, Lemma 11]).

Keeping in mind that in view of Lemma 4.4, v_n is a continuous viscosity solution of (1.1) we have

$$-\frac{\varphi_n(|\nabla\Upsilon(y_n)|)}{|\nabla\Upsilon(y_n)|}\,\Delta\Upsilon(y_n) - \frac{|\nabla\Upsilon(y_n)|\varphi_n'(|\nabla\Upsilon(y_n)|) - \varphi_n(|\nabla\Upsilon(y_n)|)}{|\nabla\Upsilon(y_n)|^3}\,\Delta_\infty\Upsilon(y_n) \geq \lambda e^{\Upsilon(y_n)}\,. \tag{4.6}$$

Since $\lambda e^{\Upsilon(y_n)} > 0$ for any $\lambda \in (0, \lambda^*)$, it follows that $|\nabla\Upsilon(y_n)| > 0$ for each positive integer n. Recalling inequality (2.6) states

$$\min\{s^{\varphi_n^- - 1}, s^{\varphi_n^+ - 1}\}\varphi_n(t) \leq \varphi_n(st) \leq \max\{s^{\varphi_n^- - 1}, s^{\varphi_n^+ - 1}\}\varphi_n(t), \qquad \forall\, s, t \geq 0 \tag{4.7}$$

and keeping in mind (1.3), for each positive integer n sufficiently large, the functions $A_n, B_n : [0, \infty) \to \mathbb{R}$,

$$A_n(t) := \begin{cases} \dfrac{t\varphi_n'(t) - \varphi_n(t)}{t^3}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases} \qquad B_n(t) := \begin{cases} \dfrac{\varphi_n(t)}{t}, & \text{if } t > 0, \\ 0, & \text{if } t = 0 \end{cases}$$

are continuous. Moreover, function B_n is of class C^1 since $A_n(t) = t^{-1}B_n'(t)$ for $t > 0$. According to (1.2) and (1.3), we deduce that

$$\frac{|\nabla\Upsilon(y_n)|^3}{|\nabla\Upsilon(y_n)|\varphi_n'(|\nabla\Upsilon(y_n)|) - \varphi_n(|\nabla\Upsilon(y_n)|)} > 0.$$

Inequality (4.6) multiplied with the above positive quantity in both sides becomes

$$\begin{aligned} -\frac{\varphi_n(|\nabla\Upsilon(y_n)|)|\nabla\Upsilon(y_n)|^2}{|\nabla\Upsilon(y_n)|\varphi_n'(|\nabla\Upsilon(y_n)|) - \varphi_n(|\nabla\Upsilon(y_n)|)}\,\Delta\Upsilon(y_n) - \Delta_\infty\Upsilon(y_n) \\ \geq \frac{\lambda e^{\Upsilon(y_n)}|\nabla\Upsilon(y_n)|^3}{|\nabla\Upsilon(y_n)|\varphi_n'(|\nabla\Upsilon(y_n)|) - \varphi_n(|\nabla\Upsilon(y_n)|)}\,. \end{aligned} \tag{4.8}$$

On the other hand, we obtain

$$\frac{\varphi_n(|\nabla\Upsilon(y_n)|)|\nabla\Upsilon(y_n)|^2}{|\nabla\Upsilon(y_n)|\varphi_n'(|\nabla\Upsilon(y_n)|) - \varphi_n(|\nabla\Upsilon(y_n)|)} = \frac{|\nabla\Upsilon(y_n)|^2}{\dfrac{|\nabla\Upsilon(y_n)|\varphi_n'(|\nabla\Upsilon(y_n)|)}{\varphi_n(|\nabla\Upsilon(y_n)|)} - 1} \leq \frac{|\nabla\Upsilon(y_n)|^2}{\varphi_n^- - 2}, \tag{4.9}$$

where in the latter inequality we use Lieberman-type condition (1.2).

In relation (4.8) we pass to the limit as $n \to \infty$ and then using (1.3) we infer by relation (4.9) that

$$-\Delta_\infty\Upsilon(y_0) \geq \limsup_{n\to\infty} \frac{\lambda e^{\Upsilon(y_n)}|\nabla\Upsilon(y_n)|^3}{|\nabla\Upsilon(y_n)|\varphi_n'(|\nabla\Upsilon(y_n)|) - \varphi_n(|\nabla\Upsilon(y_n)|)} \tag{4.10}$$

which hints that

$$-\Delta_\infty \Upsilon(y_0) \geq 0. \tag{4.11}$$

In the following we will show that

$$|\nabla \Upsilon(y_0)| - 1 \geq 0. \tag{4.12}$$

If we assume by contradiction that is not the case of the above claim, we get $|\nabla \Upsilon(y_0)| - 1 < 0$, that implies $|\nabla \Upsilon(y_n)| < 1$ for any positive integer n sufficiently large. Taking into consideration (1.2) and then inequality (4.7) we arrive at

$$\begin{aligned}
\frac{\lambda e^{\Upsilon(y_n)}|\nabla \Upsilon(y_n)|^3}{|\nabla \Upsilon(y_n)|\varphi_n'(|\nabla \Upsilon(y_n)|) - \varphi_n(|\nabla \Upsilon(y_n)|)} &= \frac{|\nabla \Upsilon(y_n)|^3}{\dfrac{|\nabla \Upsilon(y_n)|\varphi_n'(|\nabla \Upsilon(y_n)|)}{\varphi_n(|\nabla \Upsilon(y_n)|)} - 1} \cdot \frac{\lambda e^{\Upsilon(y_n)}}{\varphi_n(|\nabla \Upsilon(y_n)|)} \\
&\geq \frac{|\nabla \Upsilon(y_n)|^3}{\varphi_n^+ - 2} \cdot \frac{\lambda e^{\Upsilon(y_n)}}{\varphi_n(|\nabla \Upsilon(y_n)|)} \\
&\geq \frac{|\nabla \Upsilon(y_n)|^3}{\varphi_n^+ - 2} \cdot \frac{\lambda e^{\Upsilon(y_n)}}{\varphi_n(1)|\nabla \Upsilon(y_n)|^{\varphi_n^- - 1}} \\
&= \left[\left(\frac{\lambda e^{\Upsilon(y_n)}}{(\varphi_n^+ - 2)\varphi_n(1)}\right)^{1/(\varphi_n^- - 4)} \frac{1}{|\nabla \Upsilon(y_n)|}\right]^{\varphi_n^- - 4}.
\end{aligned}$$

Since by (1.5) we have $\lim_{n\to\infty} \varphi_n(1)^{1/\varphi_n^-} = 1$ we get using (1.4) that

$$\lim_{n\to\infty} \left(\frac{\lambda e^{\Upsilon(y_n)}}{(\varphi_n^+ - 2)\varphi_n(1)}\right)^{1/(\varphi_n^- - 4)} = 1.$$

Next, taking into account that $\lim_{n\to\infty} \dfrac{1}{|\nabla \Upsilon(y_n)|} = \dfrac{1}{|\nabla \Upsilon(y_0)|} > 1$ we obtain

$$\lim_{n\to\infty} \left(\frac{\lambda e^{\Upsilon(y_n)}}{(\varphi_n^+ - 2)\varphi_n(1)}\right)^{1/(\varphi_n^- - 4)} \frac{1}{|\nabla \Upsilon(y_n)|} = \frac{1}{|\nabla \Upsilon((y_0)|} > 1$$

and then, we deduce that there exists $\epsilon_0 > 0$ such that

$$\left(\frac{\lambda e^{\Upsilon(y_n)}}{(\varphi_n^+ - 2)\varphi_n(1)}\right)^{1/(\varphi_n^- - 4)} \frac{1}{|\nabla \Upsilon(y_n)|} \geq 1 + \epsilon_0 \quad \text{for all positive integer } n \text{ sufficiently large,}$$

which yields to

$$\limsup_{n\to\infty} \frac{\lambda e^{\Upsilon(y_n)}|\nabla \Upsilon(y_n)|^3}{|\nabla \Upsilon(y_n)|\varphi_n'(|\nabla \Upsilon(y_n)|) - \varphi_n(|\nabla \Upsilon(y_n)|)} \geq \lim_{n\to\infty} (1 + \epsilon_0)^{\varphi_n^- - 4} = +\infty,$$

a contradiction with (4.10). Thus, inequality (4.12) holds true.

Assembling relations (4.11) and (4.12) we have $\min\{-\Delta_\infty \Upsilon(y_0), |\nabla \Upsilon(y_0)| - 1\} \geq 0$ which leads to the fact that v_∞ is a viscosity supersolution of (4.5).

Now, it remains to see that in fact v_∞ is a viscosity subsolution of (4.5). We take a test function $\Psi \in C^2(\Omega)$ that touches the graph of v_∞ from above in a point $x_0 \in \Omega$, that means $v_\infty(x_0) = \Psi(x_0)$ and $v_\infty(x) < \Psi(x)$ for every x in a punctured neighborhood of x_0 and we have to establish that $\min\{-\Delta_\infty\Psi(x_0), |\nabla\Psi(x_0)|-1\} \leq 0$. We notice that if $|\nabla\Psi(x_0)| = 0$ then we have $\Delta_\infty\Psi(x_0) = 0$ and everything is clear. Then, it is sufficient to check that if $|\nabla\Psi(x_0)| > 0$ and also

$$|\nabla\Psi(x_0)| - 1 > 0, \tag{4.13}$$

we get $-\Delta_\infty\Psi(x_0) \leq 0$. Actually, the uniform convergence of subsequence of $\{v_n\}$ ensures again, as in the first part of this proof, the existence of a sequence $x_n \to x_0$ as $n \to \infty$ such that $v_n - \Psi$ has a strict local maximum point at x_n and

$$-\frac{\varphi_n(|\nabla\Psi(x_n)|)|\nabla\Psi(x_n)|^2}{|\nabla\Psi(x_n)|\varphi_n'(|\nabla\Psi(x_n)|) - \varphi_n(|\nabla\Psi(x_n)|)}\Delta\Psi(x_n) - \Delta_\infty\Psi(x_n) \leq \frac{\lambda e^{\Psi(x_n)}|\nabla\Psi(x_n)|^3}{|\nabla\Psi(x_n)|\varphi_n'(|\nabla\Psi(x_n)|) - \varphi_n(|\nabla\Psi(x_n)|)}. \tag{4.14}$$

Passing to the limit as $n \to \infty$ in the above relation and using (4.13), inequality (4.7), and assumptions (1.3) and (1.5), we deduce that

$$-\Delta_\infty\Psi(x_0) \leq \liminf_{n\to\infty}\left[\left(\frac{\lambda e^{\Psi(x_n)}}{(\varphi_n^- - 2)\varphi_n(1)}\right)^{1/(\varphi_n^- - 4)}\frac{1}{|\nabla\Psi(x_n)|}\right]^{\varphi_n^- - 4} = 0$$

which implies that $-\Delta_\infty\Psi(x_0) \leq 0$. Thus, we conclude that v_∞ is a viscosity solution of problem (4.5). The proof of Theorem 4.5 is complete. □

Next, we identify the limit of the entire sequence of weak solutions $\{v_n\}$ of problem (1.1).

Proof of Theorem 4.1 (concluded). It is well-known that problem (4.5) has as unique viscosity solution $\operatorname{dist}(\cdot, \partial\Omega)$, namely the distance function to the boundary of Ω (see Jensen [25], or Juutinen [26, Lemma 6.10], or Ishibashi & Koike [24, p. 546]). As a consequence, Lemma 4.2 and Theorem 4.5 allow us to reach to the conclusion that the entire sequence $\{v_n\}$ converges uniformly to $\operatorname{dist}(\cdot, \partial\Omega)$ in Ω, as $n \to \infty$. □

Acknowledgements

The authors have been partially supported by CNCS-UEFISCDI Grant No. PN-III-P1-1.1-TE-2019-0456.

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